Development and Analysis of an Inventory System for Exponential Demand Items with Perishability and Life Time

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Abstract: In this paper an inventory model is developed and analyzed with exponential demand under the condition of permissible delay in payments. Deterioration rate is taken to be a constant fraction of the on hand inventory. In deterioration the realistic assumption of life period is taken into account. Shortages are allowed and they are taken as completely backlogged. Interest is earned on the unit cost of generated sales revenue which is to be deposited in an interest bearing account. Interest charges are paid on the stock held beyond the permissible period. Three different cases depending upon the permissible period are discussed.

Keywords: inventory, permissible, perishability. Exponential

I. Introduction

In most of the literature dealing with inventory problems, either in deterministic or probabilistic model, it is often assumed that payments will be made to the supplier for the goods immediately after receiving the consignment. However under most market behaviours, a vendor often provided buyers with a credit period to stimulate the demand, boost market share or decrease inventories of certain items. Vendor allows the buyers to pay at the end of specified period of time rather than immediately after receipt of the goods. Goyal (1985) first studied the EOQ model under conditions of permissible delay in payments. He assumed that the unit purchase cost is the same as the selling price per unit and concluded that the economic replenishment interval and order quality generally increases marginally under the permissible delay in payments. To accommodate more practical features of the real inventory system, Aggarwal and Jaggi (1995) extended the previous authors work. Shahs N.H (1993) present a lot size inventory model for exponentially decaying inventory when delay in payments is permissible.

Chung K.J. (2000) explores the inventory replenishment policy for deteriorating items under permissible delay in payments. The inventory system discussed by this paper is the same as that of Shah (1993). This paper shows that total variable cost per unit time is convex. A simple optimization procedure is developed to improve that described by Shah.

Dye (2002) presented a model in which demand is taken stock dependent and shortages are permitted and partially backlogged. Teng amended the Goyals model by considering the difference between unit price and unit cost. In Chung’s model account is settled at the end of the credit period while in Dye’s model account is settled when the shortages starts.

Chang (2004) developed a deterministic inventory model for deteriorating items with stock dependent demand and shortages. The replenishment number and fraction of each cycle in which there is no shortage are both determined so as to minimize the present value of inventory cost over a finite planning horizon.

Chandand Werd (1987) reinvestigated the Goyal’s model ( 1985 ) within the framework of the classical EOQ model. In the model considered by ‘Goyal’ it is assumed that no shortage of the stock is allowed to occur, replenishment of stock is instantaneous, the sales revenue is invested to earn interest only during the permissible settlement period and beyond this period, if stock remains it has to be financed at some interest rate, which in general differ from the rate of interest earned from the sales revenue. If the settlement period is less than the length of inventory cycle, than from remaining beyond settlement period, the sales revenue should also earn interest as it earned during the settlement period.

In this paper an inventory model is developed with exponential demand rates. Deterioration starts after a fix time. Such deterioration is termed as non instantaneous deterioration. Some credit period is provided in payments. Expressions are obtained for different parameters. Cost minimization technique is used to obtain the solution.
NOTATIONS:
C: Ordering cost of inventory per order.
C₁: Holding cost excluding interest charge per unit per unit time.
C₂: Shortage cost per unit per unit time.
C₃: Unit purchase cost.
Ir : Interest paid per rupee invested in stock s per year Ir>Ie.
Ie : Interest which can be earned per rupee per year.
q(t) : Inventory level at time t.
M : Permissible delay period for settling accounts in time 0<M<1.
t₁ : Time at which shortages starts.
T : Length of replenishment cycle.
μ: Life period of item at the end of which deterioration starts.
Q : Total amount of inventory produced or purchased at the beginning of each production cycle.
S(S<Q) : Initial amount of inventory after fulfilling back orders.
TC(t₁,T) : The total average cost of the inventory system per unit time.
TC₁(a)(t₁,T) : The total average cost of the inventory system per unit time for M≤t₁ and M≤μ.
TC₁(b)(t₁,T) : The total average cost of the inventory system for M≤t₁ and M≥μ.
TC₂(t₁,T) : The total average cost of the inventory system per unit time for M>t₁.
d: Total demand during the production cycle T.

ASSUMPTIONS:
The inventory system consists of single item only.
There is no repair or replacement of the deteriorated unit.
The replenishment occurs instantaneously at an infinite rate.
A constant fraction θ of the on hand inventory deteriorates per unit time only after the expiry of the life period μ of the item.
Hence the deterioration function θ can be taken in the following form-
θ = θ₀ H(t - μ), 0 < θ₀ << 1, t, μ > 0

Where H(t-μ) is heaviside unit function defined as follows

H(t - μ) \begin{cases} 
1, & t ≥ μ \\
0, & t < μ 
\end{cases} 

The demand rate function D(t) is deterministic and is exponential function of time. Demand rate is given by-

D(t) = \frac{d}{(e - 1)} e^{t/T} 

Shortages are allowed and they are completely backlogged.
During the fixed credit period M the unit cost of generated sales revenue is deposited in an interest bearing account. The difference between sales price and unit cost is retained by the system to meet the day to day expenses of the system. At the end of the credit period account to settled. Then interest is again earned during the period (M, t₁). If M≤t₁, interest charges are paid on the stock held beyond the permissible period.

