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Review of Controllability Results of Dynamical System

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Abstract: *This paper contains the reviews and descriptions of fundamental results concerning the solutions of some popular linear continuous-time control models with constant coefficients. Different types of stability results are discussed. Fundamental definitions on controllability for finite-dimensional systems are recalled and necessary and sufficient conditions for different types of controllability were given. The paper was concluded, by remarks and comments concerning possible extensions.*

Key words: *Distributed parameter systems, controllability, linear system, stability.*

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I. Introduction

Controllability is one of the fundamental concepts in modern Mathematical control theory. Controllability is a qualitative property of control systems and is of particular importance in control theory. Serious study of controllability started many years ago and theory of controllability was based on the Mathematical description of dynamical systems. Many dynamical systems are such that the control does not affect the complete state of the dynamical system, but only a part of it. On the other hand, very often in the real industrial processes, it is possible to observe only a certain part of the complete state of a dynamical system.

Roughly speaking, controllability generally means that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability plays an essential role in the development of the modern Mathematical control theory. There are important relationships between controllability, stability and stabilizability of linear control systems [1] [2].

Moreover, it should be pointed out that there is a formal duality relationship between the concept of controllability and observability [3].

In the literature, there are many different definitions of controllability and stability which depend on the type of dynamical control systems. The main aim of this work is to review most of the existing controllability results for the linear continuous-time invariant control systems. It should be pointed out, that for linear control dynamical systems, the most popular controllability conditions are purely algebraic forms and hence are rather easily computable. In this case, the conditions require verification of the rank conditions for suitable defined constant controllability matrix.

We want to prosecute this review work in the following order. In section 2, systems descriptions and fundamental results concerning the solution of the most popular used linear continuous-time control model with constant coefficient were discussed. In section 3, we present fundamental definitions of controllability and the most frequently use necessary and sufficient conditions for different types of controllability. In section 4, under suitable assumptions, minimum energy control problem is analytically solved. For economy of space, full review on the subject is impossible. So, only selected fundamental results without proofs are presented.

II. The Mathematical Model.

In the theory of linear time invariant dynamical control systems, the most popular and most frequently used Mathematical model is given in the following differential state equation and algebraic output equations.

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

and

$$y(t) = Cx(t) \tag{2.2}$$

where $x(t)\epsilon E^n$ is the state vector, $u(t)\epsilon E^m$ is the input vector, $y(t)\epsilon E^p$ is the output vector, $t \geq 0, A, B\ and\ C$ are constant matrices of appropriate dimensions.

It is known [4] that for a given initial state $x(0) = x_0\epsilon E^n$ and control $u(t)\epsilon E^m, t \geq 0$, there is a unique solution $x(t, x_0 u)\ \epsilon E^n$ of the differential state equation (2.1) of the following form.

$$x(t, x_{0,} u) = e^{tA}x_0 + \int_0^t e^{(t-s)A}\, Bu(s)ds \tag{2.3}$$

Note that if P is $n\,x\,n$ constant non singular transformation matrix and if we define the equivalence transformation $z(t) = Px(t)$, then the differential state equation (2.1) and output equation (2.2) become

$$\dot{z}(t) = Jz(t) + Gu(t) \tag{2.4}$$

$$y(t) = Hz(t) \tag{2.5}$$

where $J = PAP^{-1}, G = PB \; and \; H = CP^{-1}$

Dynamical systems (2.1), (2.2) and (2.5), (2.5) are said to be equivalence and many of their properties are invariant under equivalence transformations [5]. If we take into account controllability concept, among different equivalence transformations, special attention should be paid to the transformation, which will lead us to the so called Jordan Canonical form of dynamical system (2.1). In the case when $n \times n$ dimensional matrix J is in Jordan Canonical form, then equation (2.4), (2.5) are said to be in a Jordan Canonical form. However, it should be stressed that every dynamical system (2.1), (2.2) has an equivalence Jordan Canonical form [5].

III. Controllability

3.1 Fundamental Results

Now, let us recall the most popular and most frequently used fundamental definition of controllability for linear control systems with constant coefficients.

Definition 3.1

Dynamical system (2.1) is said to be controllable if for every initial condition $x(0)$ and every vector $x^1 \epsilon E^n$, there exist a finite time t_1 and control $u(t) \epsilon E^m, t \epsilon [0, t_1]$ such that $x(t_1, x(0), u) = x^1$.

