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Comparative Analysis of Simplex Method and the Family of Interior Point Method

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Abstract: *Linear programming plays an important role in our lives, it impact is marked as one of the most important scientific advancement since the nineteenth century. Simplex method is one of the most popular and most important methods of finding the solution to the LP problems. The simplex method in general tends to run in time linear to the number of constraints of the problem but in certain worst cases it tends to run in polynomial time algorithm. This become difficult to the researcher in the first time of its appearance and also shows poor performance in some problems (i.e. Klee and Minty problem) it also has difficulties in solving huge LP problems (i.e. airlines scheduling problems). This prompted scientists to go beyond simplex methods and interior point methods for linear programming were born. This work study the Interior point method (Karmarkar's algorithm), the principal idea behind the method and the three key concept used in development of the method. At the end we made geometrical comparison of the results obtained by the simplex and interior point methods. This demonstrated the comparative superiority of interior point method over the simplex method.*
Keywords: *Linear programming,* Simplex method, Interior point method

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I. Introduction

No science were born in a specific day[1], likewise the history of Operational Research (OR) is not an exception, it started during the World War II [12] Scientists and Engineers were asked to analyse several military problems: developing effective methods of using the newly invented radar, how to better manage convoy and antisubmarine, bombing and military operations. The applications of mathematics and scientific methods to military operations was called Operations Research (commonly referred to as OR). The term OR (or often Management Science) means a scientific approach to decision making which seeks to determine how best design and coordinate a system usually under conditions requiring the allocation of the scarce resources. OR aims to find the optimal solution among the series of solutions in order to allocate the scarce resource effectively and use it efficiently. Among the various topics in operational research the linear programming is the most popular in all of them and also the most important.

Linear programming (LP) [3][1] is among the most important scientific advances since the middle of nineteenth century, the LP is applied to almost all aspect of human life and its uses in various sector of human endeavour are spreading rapidly. Nowadays scientist has devoted many portions on computer computation in development of LP, hundreds of books, research papers, published articles, M.Sc. and Ph.D. thesis had been written which are describing the important application of linear programming. What is the nature of this remarkable tool, and what kind of problems does it address? Briefly, the most common type of application involve the general problems of allocating limited resources among competing activities in a best possible way (i.e., optimal). More precisely, these problems involve selecting the level of certain activities that compete for scarce resources that area necessary to perform those activities. The choice of activity levels than dictate how much of each resources will be consumed by each activities. The variety of situation to which this description applies is diverse, indeed, ranging from the allocation of production facilities to product to the allocation of national resources to domestics need, from portfolio selection to the selection of shipping patterns, from Agricultural planning to the design of radiation therapy and so on. However, the one common ingredient in each of these situations is the necessity for allocating resources to activities by chosen the level of those activities. [5]. Linear programming uses a Mathematical model to describe the problem of concern. The adjective linear means that all the Mathematical function in these models is require being linear functions. And the word programming does not refer here to computer programming; rather, it is essentially a synonym for planning. Thus Linear programming involve the planning of activities to obtain an optimal result, i.e., a result that reaches the specified goal best (according to the mathematical model) among all feasible alternatives.

This research work was aim to study the interior point method (Karmarkar Method), the principle idea behind the method, the basics concept that is used in the development of the method and finally to make comparative analysis between the Interior Point method and the Simplex method by showing geometrically the path of a solution to a Linear Programming problem obtained by the both two method.

II. The Simplex method

For us to be able to understand the interior point method and its family we have to first understand the concept and the main idea of the simplex method. The Simplex method for solving linear programs is based on this principle:

Theorem: Let a, b and c is an instance of the LPP, defining a convex polytope in R^n. Then there exists an optimal solution to this program at one of the vertices of the polytope.

The simplex algorithm works roughly as follows. It begins with a feasible point at one of the vertices of the polytope. Then it walks along the edges of the polytope from vertex to vertex, in such a way that the value of the objective function monotonically decreases at each step. When it reached a point in which the objective value can decrease no more, it terminated the process. Each step along an edge of the polytope is determined by a pivoting operation, the definition of which is a key to the performance of the resulting algorithm. If the pivoting operation is not defined carefully then a problem known as “cycling” can occur, where the algorithm gets trapped walking around on the same cycle of vertices without decreasing the value of the objective function.[7].[9].[10].

In fact, the simplex method tends to run in time linear in the number of constraints of the problem. However, there is a simple LP known as the “twisted cube" for which simplex can run in exponential time in the worst case. This become difficult to the researcher in the first time of its appearance, there is also another worst case in which it can run polynomial time algorithm. At that time, researchers were searching for a pivoting rule that was strong enough to guarantee polynomial time performance in the worst case, but this is still an open problem. It turns out that the answer lied beyond Simplex, and interior point methods for linear programming were born.[6].[13].[14].