II. Mathematical Model And Analysis Of The System:
Let Q be the total amount of inventory produced or purchased at the beginning of each production cycle. Let S(s< Q) be the initial inventory after fulfilling back orders. During the period [ μ, t₁] the inventory level decreases due to the market demand only. After this during the period [μ, t₁] the inventory level further decreases due to the combined effect of market demand and deterioration. At time t₁ the inventory level falls to zero and shortages starts. Demand is backlogged in the interval [ t₁, T ]. At time T when Q amount inventory is remaining amount S of inventory is left for the next replenishment cycle.
The differential equations showing variations of inventory level during the period [ 0, T ] are as follows-

\frac{d q(t)}{dt} = \frac{-d}{(e - 1) T} e^{t/T}, 0 ≤ t ≤ μ 

……(1)
\[
\frac{dq(t)}{dt} + \theta_0 q(t) = -\frac{d}{(e-1)T} e^{\frac{t}{\theta}} , \mu \leq t \leq t_1
\] 
\[\text{..(2)}\]
\[
\frac{d q(t)}{dt} = -\frac{d}{(e-1)T} e^{\frac{t}{\mu}} , \ t_1 \leq t \leq T
\] 
\[\text{...(3)}\]

The boundary conditions are:

\[
q(t) = S \quad \text{at} \quad t = 0,
\]
\[
q(t) = q(S), \quad \text{at} \quad t = \mu
\]
and
\[
q(t) = 0 \quad \text{at} \quad t = t_1
\]

The solution of equation (1) is given by

\[
q(t) = -\frac{d}{(e-1)} e^{\frac{t}{\mu}} + A
\]

When A is a constant of integration. Using boundary condition at t=0, q (t) = s, we get-

\[
A = S + \frac{d}{(e-1)}
\]

\[
q(t) = S + \frac{d}{(e-1)} [-e^{\frac{t}{\mu}} + 1], \quad 0 \leq t \leq \mu
\] 
\[\text{.....(4)}\]

Apply another boundary condition at t = \mu

\[
q(\mu) = S + \frac{d}{(e-1)} (1 - e^{\frac{\mu}{T}})
\] 
\[\text{.....(5)}\]

The solution of equation (2) is given by-

\[
q(t)e^{\theta_0 t} = \left[ -\frac{d}{(e-1)(1 + \theta_0 T)} e^{(1+\theta_0 T)\frac{t}{\mu}} \right] + B
\]

here B is constant of integration.

Using boundary condition at t = \mu, we get

\[
B = \frac{d}{(e-1)} e^{\theta_0 \mu} (1 - e^{\frac{\mu}{T}}) + Se^{\theta_0 \mu}
\]

Therefore

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\[ q(t)e^{\theta_0 t} = \frac{-d}{(e-1)(1+\theta_0 T)} \left[ e^{(1+\theta_0 T)\frac{t}{T}} - e^{(1+\theta_0 T)\frac{\mu}{T}} \right] + \frac{d}{(e-1)} e^{\theta_0 \mu} (1-e^{\frac{\mu}{T}}) + Se^{\theta_0 \mu} \]

\[ ......(6) \]

Now applying the boundary condition at \( t = t_1 \), \( q(t) = 0 \) in equation (6), we get

\[ S = \frac{d}{(e-1)(1+\theta_0 T)} \left[ e^{(1+\theta_0 T)\frac{t_1}{T}} - e^{(1+\theta_0 T)\frac{\mu}{T}} \right] - \frac{d}{(e-1)} (1-e^{\frac{\mu}{T}}) \]

\[ ......(7) \]

By the substitution of the value of \( S \) in the equation (6) reduces to-

\[ q(t)e^{\theta_0 t} = \frac{d}{(e-1)(1+\theta_0 T)} \left[ e^{(1+\theta_0 T)\frac{t}{T}} - e^{(1+\theta_0 T)\frac{\mu}{T}} \right] \]

Hence

\[ q(t) = \frac{d}{(e-1)(1+\theta_0 T)} \left[ e^{(1+\theta_0 T)\frac{t}{T}} - e^{\frac{\mu}{T}} \right], \quad \mu \leq t \leq t_1 \]

\[ ......(8) \]

Solution of equation (3) using boundary condition at \( t = t_1 \), \( q(t) = 0 \) is given by

\[ q(t) = \frac{d}{(e-1)} \left( e^{\frac{t}{T}} - e^{\frac{\mu}{T}} \right), \quad t_1 \leq t \leq T \]

\[ ......(9) \]

