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Remodelling of Equations of Arithmetic Progressions (S_n & T_n)

R Subramanya,
Mysuru, India
Corresponding Author: R Subramanya,

Abstract: *Advanced and more generalised equations for Arithmetic Progression are derived in this article. A thorough survey of field of Arithmetic Progressions is made and shortcut route is picked up and produced in this article for the benefit of students.*

Keywords: *Arithmetic progressions, sequence, series, conjugate term, remodelled.*

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I. Inroduction

This is a research article wherein present equations of Arithmetic Progressions are modified such that they are more useful to the students. In most of the prevailing solutions to the problems of Arithmetic Progression, before finding the required result, finding the first term is very essential. Whereas, using remodelled equations, most of the solutions to same problems of A.P are found, without the help of first term. By this, several steps are skipped out; hence it is possible to arrive directly at the result. Solution to example2.5 is found using (3), hence it can be solved within a very few steps. At section IV of this article, theory of conjugate pairs is newly introduced. This idea helps to solve some of the problems in a different and better way. Example 3.4, shows that in such case data ‘d’ and ‘a’ are not required to find the solution and solution is drawn within a very few steps. Solution to example.4.2, is noteworthy because it unearths family of solution and required elements like ‘d’ , ‘a’, etc, from minimum available data.

II. Modified Equation For n^{th} Term Of Arithmetic Progressions

2. We know that prevailing equation for general term of Arithmetic Progression is

$$T_n = T_1 + (n-1)d \;-----\; (1.1)$$

(1.1) *can also be written as*

$$T_n = a + (n-1)d \;;[since\ T_1 = a\ ] ----- (1.2)$$

$Where\ T_n\ denotes\ the\ general\ term\ for\ \ n^{th}\ term\ of\ an\ Arithmetic\ Progression,$
$'n'\ denotes\ the\ number\ of\ terms\ and\ 'a'\ or\ T_1\ denotes\ first\ term.$
$(1.1)\ \&\ (1.2)\ facilitate\ to\ solve\ the\ problem\ for\ T_n, usually\ when\ T_1\ or\ 'a'\ is\ known.$
$To\ improve\ the\ generalisation\ of\ (1.1)\ or\ (1.2), following\ method\ is\ followed.$
$To\ start\ with, neutral\ pair\ of\ \ 'd'\ is\ added\ to\ (1.1).$
$Then, T_n = [T_1 + (n-1)d] + d - d$

Rearranging them conveniently, we have

$$T_n = (T_1 + d) + (n-1)d - d$$
$$= T_2 + (n-2)d\ ;\ [since\ T_2 = T_1 + d]$$

Repeating the same steps, we have

$$T_n = (T_2 + d) + (n-2)d - d$$
$$= T_3 + (n-3)d$$

⋮

Similarly repeating steps up to ‘p’, we have

$$\boldsymbol{T_n = T_p + (n-p)d\ ; whence\ p \le n\ -----\ (2)}$$

Proof of $\boldsymbol{T_n = T_p + (n-p)d\ ; whence\ p \le n}$ **,**

$$T_n = T_p + (n-p)d$$
$$T_n - T_p = (n-p)d$$
$$d = \frac{T_n - T_p}{n-p}\ ; this\ is\ an\ already\ known\ relation\ for\ 'd'$$

Thus proved.

$Example. 2.1. If\ 3^{rd}\ and\ 9^{th}\ terms\ are\ 4\ and\ -8\ respectively, which\ term\ of\ this\ A.P.is\ zero?$ [7]

Method.1. Solution to this problem as given in, [8].

$Here, T_3 = 4, and\ T_9 = -8.$

$\therefore using\ T_n = a + (n-1)d$

$\Longrightarrow \quad T_3 = a + 2d = 4$

$T_9 = a + 8d = -8$

Subtracting (1)from (2) we get,

$(a + 8d) - (a + 2d) = -8 - 4$

$\Longrightarrow \quad 6d = -12$

$\Longrightarrow \quad d = -2$

Now from (1) we have,

$a + 2d = 4$

$\Longrightarrow \quad a + 2(-2) = 4$

$\Longrightarrow a = 8$

$Let\ the\ n^{th}\ term\ of\ the\ A.P\ be\ 0.$

$\therefore\ T_n = a + (n-1)d = 0$

$\Longrightarrow 8 + (n-1) \times (-2) = 0$

$\Longrightarrow \quad n - 1 = 4$

$\Longrightarrow \quad n = 5$

$Thus\ the\ 5^{th}\ term\ of\ A.P\ is\ 0.$

Method.2. Solution using remodelled equation.