Interior Point Methods (Karmarkar’s Method)

The above question about the complexity of the LPP was answered by Khachiyan. He demonstrated a worst-case polynomial time algorithm for linear programming introduced the ellipsoid in which the algorithm moved across the interior of the feasible region, and not along the boundary like simplex. Unfortunately the worst case running time for the ellipsoid method was high. Moreover, this method tended to approach the worst-case complexity on nearly all inputs, and so the simplex algorithm remained dominant in practice.[8].[10].[13]. This algorithm was only partially satisfying to researchers: was there a worst-case polynomial time algorithm for linear programming which had a performance that rivalled the performance of simplex on day-to-day problems? This question was answered by Karmarkar in 1984. He produced a polynomial-time algorithm called the projective algorithm for linear programming that ran in much better time. Moreover, in practice, the algorithm actually seemed to be reasonably efficient. This algorithm also worked by moving across the interior of the feasible region, but was far more complicated than the ellipsoid method.[15].
Karmarkar assumes that the LP is given in the conical form of the problems

Min Z = CX

Such that

$AX = 0, \quad 1X = 1, \quad X \geq 0.$

Assumption

- $X_0 = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$ is a feasible solution
- $Minimum(Z) = 0$

Example of standard form versus canonical form

Standard form

$Min\ Z = y_1 + y_2 \quad (Z = CY)$

Subjected to the constraint

- $y_1 + 2y_2 \leq 2 \quad (AY \leq b)$
- $y_1, y_2 \geq 0$

Canonical form

$Min\ Z = 5x_1 + 5x_2 \quad (Z = CX)$

Subjected to the constraints

- $3x_1 + 8x_2 + 3x_3 - 2x_4 = 0 \quad (AX = 0)$
- $x_1 + x_2 + x_3 + x_4 = 1 \quad (1X = 1)$

- $x_j \geq 0,\ j = 1, 2, 3, 4$

To apply the algorithm to LP problem in standard form transformation is needed

The principle Idea in Karmarkar's Algorithm

Create a sequence of points $x^{(0)}$, $x^{(1)}$, $x^{(2)}$,…, $x^{(k)}$ having decreasing values of the objective function in the k^{th} step, the point $x^{(k)}$ is brought into the center of the simplex by projective transformation.[11].

Basic key concepts

There are three basic key concepts that are used in developing of Karmarkar's algorithm [11] those include:

i. Projection of a vector onto the set of X satisfying AX = 0
ii. Karmarkar's Centering Transformation
iii. Karmarkar's Potential Function

Projection

- We want to move from a feasible point X^0 to another feasible point X^1, that for some fixed vector v, will have a larger value of vx
- If we choose to move in direction d = (d_1, d_2, … d_n) that solves the optimization problem

  Max vd such that Ad = 0, $d_1 + d_2 + \ldots d_n = 0$ (so that Ad remains feasible)

  And ||d|| = 1

  Then we will be moving in a feasible direction that maximizes the increase in vx per unit length moved.
- The direction d that solves this optimization problem is given by the projection of v onto X satisfying

  $Ax = 0, \quad x_1 + x_2 + \cdots + x_n = 0$

  And is given by

  $[1 - B^T(BB^T)^{-1}B]v$, Where

  $B = \begin{bmatrix} A \\ 1 \end{bmatrix}$

Karmarkar's Centering Transformation

- If x^k is a point in S, then $f([x_1, x_2, \ldots\ x_n]|x^k)$ transforms a point $[x_1, x_2, \ldots\ x_n]^T$ in S into a point [y_1, $y_2, \ldots\ y_n$] in S, where

$$y_j = \frac{\frac{x_j}{x_j^k}}{\sum_{r=1}^{r=n} \frac{x_r}{x_r^k}}$$

- Consider the LP

  Min $z = x_1 + 3x_2 - 3x_3$ such that

  $x_1 - x_3 = 0$

  $x_1 + x_2 + x_3 = 1$

  $x_i \geq 0$

The LP has a feasible solution $\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]^T$ and the optimal value of Z is 0. The feasible point $\left[\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right]^T$ yield the following transformation

$$f([x_1, x_2, \ldots\ x_n]| \left[\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right]) =$$

$$[x_1, x_2, \ldots\ x_n]|[\frac{4x_1}{4x_1 + \frac{8x_2}{3} + \frac{8x_3}{3}}, \frac{\frac{8x_2}{3}}{4x_1 + \frac{8x_2}{3} + \frac{8x_3}{3}}, \frac{\frac{8x_3}{3}}{4x_1 + \frac{8x_2}{3} + \frac{8x_3}{3}}]$$

For example, $f\left(\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right] \middle| \left[\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right]\right) = [\frac{12}{28}, \frac{8}{28}, \frac{8}{28}]$

We now refer to the variable $x_1, x_2, \ldots\ x_n$ as being the **original space** and the variables $y_1, y_2, \ldots\ y_n$ as being the **transformed space** and the unit simplex involving variables $y_1, y_2, \ldots\ y_n$ will be called **transformed unit** simplex.