Now the total amount of carrying units (\( q_H \)) during the period \((0, t_1)\) is given by-

\[ q_H = \int_0^\mu q(t)dt + \int_{\mu}^{t_1} q(t)dt \]

\[ = \int_0^\mu \left\{ \frac{d}{(e-1)} (1-e^{\frac{t}{T}}) + S \right\} dt + \]

\[ \int_{\mu}^{t_1} \left[ \frac{d}{(e-1)(1+\theta_0 T)} \left( e^{(1+\theta_0 T)\frac{t}{T}} - e^{(1+\theta_0 T)\frac{\mu}{T}} \right) \right] dt \]

or
\[ q_H = \frac{d}{(e-1)} (T + e^{\mu/T}(\mu - T)) + \frac{d}{(e-1)(1 + \theta_0 T)} \]

\[ e^{\left\{ (1+\theta_0 T)\frac{t_1}{T} - \frac{\theta_0 \mu}{\theta_0} \right\}} (\mu + \frac{1}{\theta_0} ) + e^{\mu/T}(T - \mu) - e^{\frac{t_1}{T}}(T + \frac{1}{\theta_0}) \]

\[ \ldots \ldots (10) \]

The total amount of deterioration units \( q_D \) during the period \((0, t_1)\) is given by:

\[ q_D = q(\mu) - \int_{\mu}^{t_1} \frac{d}{(e-1)T} e^{\mu/T} dt \]

\[ = \frac{d}{(e-1)} \left( 1 - e^{\mu/T} \right) + S \frac{d}{(e-1)} \left( e^{\frac{t_1}{T}} - e^{\mu/T} \right) \]

Using the value of \( S \), one can get

\[ q_D = \frac{d}{(e-1)} \left[ e^{-\theta_0 \mu} \left( e^{(1+\theta_0 T)\frac{t_1}{T}} - e^{(1+\theta_0 T)\frac{\mu}{T}} \right) \right. \]

\[ \left. - e^{\frac{t_1}{T} + e^{\frac{\mu}{T}}} \right] \ldots \ldots (11) \]

Total Amount of the shortage units \( q_S \) during the period \((t_1, T)\) can be obtained as

\[ q_S = -\int_{t_1}^{T} q(t) \ d(t) \]

\[ = -\int_{t_1}^{T} \frac{d}{(e-1)} \left[ e^{\frac{t_1}{T}} - e^{\frac{t}{T}} \right] d(t) \]

\[ = \frac{d}{(e-1)} \left[ T \left( e^{1/t} - e^{t_1/T} \right) - e^{\frac{t_1}{T}}(T - t_1) \right] \ldots \ldots (12) \]

Now there are two possibilities regarding the period \( M \) of permissible delay in payments

**CASE(I)**  \( M \leq t_1 \)

**CASE(II)**  \( M > t_1 \)

We first discuss case(I)

**CASE(I)**  \( M \leq t_1 \)

In this case there are further two possibilities as follows

**CASE I(A)**  \( M \leq \mu \leq t_1 \)
CASE I(B) \( \mu \leq M \leq t_1 \)

We first discuss case I(A)

CASE I(A):
Since here the length of period with positive inventory stock is larger than the credit period \( M \), the buyer can use sale revenue to earn the interest with an annual rate \( I_e \) during the period \([0, M]\). The unit cost of the generated sales revenue is deposited in an interest bearing account. The difference between sales price and unit cost is retained by the system to meet the day to day expenses of the system. At the end of the credit period, the account is settled. After setting the account at time \( M \) again the unit cost of generated sales revenue is deposited in an interest bearing account to earn interest with an annual rate \( I_e \) during the period \([M, t_1]\). Beyond the fixed credit period product still in stock is assumed to be financed with an annual rate \( I_r \). Now the total interest earned \( IE_{1(a)} \) during the period \([0, t_1]\) is given by-

\[
IE_{1(a)} = C_3 I_e \left[ \int_0^M (M-t) \frac{d}{(e-1)T} e^{t/T} dt + \int_M^{t_1} (t_1-t) \frac{d}{(e-1)T} e^{t/T} dt \right]
\]

\[
= C_3 I_e \frac{d}{(e-1)T} \left[ \left\{ MT e^{t_1/T} - T t e^{t_1/T} + T^2 e^{t_1/T} \right\}^M_0 + \right.
\]

\[
\left. \left\{ t_1 T e^{t_1/T} - T t e^{t_1/T} + T^2 e^{t_1/T} \right\}^M_{t_1} \right]
\]

\[
= C_3 I_e \frac{d}{(e-1)T} \left[ T(e^{M/T} - 1) + T(e^{t_1/T} - e^{M/T}) \right]
\]

\[
= C_3 I_e \frac{d}{(e-1)} \left[ e^{t_1/T} - M - 1 \right]
\] ....(13)