This definition requires only that any initial state $x(0)$ can be steered to any final state x^1 at finite time t_1. However, the trajectory of the dynamical system (2.1) between time t equal to 0 and t_1, is not specified. Furthermore, there is no constraints posed on the control vector $u(t)$ and the state vector $x(t)$.

In order to formulate easily computable algebraic controllability criteria, let us introduce the so called controllability matrix W, which is known as Kalman matrix and defined as

$W = [B \; AB \; A^2B \; \cdots A^{n-1}B]$.

It should be pointed out that the controllability matrix W is an n x m-dimensional constant matrix and depends only on system parameters.

Theorem 3.1

Dynamical system (2.1) is controllable if and only if

$rank\; W = n$

Proof;

First, note that from definition 3.1 and the form of solution of the state equation, it follows that dynamical system (2.1) is controllable, if and only if for certain time $t = t_1$, the range of integral operator

$$\int_0^{t_1} e^{(t_1 - s)A} Bu(s)ds$$

is the whole state space E^n. However, since $e^{\,t_1 A}$ is non singular for any t_1, then this is true if and only if symmetric nxn constant matrix.

$$\int_0^{t_1} e^{-SA} BB^T e^{-SA^T} ds$$

is non-singular. Taking into account Taylor series expansion of $e^{(-SA)}$ and the well known Cayley – Hamilton theorem, we get that dynamical system is controllable if and only if $rank\; W = n$.

Corollary 3.1

Dynamical system (2.1) is controllable if and only if the n x n-dimensional symmetric matrix WW^T is non-singular.

Since the controllable matrix W does not depend on time t_1, then from Theorem 3.1 and corollary 3.1, it follows that controllability of dynamical system does not depend on the length of time of control interval. However, this statement is valid only for dynamical systems (2.1) without any constraints posed on the control vector y(t) and the state vector x(t).

If we observe that in many cases, in order to check controllability, it is not necessary to calculate whole controllability matrix W, but only a matrix with the same number of rows but with a smaller number of columns. It depends on the rank of the matrix B and the degree of the minimal polynomial for the matrix A, where the minimal polynomial is the polynomial of the lowest degree, which annihilates the matrix A. T his observation is based on the following corollary.

Corollary 3.2

Let rank B = r, and q the degree of the minimal polynomial of the matrix A the dynamical system (2.1) is controllable if and only if

$$rank[B \quad AB \quad A^2B \ \cdots A^K B \cdots A^{n-k}B] = N$$

where the integer $K \leq \min(n-r, q-1)$

If the dynamical system (2.1) is controllable, then the system remains controllable after the equivalence transformation in the space E^n. This is natural and very clear because an equivalence transformation changes only the basis of the state space and does not change the properties of the dynamical system (2.1). So, we have the following corollary.

Corollary 3.3

Controllability of the dynamical system (2.1) is invariant under any equivalence transformation in the state space E^n.

Since controllability of dynamical system (2.1) is preserved under any equivalence transformation, then it is possible to obtain simpler controllability criteria by transforming the original differential state equation (2.1) into its special Canonical form (2.3). For example, if we transform dynamical system (2.1) into Jordan Canonical form, then controllability can be determined very easily, almost by inspection [3].

3.2 Stabilizability;

It is well known that for linear dynamical system (2.1), there are certain relationships between controllability and stability. In order to explain these relationships; let us introduce stability definitions. Let us first of all check the concept of equilibrium state.

Definition 3.2

An equilibrium state x^e is said to be stable if and only if for any positive number ε, there exists a positive number $\delta(\varepsilon)$ such that the inequality

$\|x(0) - x^e\| \leq \delta$

implies

$\|x(t), x(0), 0) - x^e\| \leq \varepsilon \ for \ all \ t \geq 0$

Roughly speaking, an equilibrium state x^e is stable if the response due to any initial state that is sufficiently near x^e will not move far away from x^e. If the response will, in addition, go back to x^e, then x^e is said to be asymptotically stable.

Definition 3.3

A state x^e of a dynamical system (2.1) is said to be an equilibrium state if and only if $x^e = x(t, x^e, 0)$ for all $t \geq 0$.