$Data:\ T_3 = 4;\ T_9 = -8;\ T_n = 0; n = ?$

$Initially, 'd'\ is\ found\ by\ known\ relation, d = \dfrac{T_q - T_p}{q - p}$

$Putting\ the\ known\ values, we\ have, \quad d = \dfrac{-8-4}{9-3}$

$d = -2$

$Using\ (2), n\ is\ found, i.e., \quad T_n = T_p + (n-p)d$

$Putting\ the\ values\ we\ have, \quad 0 = 4 + (n-3)(-2)$

$After\ simplification\ we\ have, n = 5$

$\therefore\ 5^{th}\ term\ of\ the\ A.P.is\ 0.$

$Note$: $In\ method.1, prevailing\ equation\ to\ find\ the\ n^{th}\ term\ is\ applicable\ only\ when\ initial\ term\ 'a'\ is\ found. Whereas\ in\ method.2. answer\ is\ found\ by\ using\ remodelled\ equation\ without\ the\ help\ of\ initial\ term\ 'a'.$

$Example.2.2. An\ A.P\ consists\ of\ 50\ terms\ of\ which\ 3^{rd}\ term\ is\ 12\ and\ last\ term\ is\ 106.\ Find\ 29^{th}\ term.$ [9]

Solution- Method.1: Prevailing method. [10]

$Given\ T_3 = 12, and\ T_{50} = 106,$

$When\ two\ terms\ of\ A.P\ are\ given\ common\ difference\ can\ be\ found\ by\ the\ equation\ , d = \dfrac{T_p - T_q}{p - q}$

$$\therefore d = \frac{T_{50} - T_3}{50 - 3}$$

$$= \frac{106 - 12}{47}$$

$\Rightarrow \quad d = 2$

$In\ an\ A.P., T_n = a + (n-1)d - - - - - (1)$

$Put, n = 3, \quad T_3 = a + (3-1)2$

$\Rightarrow \quad 12 = a + 4$

$Hence, \quad a = 8$

$Put, d = 2; a = 8\ , and\ n = 29\ in\ (1)$

$T_{29} = 8 + (29-1)2$

$T_{29} = 64$

$\therefore The\ 29^{th}\ term\ is\ 64.$

$Solution - Method.2. solution\ to\ same\ problem\ using\ remodelled\ equation.$

$Initially\ common\ difference\ is\ found\ using\ the\ relation, d = \dfrac{T_p - T_q}{p - q}$

$$\therefore d = \frac{T_{50} - T_3}{50 - 3}$$

$$= \frac{106 - 12}{47}$$
$\Rightarrow\ d = 2$

$Using\ (2)\ the\ remodelled\ equation, T_n = T_p + (n-p)d, solution\ can\ be\ found\ without\ finding\ first\ term.$

$Substituting\ the\ given\ values, we\ have\ \ T_{29} = 12 + (29-3)2$

$$\Rightarrow \quad T_{29} = 64$$

Note: This method skips out, steps to find $'a'$, which is done in method.1.

$Example.$ 2.3: $Determine\ the\ A.P\ whose\ third\ term\ is\ 16\ and\ the\ 7^{th}\ term\ exceeds\ the\ 5^{th}\ term\ by\ 12.$ [11]

Method.1: Current method of solution: [12].

$Let\ the\ first\ term = a\ and\ the\ common\ difference = d.$

$\therefore\ T_n = a + (n-1)d, we\ have$

$T_3 = a + 2d$

$\Rightarrow a + 2d = 16$

$And, T_7 = a + 6d\ ;\ T_5 = a + 4d$

According to the condition,

$$T_7 - T_5 = 12$$
$$\Rightarrow \quad (a+6d) - (a+4d) = 12$$
$$\Rightarrow \quad a + 6d - a - 4d = 12$$
$$\Rightarrow \quad 2d = 12$$
$$\Rightarrow \quad d = \frac{12}{2} = 6$$

Now, from (1) and (2), we have,

$a + 2(6) = 16$

$\Rightarrow\ a + 12 = 16$

$\Rightarrow\ a = 4$

$\therefore The\ required\ A.P\ is$ 4, [4+6], [4+2(6)], [4+3(6)],…. Or. 4,10,16,22,…

Method.2: Solution using remodelled equation:

$Remodelled\ equation,\quad T_n = T_p + (n-p)d$

$$T_7 = T_5 + (7-2)d$$
$$\Rightarrow \quad (7-2)d = 12$$
$$\Rightarrow \quad d = 6$$

$The\ sequance\ is:\ T_3 - 2d, T_3 - d, T_3, \ldots$

$i.e., 4{,}12{,}16, \ldots$

Comparatively, this method is easier than method.1.

$\boldsymbol{Case\ study\ of\ equation, T_n = T_p + (n-p)d}$ **.**

$\boldsymbol{Case. 1}.\ When\ p = 1,\quad remodelled\ equation, T_n = T_p + (n-p)d\ will\ be$