Total interest payable \( IP_{1(a)} \) is given by

\[
IP_{1(a)} = C_3 I_r \int_{M}^{t_1} q(t) dt
\]

\[
= C_3 I_r \left[ \int_{M}^{\mu} q(t) dt + \int_{M}^{t_1} q(t) dt \right]
\]

\[
= C_3 I_r \left[ \int_{M}^{\mu} \left\{ \frac{d}{(e-1)} (1 - e^{t_1/T}) + S \right\} dt + \right]
\]
\[
\int_{\mu}^{t_1} \frac{d}{(e-1)(1 + \theta_0 T)} \left( e^{(1+\theta_0 T)\frac{t_1}{T}} - e^{(1+\theta_0 T)\frac{T}{T}} \right) dt \\
= C_3 I_r \left\{ \frac{d}{(e-1)} \left( (\mu - M) - T \left( e^{\frac{\mu}{T}} - e^{\frac{M}{T}} \right) + S(\mu - M) \right) \right\} + \\
\left\{ \frac{d}{(e-1)(1 + \theta_0 T)} \left[ \frac{\mu}{T} - e^{\frac{\mu}{T}} \left( \frac{1}{\theta_0} + T \right) + \frac{e^{(1+\theta_0 T)\frac{t_1}{T}} - \theta_0 \mu}{\theta_0} \right] \right\} \\
\text{The substitution of the value of } S \text{ it reduces to-} \\
I_{P_1(a)} = C_3 I_r \left\{ \frac{d}{(e-1)} \left( (\mu - M) - T \left( e^{\frac{\mu}{T}} - e^{\frac{M}{T}} \right) \right) \right\} + (\mu - M) \\
\left\{ \frac{d}{(e-1)(1 + \theta_0 T)} \left[ e^{(1+\theta_0 T)\frac{t_1}{T}} - e^{(1+\theta_0 T)\frac{T}{T}} - \frac{d}{(e-1)} \left( 1 - e^{\frac{\mu}{T}} \right) \right] \right\} + \\
\left\{ \frac{d}{(e-1)(1 + \theta_0 T)} \left[ \frac{T}{e^{\frac{\mu}{T}}} - e^{\frac{T}{T}} \left( \frac{1}{\theta_0} + T \right) + \frac{e^{(1+\theta_0 T)\frac{t_1}{T}} - \theta_0 \mu}{\theta_0} \right] \right\} \\
\text{......(14)} \\
\text{Now the total average cost of the system in consideration is given by} \\
TC_{1(a)}(t_1, T) = \frac{C + C_1 q_H + C_3 q_D + C_2 q_S + I_{P_1(a)} - I E_{1(a)}}{T} \\
= \frac{C}{T} + \frac{C_1}{T} \left\{ \frac{d}{(e-1)} \left( T + e^{\frac{\mu}{T}} (\mu - T) \right) \right\} + \frac{d}{(e-1)(1 + \theta_0 T)} \left\{ e^{(1+\theta_0 T)\frac{t_1}{T}} - \theta_0 \mu \left( \mu + \frac{1}{\theta_0} \right) + e^{\frac{\mu}{T}} (T - \mu) - e^{\frac{t_1}{T}} \right\}
\]
To minimize the total average cost per unit time $T_{C_1(a)}(t_1, T)$ the optimal values of $t_1$ and $T$ (say $t_1^*$ and $T^*$) can be obtained by solving the following two equations simultaneously:

$$
\frac{\partial T_{C_1(a)}(t_1, T)}{\partial t_1} = 0
$$

and

$$
\frac{\partial T_{C_1(a)}(t_1, T)}{\partial T} = 0
$$

provided they satisfy the sufficient conditions: 

$$
... (15)
$$

$$
... (16)
$$

$$
... (6.17)
$$
\[
\frac{\partial^2 TC_{1(a)}(t_1, T)}{\partial t_1^2} > 0
\]
\[
\frac{\partial^2 TC_{1(a)}(t_1, T)}{\partial T^2} > 0
\]
and
\[
\left(\frac{\partial^2 TC_{1(a)}(t_1, T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TC_{1(a)}(t_1, T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC_{1(a)}(t_1, T)}{\partial t \partial T}\right)^2 > 0
\]

Equation (6.16) and (6.17) are equivalent to-

\[
\frac{C_1}{T} \frac{d}{(e-1)} \left[ \frac{1}{T} e^{(1 + \theta_0 T) \frac{t_1}{T} - \theta_0 M} \left(\mu + \frac{1}{\theta_0}\right) - \frac{1}{T} e^{\frac{t_1}{T}} \left(T + \frac{1}{\theta_0}\right) \right]
\]
\[
\frac{(t_1 - T)}{T} e^{\frac{t_1}{T}} \right] + \frac{C_3 I_e d}{T(e-1)} \left[ (\mu - M) \left\{ \frac{e^{(1 + \theta_0 T) \frac{t_1}{T} - \theta_0 M}}{T} \right\} \right.
\]
\[
- \frac{1}{(1 + \theta_0 T)} \left\{ e^{\frac{t_1}{T}} \frac{(1 + \theta_0 T)}{\theta_0 T} \right\} - \frac{C_3 I_e d}{T^2(e-1)} e^{\frac{t_1}{T}} = 0
\]

\[\ldots(18)\]