From this definition 3.3, we see that if a trajectory reaches an equilibrium state, and if no input is applied, the trajectory will stay at the equilibrium state forever. For linear dynamical systems, the zero state is always an equilibrium state.

Definition 3.4

An equilibrium state x^e is said to be asymptotically stable if it is stable in the sense of Lyapunov and if every motion starting sufficiently close to x^e converges to $x^e \ \ as \ t \ \to \infty$.

Let us denote the eigen values of $A \ by \ \lambda_i$. Then

$$\lambda_i = R_e\ (s_i) + Im\ (s_i), i = 1,2, \cdots, r; r \leq n,$$

are the distinct eigenvalues of the matrix A and R_e and Im stand for the real part and the imaginary part of the eigenvalue λ_i, respectively.

Theorem 3.2

Every zero state of the dynamical system (2.1) is stable if and only if all the eigenvalues of the matrix A have non positive real parts. i.e. $R_e(\lambda_i) \leq 0$. for all $i = 1,2, \cdots r$ and those with zero real parts are simple zeros of the minimal polynomial of the matrix A.

Theorem 3.3

The zero state of the dynamical system (2.1) is asymptotically stable if and only if all eigenvalues of the matrix A have negative real parts [6] i.e.

$$R_e(\lambda_i) \leq 0. \text{ for all } i = 1,2, \cdots r$$

From the above Theorem, it directly follows that stability and asymptotic stability of a dynamical systems depend only on the matrix A and are independent of the matrices B, and C.

Suppose that the dynamical system (2.1) is stable or asymptotically stable, then the dynamical system remains stable or asymptotically stable after any arbitrary equivalence transformation.

This is natural and intuitively clear because an equivalence transformation only changes the basis of the state space. So, we have the following corollary.

Corollary 3.4.

Stability and asymptotically stability are both invariant under any equivalence transformation [6].

It is well known that the controllability concept for linear dynamical system (2.1) is strongly related to its stabilizability by the linear static state feed back of the form.

$$u(t) = Kx(t) + v(t) \quad (3.1)$$

where $v(t)\epsilon E^m$ is a new control vector and K is m x n -dimensional constant state feedback matrix.

Introducing the linear static state feedback given in (3.1), we directly obtain new linear differential state equation for the feedback linear dynamical system of the form.

$$\dot{x}(t) = (A + BK)x(t) + Bv(t) \quad (3.2)$$

which is characterized by the pair of constant matrices (A+BK, B).

An interesting result is the equivalence between controllability of the dynamical systems (2.1) and (3.2), which is explained in the following corollary.

Corollary 3.5

Dynamical system (2.1) is controllable if and only if for any arbitrary matrix K, the dynamical system (3.2) is controllable.

From corollary 3.5, it follows that under controllability assumption, we can arbitrarily form the spectrum of dynamical systems of the form (2.1) by the introduction of suitable defined linear static state feedback (3.2). Hence, we have the following theorem;

Theorem 3.4

The pair of matrices (A, B) represents the controllable dynamical system (2.1) if and only if for each set V consisting of n-complex numbers and symmetric with respect to real axis, there exists constant state feedback matrix K such that the spectrum of the matrix (A + BK) is equal to the set V.

Practically, in the design of the dynamical system, sometimes it is required only to change unstable eigenvalues into stable eigenvalues. This is called stabilization of the dynamical system (2.1). So, we have the following formal definition of stabilizability.

Definition 3.5

Dynamical System (2.1) is said to be stabilizable if there exists a constant static state feedback matrix K such that the spectrum of the matrix (A + BK) entirely lies in the left hand side of the complex plane.

Theorem 3.5

Dynamical system (2.1) is stabilizable if and only if all its unstable modes are controllable.

$$i.e. \ \ rank\ [\lambda_i I - A \backslash B] = n, i = 1,2, \cdots q$$

If we compare Theorem 3.5 and corollary 3.3, we see that controllability of dynamical system (2.1) always implies its stabilizability. Note that the converse is not always true. So, stabilizability concept is essentially weaker that controllability property.

3.3 Output Controllability (observability)

In a similar way to state controllability of dynamical control system, it is possible to define the so called output controllability for output vector $y(t)\epsilon E^p$ of dynamical system. Although, these two concepts are quite similar, it should be mentioned that the state controllability property is a property of the differential state equation (2.1), whereas the output controllability is a property of both of the state equation (2.1) and algebraic output equation (2.1).