$T_n = T_1 + (n-1)d\ ; this\ is\ the\ present\ form\ of\ equation\ for\ T_n.$

$\boldsymbol{Case. 2}$:

$$When\ p = \frac{n}{2}\ ;\ \left[\begin{array}{c} Note\ that\ n\ is\ a\ natural\ number\ since\ it\ is\ an\ ordinal \\ number\ that\ specify\ the\ position\ of\ the\ term, it\ should \\ be\ an\ even\ number, in\ this\ case. \end{array}\right]$$

$$T_n = T_p + (n-p)d$$

$Putting\ the\ defined\ variable\ of\ p\ in\ the\ above\ equation, T_n = T_{\left(\frac{n}{2}\right)} + \left(n - \frac{n}{2}\right)d$

$$T_n = T_{\left(\frac{n}{2}\right)} + \left(\frac{n}{2}\right)d$$

Example.2.4: The A.P is 3, 8, 13, 18, … find $8^{th}\ term.$

$Data: d = T_2 - T_1 = 8 - 3 = 5$

$T_{\frac{n}{2}} = T_{\frac{8}{2}} = T_4 = 18\ ; therefore\ equation\ derived\ in\ case\ 2\ is\ applicable.$

$Then, T_n = T_{\left(\frac{n}{2}\right)} + \left(\frac{n}{2}\right)d$

$$T_8 = T_{\left(\frac{8}{2}\right)} + \left(\frac{8}{2}\right)5$$

T_8 =18+20=38

Case.3:

$When\ p = \frac{n \pm 1}{2}$ $, relation\ (2) i.e. T_n = T_p + (n-p)d, will\ be\ ;$ $\left(\because 'p'\ is\ a\ natural\ number, 'n' \atop should\ be\ an\ odd\ number \right)$

$$T_n = T_{\left(\frac{n \pm 1}{2}\right)} + \left(n - \left(\frac{n \pm 1}{2} \right) \right) d$$

$Further, cases\ p = \frac{n+1}{2}\ and\ p = \frac{n-1}{2}\ are\ dealt\ separately, as\ following,$

$a)\ When\ p = \frac{n+1}{2},$

$$T_n = T_p + (n-p)d$$

$Replacing\ the\ relation\ of\ p, in\ the\ above\ equation, T_n = T_{\left(\frac{n+1}{2}\right)} + \left[n - \left(\frac{n+1}{2} \right) \right] d$

$$\Rightarrow \qquad T_n = T_{\left(\frac{n+1}{2}\right)} + \left[\left(\frac{n-1}{2} \right) \right] d$$

$Example. 2.5: The\ 8^{th}\ term\ of\ an\ A.P\ is\ 17\ and\ the\ 19^{th}\ term\ is\ 39.\ find\ the\ 25^{th}\ term..$

Method.1.: Current solution to this problem is as following;

$Data: T_8 = 17, T_{19} = 39, T_{25} = ?.$

$First\ 'd'\ is\ found\ using\ the\ relation, d = \frac{T_p - T_q}{p - q}$

$$d = \frac{T_{19} - T_8}{19 - 8} = \frac{39 - 17}{11} = \frac{22}{11}$$

$$\therefore d = 2$$

$Consider, T_8 = 17$

$$a + 7d = 17\ ;\ [\because\ T_n = T_1 + (n-1)d]$$

$$a + 7(2) = 17$$

$$a + 14 = 17$$

$$\therefore a = 3$$

$To\ find\ T_{25},$ $\qquad T_n = T_1 + (n-1)d$

$$T_{25} = 3 + (25-1)2 = 3 + (24 \times 2) = 3 + 48$$

$$\therefore\ T_{25} = 51$$

Method.2: Same problem is solved using remodelled equation (2) $as\ following,$

$First\ 'd'\ is\ found\ using\ the\ relation, d = \frac{T_p - T_q}{p - q}$

$$i.e., \qquad d = \frac{T_{19} - T_8}{19 - 8} = \frac{39 - 17}{11} = \frac{22}{11}$$

$$\therefore d = 2$$

$Now\ using\ (2), we\ have$

$$T_n = T_p + (n-p)d$$

$$T_{25} = 39 + (25 - 19)2$$

$$T_{25} = 51$$

This method brings down several steps, in comparison with the former method.

$Replacing\ the\ relation\ of\ p, in\ the\ above\ equation, T_n = T_{\left(\frac{n-1}{2}\right)} + \left[n - \left(\frac{n-1}{2} \right) \right] d$

$$\Rightarrow \qquad T_n = T_{\left(\frac{n-1}{2}\right)} + \left[\left(\frac{n+1}{2} \right) \right] d$$

$Example. 2.6: If\ \frac{n-1}{2} = 17, and\ 17^{th}\ term\ is\ -27, and\ d = -3, find\ n^{th}\ term.$

$Solution: It\ is\ given\ that\ \frac{n-1}{2} = 17$

$\Rightarrow n = 35$

$Then\ n^{th}\ term\ can\ be\ found\ using, T_n = T_{\left(\frac{n-1}{2}\right)} + \left[\left(\frac{n+1}{2} \right) \right] d$

$Putting\ the\ known\ values, we\ have, T_{35} = T_{\left(\frac{35-1}{2}\right)} + \left[\left(\frac{35+1}{2} \right) \right] - 3$

$T_{35} = -27 + (-54)$

$T_{35} = -81$

$\boldsymbol{Case. 4}: When\ p = n - 1$

$$T_n = T_p + (n-p)d$$

$T_n = T_{n-1} + \left(n - (n-1)\right)d$

$T_n = T_{n-1} + d$

$\boldsymbol{Case}.\,\mathbf{5}: When\ p = n$

$T_n = T_p + (n-p)d$

$T_n = T_n + (n-n)d$

$T_n = T_n$

Case.6: When two A.P have same c.d, (Common difference), The difference between equidistant terms will be constant.