and

\[
\left(\mu + \frac{1}{\theta_0}\right) \left[ -\frac{T - 2\theta_0 T^2 - t(1 + \theta_0 T)}{T^3(1 + \theta_0 T)} \right] e^{(1 + \theta_0) \frac{t_1}{T} - \theta_0 M}
\]
\[
\left[ -\theta_0 T^2 + 2\mu \theta_0 T + T^2 - \mu T + \mu + \theta_0 T^3 - \mu \theta_0 T^2 \right] e^{\frac{t_1}{T}} \]
\[
\begin{align*}
&+ \left[ \frac{\theta_0^2 T^4 + \theta_0 T^3 + T + \theta_0 T^2 + t_1 T + \theta_0 T^2 t_1 + 1 + \theta_0 T}{(\theta_0 T^3)(1 + \theta_0 T)} \right] e^{t_1/T} \\
&- \frac{C_3}{T^3 (e-1)} \left[ \frac{e^{-\theta_0 \mu}}{(1 + \theta_0 \mu)} \left\{ T + 2 \theta_0 T^2 + t_1 (1 + \theta_0 T) \right\} e^{(1 + \theta_0 T) t_1/T} \\
&- \left( T + 2 \theta_0 T^2 + \mu (1 + \theta_0 T) \right) e^{(1 + \theta_0 T) \mu/T} \right) + (t - T) e^{t_1/T} \\
&+ (T - \mu) e^{\mu/T} \right] + \frac{C_2 d}{(e-1)T^2} (T - 1) e^{t_1/T} + \frac{C_3 I_r d}{(e-1)} \\
&\left[ \frac{1}{T^2} \left( \mu e^{\mu/T} - M e^{M/T} \right) \right] + \left( \mu - M \right) \left[ \frac{e^{-\theta_0 \mu}}{T^3 (1 + \theta_0 T)^2} \\
&\left\{ -2 \theta_0 T^2 - T - t_1 (1 + \theta_0 T) \right\} e^{(1 + \theta_0 T) \mu/T} + \left( 2 \theta_0 T^2 + T + \mu (1 + \theta_0 T) \right) \right] \\
&e^{(1 + \theta_0 T) \mu/T} \right] + \frac{1}{T^2 (1 + \theta_0 T)^2} \left\{ (T - \mu) (1 + \theta_0 T)^2 - \theta_0 T - \mu (1 + \theta_0 T) \right\} e^{\mu/T} \\
&+ \frac{1}{\theta_0 T^3} \left\{ T (\theta_0 T + 1) + t_1 (1 + \theta_0 T) \right\} e^{t_1/T} + \left( \theta_0^2 T^2 - 1 - \theta_0 T \right) \\
&e^{(1 + \theta_0 T - \theta_0 \mu)} \right] + \frac{C_3 I_e d}{T^2 (e-1)} \left[ e^{t_1/T} (T - t) e^{t_1/T} - M - 1 \right] = 0 \\
\end{align*}
\]

To obtain the optimal values of \( t_1 \) and \( T \) that minimizes \( TC_{1(a)} \) \(( t_1, T)\) one have to develop the following algorithm.

**Algorithm 1(A)**

**STEP 1-**
(I) Start with \( t_1 = M \)
(II) Substitute \( t_1(1) \) in equation (18) to obtain \( T(1) \)
(III) Using \( T(1) \) determines \( t_1(2) \) from equation (19)
(IV) Repeat (II) and (III) until no change occurs in the value of \( t_1 \) and \( T \).

**STEP 2-To compare \( t_1 \) and \( M \)**
(I) If \( M \leq t_1 \), \( t_1 \) is feasible than go to step (3).
(II) If \( M > t_1 \), \( t_1 \) is not feasible set \( t_1 = M \) and evaluate the corresponding values of \( T \) from equation (19) and then go to the step (3).
STEP 3: Calculate the corresponding.

\[ TC_{(b)} ( t_1^*, T^* ) \]

CASE I(B): \( \mu \leq M \leq t_1 \)

When \( M \geq \mu \) and \( M \leq t_1 \)

This case is similar to case 1(a). But as \( M > \mu \) the interest earned during \( IE_{1(a)} [0, t_1] \) is given by:

\[
IE_{1(b)} = C_3 I e \left[ \int_0^M (M-t) \frac{d}{(e-1)T} e^{\frac{t}{T}} dt + \int_M^{t_1} \frac{d}{(e-1)T} e^{\frac{t}{T}} dt \right]
\]

\[
= \frac{C_3 I e}{(e-1)} \left[ e^{\frac{t_1}{T}} - M - 1 \right]
\]

.....(20)

Also the interest payable \( IP_{t_{(a)}} \) for the period \( (M, t_1) \) is given by

\[
IP_{1(b)} = C_3 I e \int_M^{t_1} q(t) dt
\]

\[
= C_3 I e \int_M^{t_1} \frac{d}{(e-1)(1+\theta_0 T)} \left[ \left( e^{(1+\theta_0 T)\frac{t_1}{T} - \theta_0 T} - e^{\frac{t}{T}} \right) \right] dt
\]

\[
= C_3 I e \left[ \frac{d}{(e-1)(1+\theta_0 T)} \left[ e^{(1+\theta_0 T)\frac{t_1}{T} - \theta_0 T} - \theta_0 (e^{\frac{t_1}{T}} - e^{\frac{t_1}{T} + \theta_0(t_1-M)}) \right] \right]
\]

\[
- T \left[ e^{\frac{t_1}{T}} - e^{\frac{M}{T}} \right]
\]