Definition 3.6

Dynamical system (2.1), (2.2) is said to be output controllable if for every y (0) and every vector $y^1\epsilon E^p$, there exists a finite time t_1 and control $u^1(t)\epsilon E^m$, that transfers the output from y(0) to $y^1(t) = y(t_1)$.

Output controllability generally means that we can steer output of dynamical system independently of its state vector.

Theorem 3.6.

Dynamical system (2.1),(2.2) is output controllable if and only if

$$rank[CB \ \ CAB \ \ CA^2B \ \cdots CA^KB \cdots CA^{n-1}B] = P$$

It should be pointed out that the state controllability is defined only for linear differential state equation (2.1), whereas the output controllability is defined for the output description. So, these two concepts are not necessarily related closely.

Let us recall that if the control system is output controllable, its output can be transferred to any desired vector at certain instant finite time. However, a related problem is whether it is possible to steer output so that it followed a given curved over any interval of finite time. A control system whose output can be steered along the arbitrary given curve over any interval finite time is said to be output function controllable.

3.4 Controllability with Constrained Controls

In practice, admissible controls are required to satisfy additional constraints. Let UCE^m be an arbitrary set and let the symbol M (U) denote the set is admissible controls. We have the following definitions.

Definition 3.7

Dynamical System (2.1) is said to be U-controllable to zero if for any initial state $x(0)\epsilon E^m$, there exists a finite time $t_1 > 0$ and admissible control $u(t)\epsilon M(u), t\epsilon[0, t_1]$, such that $x(t_1, x(0), u) = 0$.

Definition 3.8

Dynamical System (2.1) is said to be U-controllable from zero if for any finial state $x^1\epsilon E^n$, there exists a finite time $t_1 > 0$ and an admissible control $u(t)\epsilon M(u), t\epsilon[0, t_1]$, such that $x(t_1, 0, u) = x^1$.

Definition 3.9

Dynamical System (2.1) is said to be U-controllable if for any initial state $x(0)\epsilon E^n$, and any final state $x^1\epsilon E^n$, there exists a finite time $t_1 > 0$ and admissible control $u(t)\epsilon M(u), t\epsilon[0, t_1]$, such that $x(t_1, x(0), u) = x^1$.

Theorem 3.6

Dynamical System (2.1) is U-controllable to zero if and only if all the following conditions are satisfied simultaneously;

1) There exist $w\epsilon U$ such that Bw = 0,
2) The convex hall conv. (u) of the set U has nonempty interior in the space E^m.
3) $rank[B \;\; AB \;\; A^2B \; \cdots A^K B \cdots A^{n-1}B] = n$.
4) There is no real eigenvector $v\epsilon E^n$ of the matrix A^T satisfying $V^T Bw \leq 0$ for all $w\epsilon U$.
5) No eigenvalue of the matrix A has a positive real part.

Note that if m =1 i.e. singular input system.

Theorem 3.6 reduces to the following corollary;

Corollary 3.7

Suppose that m =1 and U = [0, 1]. Then dynamical system (2.1) is U-controllable to zero if and only if it is controllable without any constraints.

i.e. $rank[B \;\; AB \;\; A^2B \; \cdots A^K B \cdots A^{n-1}B] = n$.

and matrix A has no real eigenvalues.

Theorem 3.7

Suppose the set U is a cone with vertex at zero and a nonempty interior in the space E^m. Then dynamical system (2.1) is U-controllable from zero if and only if

1) $rank[B \;\; AB \;\; A^2B \; \cdots A^K B \cdots A^{n-1}B] = n$.
2) there is no real eigenvector $v\epsilon E^n$ of the transposed matrix A^T satisfying

$$V^T Bw \leq 0 \text{ for all } w\epsilon U.$$

Note that in the special case for single input system, i.e. m =1, Theorem 3.7 reduces to the following simple corollary.

Corollary 3.8

Suppose m =1and U = [0, 1]. the dynamical system (2.1) is U-controllable from zero if and only if it is controllable without any constraints. In other words

$rank[B \;\; AB \;\; A^2B \; \cdots A^{n-1}R] = 1$. and

matrix A has no real eigenvalues.

In conclusion, we stated earlier that we cannot exhaust the review of controllability results. We request to stop so far with the intention that many more review may be continued.

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