$i.e., If\ T_1, T_2, T_3, \ldots, T_n;\ and\ T_1', T_2', T_3', \ldots, T_n'\ are\ the\ two\ A.Ps\ that\ have\ same\ c.d$,

$i.e., d = d'$

$Then\ differance\ between\ same\ terms\ in\ both\ A.Ps\ will\ be\ constant.$

$T_{p\pm a} - T_{p\pm a}' = T_1 - T_1' = c\ ; a\ constant -------- (3)$

Proof:

$Let\ T_p and\ T_p'\ be\ the\ chosen\ terms\ to\ find\ the\ difference. i.e., value\ of\ T_p - T_p'$

$Then, T_p - T_p' = T_1 + (p-1)d - [T_1' + (p-1)\ d]\ ; \because\ d = d'$

$T_p - T_p' = T_1 - T_1'$

$For\ (p+1)^{th}\ term$

$T_{p+1} - T_{p+1}' = T_1 + \left((p+1)-1\right)d - \left[T_1' + \left((p+1)-1\right)d\right]$

$T_{p+1} - T_{p+1}' = T_1 - T_1'$

Similarly

⋮

$T_{p\pm a} - T_{p\pm a}' = T_1 - T_1'$

This shows that difference of all equidistant terms is a constant.

Thus proved

$Example.$ 2.7: Two A.Ps. have the same common difference. The difference between their 100 th terms is 100. What is the difference between their 1000^{th} terms? [13]

Method.1: Solution to this problem given in the book. [14]

$Let\ for\ the\ 1^{st}\ A.P, first\ trem = a$

$T_{100} = a + 99d$

$And\ for\ 2^{nd}\ A.P, the\ first\ term = a'$

$\therefore\ T_{100}' = a' + 99d$

$It\ is\ given\ that, T_{100} - T_{100}' = 100$

$\Rightarrow\quad a + 99d - (a' + 99d) = 100$

$\Rightarrow\qquad a - a' = 100$

$Let\qquad T_{1000} - T_{1000}' = x$

$\therefore a + 999d - (a' + 999d) = x$

$a - a' = x\ \Rightarrow 100$

$\therefore The\ difference\ between\ the\ 1000\ th\ term\ is\ 100$

$Method.\,2: Soulution\ to\ this\ problem\ is\ found\ directly\ using\ (3)$

$i.e., T_{p\pm a} - T_{p\pm a}' = T_1 - T_1' = c\ ; a\ constant$

$Given\ T_{100} - T_{100}' = 100$

$\therefore T_{1000} - T_{1000}' = 100$

III. Sum Of First 'n' Terms Of Arithmetic Progressions

2. $Further\ we\ know\ that\ S_n\ is\ the\ algebric\ notation\ for\ sum\ of\ series\ of\ Arithmetic\ Progressions\ for\ n\ terms.$ And current equations for sum of A.P are

$S_n = \frac{n}{2}[T_1 + T_n]\ ------(4.1)$

$S_n = \frac{n}{2}[a + l]\ ------- (4.2)$

$S_n = \frac{n}{2}[2a + (n-1)d]\ --- (4.3)$

$Where, Initial\ term = T_1 = a\ ; Last\ term = T_n = l\ ; number\ of\ terms = n\ and\ common\ difference = d$

3.1. $Remodelling\ of\ (4.1), (4.2), (4.3)\ are\ done\ as\ following.$

$Consider\ prevailing\ equation\ , S_n = \frac{n}{2}[T_1 + T_n]$

Using (2), above equation can be written as,

$$S_n = \frac{n}{2}\left[T_1 + \left\{T_p + (n-p)d\right\}\right] ---------(5)$$

Let $n - p = x$; substituting this relation in the above equation and rearranging conveniently we obtain.

$$S_n = \frac{n}{2}\left[T_p + (T_1 + xd)\right]$$

$$We\ know\ that\ T_1 + d = T_2$$
$$T_1 + 2d = T_3$$
$$\vdots$$
$$T_1 + xd = T_{(x+1)} ----- (6)$$

In view of (6) and (5) the equation, $S_n = \frac{n}{2}\left[T_p + (T_1 + xd)\right]$ *can be written as,* $S_n = \frac{n}{2}\left[T_p + T_{(x+1)}\right]$

Once again substituting the relation of 'x' in the above equation, we get

$$\boldsymbol{S_n = \frac{n}{2}\left[T_p + T_{(n-p)+1}\right] ----- (7)}$$

This is the remodelled equation for sum of A.P for first n terms and this resembles (4.1) and (4.2).