.....(21)

Hence total average cost \( TC_{1(b)} ( t_1, T ) \) in this case is given by:

\[
TC_{1(b)} ( t_1, T ) = \frac{C + C_1 q_H + C_3 q_D + C_2 q_S + IP_{1(b)} - IE_{1(b)}}{T}
\]

\[
= \frac{C + C_1 \frac{d}{T (e-1)}}{T} \left[ T + e^{\frac{\mu}{T}(\mu-T)} \right] + \frac{1}{1+\theta_0 T} \left( \mu + \frac{1}{\theta_0} \right)
\]
\[
e^{\left(1+\theta_0 T\right)\frac{t_1}{T}-\frac{\theta_0 T}{\theta_0}} + e^{\frac{\mu}{T}(T-\mu)} - e^{\frac{t_1}{T}\left(T + \frac{1}{\theta_0}\right)}
\]
\[
+ \frac{C_3d}{T(e-1)}\left[ e^{-\frac{\theta_0 T}{\theta_0}} \right] \left\{ e^{\left(1+\theta_0 T\right)\frac{t_1}{T}} - e^{\left(1+\theta_0 T\right)\frac{\mu}{T}} \right\}
\]
\[
- e^{\left(\frac{t_1}{T} + \frac{\mu}{T}\right)} + \frac{C_2d}{T(e-1)}\left[ e^{\frac{t_1}{T}} - e^{\frac{t_1}{T}(T-t_1)} \right]
\]
\[
+ \frac{C_3I_e d}{T(e-1)(1+\theta_0 T)}\left[ e^{\frac{t_1}{T} + \frac{\theta_0 (T-M)}{T} - \frac{t_1}{T}} \right]
\]
\[
- T\left( e^{\frac{t_1}{T} - e^{\frac{M}{T}}} \right) - \frac{C_3I_e d}{T(e-1)}\left[ e^{\frac{t_1}{T} - M-1} \right]
\]

Now to minimize the total average cost per year per unit time \(TC_{1(b)}(t_1, T)\) the optimal values of \(t_1\) and \(T\) (say \(t_{1*}\) and \(T^*\)) can be obtained by solving the following two equations simultaneously.

\[
\frac{\partial TC_{1(b)}(t_1, T)}{\partial t_1} = 0 \quad \text{......(23)}
\]

and

\[
\frac{\partial TC_{1(b)}(t_1, T)}{\partial T} = 0 \quad \text{......(24)}
\]

Provided they satisfy the sufficient conditions-

\[
\frac{\partial^2 TC_{1(b)}(t_1, T)}{\partial t_1^2} > 0
\]

\[
\frac{\partial^2 TC_{1(b)}(t_1, T)}{\partial T^2} > 0
\]
\[
\left( \frac{\partial^2 TC_{1(b)}(t_1, T)}{\partial t_1^2} \right) \left( \frac{\partial^2 TC_{1(b)}(t_1, T)}{\partial T^2} \right) - \left( \frac{\partial^2 TC_{1(b)}(t_1, T)}{\partial t_1 \partial T} \right)^2 > 0
\]

Again equation (6.23) and (6.24) are equivalent to

\[
\frac{d}{T(e-1)} e^{t_1/T} \left[ \frac{C_1}{\theta_0 T} \left\{ T (1 + \theta_0 \mu) e^{\theta_0 (t_1 - \mu)} - 1 \right\} + \frac{C_3 (\theta_0 + I_r)}{\theta_0} \left\{ e^{\theta_0 (t_1 - \mu)} - 1 \right\} + \frac{C_2 (t_1 - T)}{T} - \frac{C_3 I e}{T} \right] = 0
\]

......(25)

And

\[
-\frac{C}{T_2} + \frac{dC_1}{T(e-1)} \left[ \frac{(1 + \theta_0 \mu)}{(1 + \theta_0 T)} e^{(1 + \theta_0 T) t_1/T - \theta_0 \mu}
\right.
\]

\[
\left. \left\{ \frac{-(1 + \theta_0 T)T - \theta_0 T^2}{(1 + \theta_0 T)T \cdot \theta_0 T^2} - t_1 (1 + \theta_0 T) \right\} + e^{t_1/T} \left\{ \frac{(T + t_1)}{\theta_0 T^2} \right\} \right.
\]

\[
\left. + \frac{\theta_0 e^{\mu/T}}{T(1 + \theta_0 T)^2} \left[ T(M - T) - \mu^2 (1 + \theta_0 T) \right] + \frac{C_3 d}{T(e-1)} \left[ \frac{-e^{-\theta_0 \mu}}{T^2(1 + \theta_0 T)^2} \right] \right.
\]