Karmarkar's Potential Function

Karmarkar Potential Function f(x) is defined as

$$f(x) = \sum_{j=1}^{j=n} \ln\left(\frac{CX^T}{x_j}\right),\ \ X = [x_1, x_2, \ldots, x_n]^T$$

Karmarkar showed that if we project $(CD_k)^T\ not\ C^T$ onto the feasible region in the transformed space, then for some $\sigma > 0$, it will be true that for k = 0, 1, 2, …

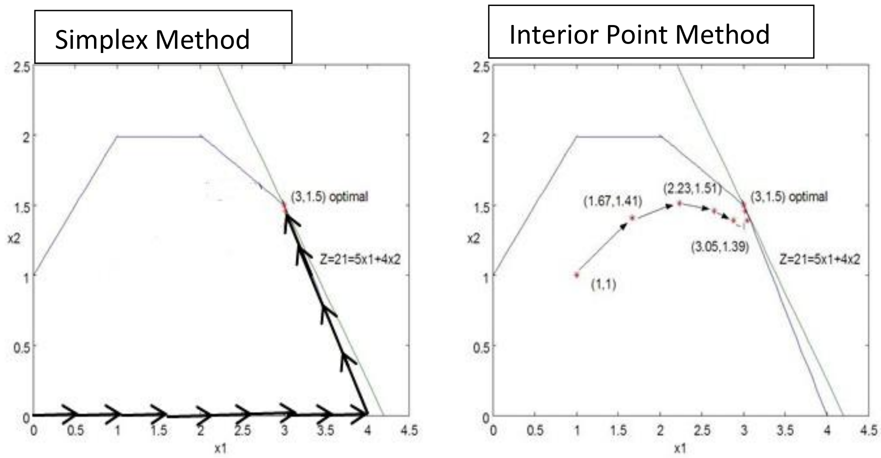
$$f(X^k) - f(X^{k+1}) \geq \sigma$$

- So that, each of the iterations of karmarkar's algorithm decreases the potential function by an amount bounded away from 0.
- Karmarkar showed that if the potential function evaluated at X^k is small enough, the $Z = CX^K$ will be near 0. Because $f(x^k)$ is decreased by at least σ per iteration, it follows that by choosing k sufficiently large, we can ensure that the Z-value for X^k is less than ϵ

Here we are going to demonstrate the result of the LP problem obtained from both the method graphically and show the path of solution from initial solution up to optimal solution.

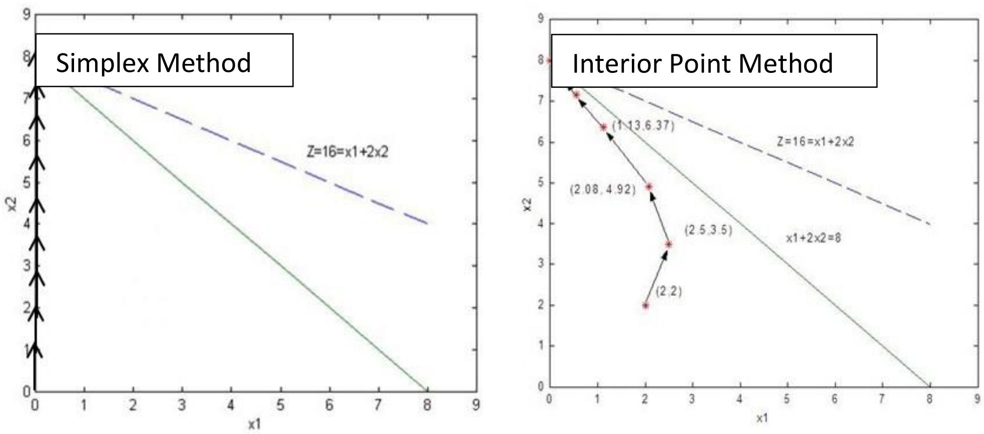
Problem 1 $Max\ Z = 5x_1 + 4x_2$

Such that $6x_1 + 4x_2 \leq 24$; $x_1 + 2x_2 \leq 6$; $-x_1 + x_2 \leq 1$; $x_2 \leq 2$; $x_1, x_2 \geq 0$;



Problem 2 $Max\ Z = x_1 + 2x_2$

Such that $x_1 + x_2 \leq 8$; $x_j \geq 0, \quad j = 1, 2$



Above is the comparison of the LP problem's solution obtained from both the Simplex and Interior point method that show the path in which the solution is moving from the initial basic feasible solution to the optimal solution, so as to demonstrate the basic idea of each method and also the nature on which solution is move from one point to another.

III. Summary

We first discussed on the introduction and the history of Operational Research and the Linear Programming, we then discussed on the Simplex method and Interior Point method of solving linear programming, the main idea behind the methods and the basic concept that was used in the developing of the methods. We then finally compare geometrically the path of the result obtained by the two methods.

IV. Conclusion

We conclude that for a huge linear programming problem with large numbers of decision variables and constraints Interior point method is more suitable and for less complicated problem Simplex method suit to give better result.

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