3.2. *Further, (4.3) is remodelled as following,*

$$i.e., S_n = \frac{n}{2}[2a + (n-1)d]$$

Replacing 'a' by T_1, we have

$$S_n = \frac{n}{2}[2T_1 + (n-1)d]$$

Adding neutral pair 2d , −2d, to the above equation, we have,

$$S_n = \frac{n}{2}[2T_1 + (n-1)d + 2d - 2d]$$

Rearranging them conveniently we have

$$= \frac{n}{2}[2T_1 + 2d + (n-1)d - 2d]$$

$$S_n = \frac{n}{2}[2(T_1 + d) + (n-3)d]$$

Similarly, repeating same steps, we have

$$S_n = \frac{n}{2}[2T_2 + 2d + (n-3)d - 2d]$$
$$S_n = [2T_3 + (n-5)d]$$
$$\vdots$$
$$S_n = \frac{n}{2}\left[2T_p + \{n - (2p-1)\}d\right]$$

Or

$$\boldsymbol{S_n = \frac{n}{2}\left[2T_p + (n + 1 - 2p)d\right] ----- (8)}$$

This equation is equivalent to (4.3)

Case Study.

Case.I:

Putting $p = 1$ in (7), we get

$$S_n = \frac{n}{2}\left[T_p + T_{(n-p)+1}\right]$$
$$S_n = \frac{n}{2}\left[T_1 + T_{(n-1)+1}\right]$$
$$S_n = \frac{n}{2}[T_1 + T_n]$$

This shows that from (7), prevailing equation is derivable.

Example. 3.1: *Given $a_3 = 15, S_{10} = 125$, find d and a_{10}.* [15]

Method.1: Solution to this problem, as given in [16]

Here, $a_3 = 15 = l$ and $S_{10} = 125$.

Let, first term of A.P be 'a' and common difference = d

$$\therefore\ a_3 = a + 2d$$

$\Rightarrow \quad a + 2d = 15 ----- (1)$

$$Again, S_n = \frac{n}{2}[2a + (n-1)d]$$

$\Rightarrow \quad S_{10} = \frac{10}{2}[2a + (10-1)d]$

$\Rightarrow \quad 125 = 5[2a + 9d]$

$\Rightarrow \quad 2a + 9d = 25 - - - - - (2)$

$Solving\ for\ a\ and\ d\ from\ simultaneous\ equations\ (1) and\ (2), i.e.,$

$a + 2d = 15 - - - - - - (1)$

$2a + 9d = 25 - - - - - - (2)$

$\Rightarrow \quad d = -1$

$\therefore From\ (1), a + 2(-1) = 15 \quad \Rightarrow a = 15 + 2$

$\Rightarrow \quad a = 17$

$Now, \quad a_{10} = a + (10 - 1)d$

$= 17 + 9(-1)$

$= 17 - 9 = 8$

$Thus, d = -1\ and\ a_{10} = 8$

Method.2: Solved the same problem using remodelled equations as following.

$It\ is\ given\ that, S_n = S_{10} = 125, a_3 = T_3 = T_p = 15, Then, p = 3\ and\ n = 10$

$Using\ (8), i.e., S_n = \frac{n}{2}\left[2T_p + (n + 1 - 2p)d\right]$

$Then, (8)\ is, \quad 125 = \frac{10}{2}[2 \times 15 + (10 + 1 - 2 \times 3)d]$

$5 = 6 + d$

$Consequantly, d = -1$

$Then, using\ (2), T_n = T_p + (n - p)d$

$T_{10} = T_3 + (10 - 3)(-1)$

$T_{10} = 15 - 7$

$\Rightarrow \quad T_{10} = a_{10} = 8$

$This\ is\ shorter\ and\ easier\ than\ the\ current\ method\ of\ solution.$

Method..3: Using (7), i.e., $S_n = \frac{n}{2}\left[T_p + T_{(n-p)+1}\right]$ $this\ problem\ is\ solvable.$

$$125 = \frac{10}{2}\left[T_3 + T_{(10-3)+1}\right]$$

$$125 = 5[15 + T_8]$$

$$T_8 = 10$$

$Now, d = \frac{T_8 - T_3}{8 - 3}$

$d = \frac{10 - 15}{5} = -1$

$Then, using\ (2) \quad T_n = T_p + (n - p)d$

$T_{10} = T_3 + (10 - 3)(-1)$

$T_{10} = 15 + (10 - 3)(-1)$

$T_{10} = 8$

$In\ method.2.\ and\ method.3., value\ of\ 'a'\ is\ not\ required\ to\ find\ the\ solution.$

Example.3.2: The sum of the third and seventh terms of an A.P is 6 and their product is 8.find the sum of first sixteen terms of A.P. [17]

Method.1:Present solution to above problem is, [18].