\[
\left. e^{(1 + \theta_0 T) t_1/T} \left\{ (1 + \theta_0 T)(T + T_1) + \theta_0 T^2 \right\} + \frac{e^{-\theta_0 \mu}}{T^2(1 + \theta_0 T)^2} \right.
\]

\[
\left. e^{(1 + \theta_0 T) \mu/T} \left\{ (1 + \theta_0 T)(\mu + T) + \theta_0 T^2 \right\} \right. + \frac{e^{t_1/T}}{T^2} (T + t_1)
\]

\[
- \frac{e^{\mu/T}}{T^2} (\mu + T) \right) + \frac{C_2 d}{T_3(e-1)} e^{t_1/T} (T - t_1) + \frac{C_3 I r d}{T(e-1)(1 + \theta_0 T)}
\]

\[
\left. \left[ -e^{t_1/T} \frac{\theta_0 (t_1 - M)}{\theta_0 T} \left\{ \left( \frac{1 + 2 \theta_0 T}{(1 + \theta_0 T)} \right) + \frac{t_1}{T} \right\} + \left( 1 + 2 \theta_0 T \right) \frac{t_1}{T \theta_0} e^{t_1/T} \right] \right.
\]

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To obtain the optimal values of $t_1$ and $T$ we develop the following algorithms.

**Algorithm 1(B):**

**STEP 1**

(I) Start with $t_1 = M$

(II) Substitute $t_1(1)$ into equation (25) to evaluate $T(1)$

(III) Using $T(1)$ to determine $t_1(2)$ from equation (26)

(IV) Repeat (II) and (III) until no change occurs in the value of $t_1$ and $T$.

**STEP 2**

(I) To compare $t_1$ and $M$

(II) If $M \leq t_1$, then $t_1$ is feasible than go to step (3).

(II) If $M > t_1$, then $t_1$ is not feasible. Set $t_1 = M$ and evaluate the corresponding values of $T$ from equation (26) and then go to the step (3).

**STEP 3**

Compute the corresponding $TC_{1(b)}(t_1^*, T^*)$

Now to discuss the next case when $t_1 < M$

**CASE 2**

In this situation since $t_1 < M$ the buyer pays no interest and earns the interest during the period $[0, M]$. The interest earned in this case is denoted by $IE_2$ and it is given by:

$$IE_2 = C_3 I_e \int_0^{t_1} (M - t) \frac{d}{(e - 1)T} e^{t/T} t$$

$$= C_3 I_e \frac{d}{(e - 1)} \left[ T(M - t)e^{t/T} - e^{t/T} \right]_0^{t_1}$$

$$= C_3 I_e \frac{d}{(e - 1)} \left[ T(M - t) - 1 \right] \left( e^{t_1/T} - 1 \right)$$

$$\ldots \ldots (27)$$

Hence the total average cost per unit time $TC_2(t_1, T)$ in the present case is given by:

$$TC_2(t_1, T) = \frac{C + C_1 q_H + C_3 q_D + C_2 q_S - IE_2}{T}$$
\[
C + \frac{C_1}{T} \left[ \frac{d}{e-1} \left\{ T + e^{\mu/T} (\mu - T) \right\} + \frac{d}{(e-1)(1 + \theta_0T)} \right].
\]

\[
e^{(1+\theta_0T)^{t_1/T}-\theta_0\mu} \left( \mu + \frac{1}{\theta_0} \right) + e^{\mu/T} (T - \mu) - e^{t_1/T} \left( T + \frac{1}{\theta_0} \right) \right]
\]

\[
+ \frac{C_3}{T} \left[ \frac{d}{e-1} \left\{ \frac{e^{-\theta_0\mu}}{(1 + \theta_0T)} \left( e^{(1+\theta_0T)^{t_1/T}} - e^{(1+\theta_0T)^{t_1/T}} \right) \right\} \right]
\]

\[
- e^{t_1/T} + e^{\mu/T} \right] + \frac{C_2}{T} \frac{d}{e-1} \left[ e^{t_1/T} - e^{t_1/T} (T - t_1) \right]
\]

\[
- \frac{C_3 I_c d}{(e-1)T} \left\{ T(M - t) - 1 \right\} \left\{ e^{t_1/T} - 1 \right\} \right] \]

For minimizing total average cost per year per unit time \( T_{C2}(t_1, T) \) the optimal values of \( t_1 \) and \( T \) (say \( t_1^* \) and \( T^* \)) can be obtained by solving the following two equations simultaneously.

\[
\frac{\partial T_{C2}(t_1, T)}{\partial t_1} = 0 \]

\[
\frac{\partial T_{C2}(t_1, T)}{\partial T} = 0
\]

Provided they satisfy the sufficient conditions-

\[
\frac{\partial^2 T_{C2}(t_1, T)}{\partial t_1^2} > 0
\]

\[
\frac{\partial^2 T_{C2}(t_1, T)}{\partial T^2} > 0
\]
\[ \left( \frac{\partial^2 TC_2(t_1, T)}{\partial t_1^2} \right) \left( \frac{\partial^2 TC_2(t_1, T)}{\partial T^2} \right) - \left( \frac{\partial^2 TC_2(t_1, T)}{\partial t_1 \partial T} \right)^2 > 0 \]

Equation (29) and (30) are equivalent to:

\[
\frac{C_1}{T} \frac{d}{(e-1)} \left[ \frac{1}{(1+\theta_0 t)} \left( \mu + \frac{1}{\theta_0} \right) e^{(1+\theta_0 T) t_1/T - \theta_0 \mu} - e^{t_1/T} \left( T + \frac{1}{\theta_0} \right) \frac{1}{T} \right] + \frac{C_3}{T} \frac{d}{(e-1)} \left[ \frac{e^{-\theta_0 \mu}}{(1+\theta_0 T)} \left\{ e^{(1+\theta_0 T) t_1/T} \left( 1 + \theta_0 T \right) - \frac{1}{T} e^{t_1/T} \right\} \right] + \frac{C_2 d}{T(e-1)} \left[ e^{t_1/T} \left( \frac{t_1}{T} - 1 \right) \right] - C_3 I e d \frac{T(M-T)-1}{(e-1)T} e^{t_1/T} \frac{1}{T} = 0
\]

\[ \ldots (31) \]

and

\[
- \frac{C}{T^2} - \frac{C_1}{T^2} \left[ \frac{d}{(e-1)} \left\{ T + e^{\mu/T} (\mu - T) \right\} + \frac{d}{(e-1)(1+\theta_0 T)} \right] e^{(1+\theta_0 T) t_1/T - \theta_0 \mu} \left( \mu + \frac{1}{\theta_0} \right) + e^{\mu/T} (T-\mu) - e^{t_1/T} \left( T + \frac{1}{\theta_0} \right) \]

\[
+ \frac{C_1}{T} \left[ \frac{d}{(e-1)} \left\{ 1 - e^{\mu/T} \left( \frac{\mu}{T^2} \right) (\mu - T) - e^{\mu/T} - \frac{d \theta_0}{(e-1)(1+\theta_0 T)} \right\} \right] e^{(1+\theta_0 T) t_1/T - \theta_0 \mu} \left( \mu + \frac{1}{\theta_0} \right) + e^{\mu/T} (T-\mu) - e^{t_1/T} \left( T + \frac{1}{\theta_0} \right) \]

\[
+ \frac{d}{(e-1)(1+\theta_0 T)} \left\{ e^{(1+\theta_0 T) t_1/T - \theta_0 \mu} \left( \mu + \frac{1}{\theta_0} \right) \left( -\frac{t_1}{T^2} \right) + e^{\mu/T} + (T-\mu) \right\} e^{\mu/T} \left( -\frac{\mu}{T^2} \right) - e^{t_1/T} - \left( T + \frac{1}{\theta_0} \right) \left( -\frac{t_1}{T^2} \right) \right\} \]
+ $\frac{C_3}{(e-1)} \left[ \frac{d}{T^2} \left\{ T(0+T)^T - e^{(1+T)} \right\} \right]$

Now we use the following algorithm to obtain the optimal values.

**Algorithm 2**

**STEP 1**

(I) Start with $(t_0_i) = M$

(II) Substituting $t_{(0)} = M$ into equation (31) to evaluate $T_{(i)}$

(III) Using $T_{(i)}$ to determine $t_{(2)}$ from equation (32)

(IV) Repeat (II) and (III) until no change occurs in the value of $t_1$ and $T$

**STEP 2**

Compare $t_1$ and $M$

(I) If $t_1 < M$, $t_1$ is feasible than go to step (3).

(II) If $t_1 \geq M$, $t_1$ is not feasible. Set $t_1 = M$ and evaluate the corresponding values of $T$ from equation (32) and then go to the step (3).

**STEP 3**

As stated earlier the objective of this problem is to determine the optimal values of $t_1$ and $T$ so that $TC (t_1, T)$ is minimum. As the discussion carried out so far one can get-

$$TC \left(t_1^*, T^*\right) = \text{Min} \left\{ TC_{1(a)} \left(t_1^*, T^*\right), TC_{1(b)} \left(t_1^*, T^*\right), TC_2 \left(t_1^*, T^*\right) \right\}$$
III. Conclusion

In the present discussion an inventory model is developed and analyzed with exponential demand under the condition of permissible delay in payments. Deterioration rate is taken to be a constant fraction of the on hand inventory. In deterioration the realistic assumption of life period is taken into account. Shortages are allowed and they are taken as completely backlogged. Interest is earned on the unit cost of generated sales revenue which is to be deposited in an interest bearing account. Interest charges are paid on the stock held beyond the permissible period. Three different cases depending upon the permissible period \( 0 < M < \mu \leq t_1 \), \( 0 \leq \mu \leq M \leq t_1 \), and \( M > t_1 \) are discussed. In each case equations are derived for total average cost of the system per unit time. An algorithm for each case is given to obtain the value of optimal shortage time \( t_1^* \) and total cycle period \( T^* \).

References


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