Sol: Here, $T_3 + T_7 = 6\ \ and\ T_3 \times T_7 = 8$

$Let, first\ term = a\ and\ the\ common\ difference = d$

$\therefore\ T_3 = a + 2d\ and\ T_7 = a + 6d$

$\because\ T_3 + T_7 = 6$

$(a + 2d) + (a + 6d) = 6$

$\Rightarrow 2a + 8d = 6$

$\Rightarrow \quad a + 4d = 3 - - - - - (1)$

$Again, T_3 \times T_7 = 8$

$\therefore (a + 2d) \times (a + 6d) = 8$

$\Rightarrow\ (a + 4d - 2d) \times (a + 4d + 2d) = 8$

$\Rightarrow\ [3 - 2d] \times [3 + 2d] = 8$

$\Rightarrow\ 9 - 4d^2 = 8$

$\Rightarrow\ d^2 = \frac{1}{4}$

$\Rightarrow d = \pm\frac{1}{2}$

$When\ d = \frac{1}{2}$
$From\ (1), we\ have$
$$a + 4\left(-\frac{1}{2}\right) = 3$$
$\Rightarrow \quad a = 1$
$Now, using\ S_n = \frac{n}{2}[2a + (n-1)d]$
$$S_{10} = \frac{16}{2}\left[2(1) + (16-1)\frac{1}{2}\right]$$
$$S_{10} = 76$$
$When, d = -\frac{1}{2}$
$From\ (1), we\ have,$
$$a + 4\left(-\frac{1}{2}\right) = 3$$
$\Rightarrow \quad a = 5$
$Again\ sum\ of\ first\ 16\ terms$
$$S_{16} = \frac{16}{2}\left[2(5) + (16-1)\left(-\frac{1}{2}\right)\right]$$
$$S_{16} = 20$$
Method.2: Same problem is solved by using remodelled equations.
$Given, T_3 + T_7 = 6\ and\ T_3 \times T_7 = 8$
$Sol: Multiplying\ first\ datam\ by\ T_3, we\ have$
$$T_3(T_3 + T_7) = 6T_3$$
$\Rightarrow \quad T_3^2 + T_3T_7 = 6T_3$
$\Rightarrow \quad T_3^2 - 6T_3 + 8 = 0\ ; sinceT_3 \times T_7 = 8$
$Factorised\ the\ above\ quadratic\ equation\ as\ following.$
$$T_3^2 - 4T_3 - 2T_3 + 8 = 0$$
$$T_3(T_3 - 4) - 2(T_3 - 4) = 0$$
$$(T_3 - 4)(T_3 - 2) = 0$$
$Consequantly, T_3 = 4\ or\ T_3 = 2$
$Then, T_7 = 6 - 4 = 2 \quad or\ T_7 = 6 - 2 = 4$
$To\ find\ 'd', known\ relation\ is\ used, i.e., \qquad d = \frac{T_q - T_p}{q - p}$
$Then, d = \frac{T_7 - T_3}{7-3} = \frac{2-4}{7-3} = \frac{-2}{4} = -\frac{1}{2}$
$Then, to\ find\ sum\ offirst\ 16\ terms, S_n = \frac{n}{2}[2T_p + (n + 1 - 2p)d]\ is\ used.$
$Accordingly, substituting\ known\ and\ obtained\ values, we\ have,$
$S_{16} = \frac{16}{2}\left[2 \times 4 + (16 + 1 - 2 \times 3)\left(-\frac{1}{2}\right)\right]; where, n = 16, T_p = T_3 = 4\ and\ d = -\frac{1}{2}$
$S_{16} = 8\left[8 - \frac{11}{2}\right]$
$i.e., S_{16} = 20$
Similarly, sum of sixteen terms for another value of $T_3\ can\ be\ found.$
Advantages of method.2., in comparison with prevailing method i.e, method.1.
1. Number of steps are remarkably less than the previous method.
2. In prevailing method, finding first term is essential, whereas in case of remodelled equation finding first term is not essential to find the result or required value.
3. 'd' is found directly using a little advanced method.
4. Finally, sum of sixteen terms is found without the help of first term.

$Example. 3.3: In\ an\ A.P., if\ p^{th}\ term\ is\ \frac{1}{q}\ and\ q^{th}\ term\ is\ \frac{1}{p}, prove\ that\ the\ sum\ of\ first\ pq\ terms\ is\ \frac{1}{2}(pq+1), where\ p \neq q.$ [19].
Method.1: Prevailing solution to this problem: [6]
$It\ is\ known\ that\ general\ term\ of\ an\ A.P, is\ a_n = a + (n-1)d$
According to given information,

$$p^{th}\ term\ is, a_p = a + (p-1)d = \frac{1}{q} ------ (1)$$
$$q^{th}\ term\ is, a_q = a + (q-1)d = \frac{1}{p} ------ (2)$$

Subtracting (2) from (1) we obtain.

$$(p-1)d - (q-1)d = \frac{1}{q} - \frac{1}{p}$$
$$\Rightarrow \quad (p-1-q+1)d = \frac{p-q}{pq}$$
$$\Rightarrow \quad d = \frac{1}{pq}$$

Putting the value of d in (1), we obtain

$$a + (p-1)\frac{1}{pq} = \frac{1}{q}$$
$$\Rightarrow \quad a = \frac{1}{q} - \frac{1}{q} + \frac{1}{pq} = \frac{1}{pq}$$
$$\therefore \ S_{pq} = \frac{pq}{2}[2a + (pq-1)d]$$
$$= \frac{pq}{2}\left[\frac{2}{pq} + (pq-1)\frac{1}{pq}\right]$$
$$= \frac{pq}{2} \times \frac{1}{pq}[2 + pq - 1]$$
$$= \frac{1}{2}(pq+1)$$

Thus the sum of first pq terms of the A. P is $\frac{1}{2}(pq+1)$

Method.2:Same problem is solved using remodelled equations as following:

$$Data: T_p = \frac{1}{q};\ T_q = \frac{1}{p}; n = pq;$$
$$Then, d = \frac{T_q - T_p}{q-p}$$
$$\therefore d = \frac{\frac{1}{p} - \frac{1}{q}}{q-p}$$
$$Simplyfying\ the\ above, we\ get, d = \frac{1}{pq}$$
$$Now\ using\ remodelled\ equation\ S_n = \frac{n}{2}[2T_p + (n+1-2p)d]$$
$$Then\ sum\ of\ first\ pq\ terms\ is\ , S_{pq} = \frac{pq}{2}\left[2T_p + (pq+1-2p)\frac{1}{pq}\right]$$
$$Simplifying\ the\ above, we\ have\ , S_{pq} = \frac{1}{2}(pq+1)$$

Thus proved.

In this method, direct approach towards the result is noticeable.

Case.II:

$$Putting\ p = \frac{n}{2}\ in\ (7), we\ obtain, (As\ 'p'\ is\ a\ natural\ number, 'n'\ will\ be\ an\ even\ number\ in\ this\ case)$$
$$i.e., S_n = \frac{n}{2}[T_p + T_{(n-p)+1}]$$
$$S_n = \frac{n}{2}\left[T_{\frac{n}{2}} + T_{\left(n-\frac{n}{2}\right)+1}\right]$$
$$S_n = \frac{n}{2}\left[T_{\frac{n}{2}} + T_{\frac{n}{2}+1}\right]$$

Note: $T_{\frac{n}{2}} + T_{\frac{n}{2}+1}\ \ is\ the\ sum\ of\ middle\ conjugate\ pair., refer\ section\ IV\ of\ this\ article.$

Example.3.4: If fifth term of an A.P, is 15 and sixth term is 18, find sum of first ten terms.

$$Data: n = 10, T_5 = T_{\frac{n}{2}} = 15\ and\ \ T_6 = T_{\frac{n}{2}+1} = 18, Also\ \frac{n}{2} = \frac{10}{2} = 5\ is\ a\ natural\ \ number.$$
$$Therefore\ above\ relation\ for\ S_n\ is\ applicable.$$

$$i.e., S_n = \frac{n}{2}\left[T_{\frac{n}{2}} + T_{\frac{n}{2}+1}\right]$$

$$\text{Substituting the given values, we have, } S_{10} = \frac{10}{2}[15 + 18]$$

$$S_{10} = 165$$

Case.III.

$When\ P = \frac{n \pm 1}{2}$, (7) *is simplified as following*; $\left(\begin{matrix}Since\ 'P'\ is\ a\ natural\ number\ 'n'\ should\ be \\ odd\ number\end{matrix}\right)$.

a) $Putting\ p = \frac{n+1}{2}\ we\ have\ S_n = \frac{n}{2}\left[T_p + T_{(n-p)+1}\right]$

$$= \frac{n}{2}\left[T_{\left(\frac{n+1}{2}\right)} + T_{\left\{n-\left(\frac{n+1}{2}\right)\right\}+1}\right]$$

$$= \frac{n}{2}\left[T_{\left(\frac{n+1}{2}\right)} + T_{\left(\frac{n+1}{2}\right)}\right]$$

$$S_n = \frac{n}{2}\left[2T_{\left(\frac{n+1}{2}\right)}\right]$$

$$\boldsymbol{S_n = n\left[T_{\left(\frac{n+1}{2}\right)}\right]} ------(\mathbf{9})$$

$Note$: $T_{\left(\frac{n+1}{2}\right)}$ *is the middle term since n is an odd number.*

$Example.$ 3.5: *Find the sum of the first* 111 *terms of an A.P, whose* 56^{th} *term is* $\frac{5}{37}$. [20]

Method.1. Prevailing Solution, [21]

$The\ 56^{th}\ term\ is\ \frac{5}{37} \Rightarrow T_{56} = \frac{5}{37}, n = 5$

$In\ A.P, T_n = a + (n-1)d$

$Put\ n = 56, \quad T_{56} = a + 55d$

$\therefore a + 55d = \frac{5}{37} ----- (1)$

Now we need to find the sum of 111 *terms*

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{111} = \frac{111}{2}[2a + 110d]$

$S_{111} = 111[a + 55d]$

Putting relation (1) *in the above equation, we have,*

$S_{111} = 111\left[\frac{5}{37}\right]$

$S_{111} = 15$

$Method.$ 2. *Using remodelled equation this problem is solved as following,*

$Given;\ n = 111, T_{56} = \frac{5}{37}$

Here n is an odd number, $\therefore \frac{n+1}{2} = \frac{111+1}{2} = 56$; $\therefore$ T_{56} *is the middle term of this A.P., then*

$S_n = n\left[T_{\left(\frac{n+1}{2}\right)}\right]$

$S_{111} = 111\left[T_{\left(\frac{111+1}{2}\right)}\right]$

$S_{111} = 111[T_{56}]$

$S_{111} = 111\left[\frac{5}{37}\right]$

$S_{111} = 15$

This method is easier and shorter than method.1.

Example.3.6: If sixth term of A.P is 18, find sum of first eleven terms of A.P.

$Data: n = 11, hence\ \frac{n+1}{2} = \frac{11+1}{2} = 6, T_6 = 18,$

$Using\ the\ equation\ derived\ in\ case.III.i.e, S_n = n\left[T_{\left(\frac{n+1}{2}\right)}\right]$

$S_{11} = 11[(18)]$

$S_{11} = 198$

In this case sum of first 11 terms is found without the help of d and a.
But, actual A.P is remaining unknown. Specific A.P is not possible to find in this case, although it is possible to find set of A.Ps or family of A.P.

$It\ is\ known\ that\ T_6 = 18, this\ can\ be\ written\ as\ , T_1 + 5d = 18$

$This\ equation\ T_1 + 5d = 18\ indicates\ family\ of\ solutions.$

$To\ find\ paticular\ A.P\ it\ requires\ one\ more\ criterion. For\ example, if\ T_1 = d,$

$we\ have\ T_1 + 5d = 18$

$d + 5d = 18$

$6d = 18$

$Then\ d = 3$

Now only it is possible to find particular A.P i.e., 3,6,9…18…33

$Similarly, if\ T_1 = 8,$

$putting\ value\ of\ T_1 in\ T_1 + 5d = 18\ , we\ have$

$8 + 5d = 18$

$Then, \quad d = 2$

$\therefore\ A.P\ is\ 8, 10, 12, \dots 18, \dots, 28$

$\therefore\ T_1 + 5d = 18\ indicates\ family\ of\ A.Ps$

$Example. 3.7: Find\ three\ numbers\ in\ A.P. whose\ sum\ and\ products\ are\ respectively. i)\ 21 and\ 231.$ [22]

Method.1: This is the current method of solution.[23]

$Let\ the\ three\ numbers\ be\ a - d, a, a + d$

$Given\ their\ sum = 21$

$\therefore a - d + a + a + d = 21$

$3a = 21$

$a = 7$

$And\ their\ product\ is = 231$

$(a - d)a(a + d) = 231$

$a(a^2 - d^2) = 231$

$7(7^2 - d^2) = 231$

$7^2 - d^2 = 33$

$d = \pm 4$

$\therefore The\ numbers\ are\ a - d = 7 - 4 = 3$

$a = 7$

$a + d = 7 + 4 = 11$

$They\ are, 3,7,11$

Note- In this method $'a'$ is not initial term. This may lead to confusion because algebraic notation $'a'$ in Arithmetic Progression is particularly meant to first term.

$Method. 2: Same\ problem\ is\ solved\ by\ using\ remodelled\ equations\ as\ following.$

$Given\ S_3 = 21$

$i.e., applying\ (9) we\ have,$

$21 = 3[T_2]\ ; since\ T_2\ is\ the\ middle\ term.$

$\therefore \quad T_2 = 7$

$Also, given\ that, T_1 \times T_2 \times T_3 = 231$

$This\ can\ be\ written\ as\ (T_2 - d)T_2(T_2 + d) = 231$

$(7 - d)7(7 + d) = 231$

$7^2 - d^2 = 33$

$d = \pm 4$

$\therefore numbers\ are, 3, 7, 11 \quad (\ since\ T_2 = 7\ and\ d = 4, T_1 will\ be\ = 3)$

b) $Putting\ p = \dfrac{n-1}{2}\ , we\ have$

$$S_n = \frac{n}{2}\left[T_p + T_{(n-p)+1}\right]$$

$$= \frac{n}{2}\left[T_{\left(\frac{n-1}{2}\right)} + T_{\left\{n-\left(\frac{n-1}{2}\right)\right\}+1}\right]$$

$$S_n = \frac{n}{2}\left[T_{\left(\frac{n-1}{2}\right)} + T_{\left(\frac{n+3}{2}\right)}\right]$$

$Example: If\ fifth\ term\ of\ an\ A.P\ is\ 15\ and\ seventh\ term\ is\ 21, find\ sum\ of\ first\ \ 11\ terms.$

$Data: n = 11, T_{\left(\frac{n-1}{2}\right)} = T_{\left(\frac{11-1}{2}\right)} = T_5 = 15, T_{\left(\frac{n+3}{2}\right)} = T_{\left(\frac{11+3}{2}\right)} = T_7 = 21.$

$$\begin{aligned} Then, S_n &= \frac{n}{2}\left[T_{\left(\frac{n-1}{2}\right)} + T_{\left(\frac{n+3}{2}\right)}\right] \\ &= \frac{n}{2}\left[T_{\left(\frac{11-1}{2}\right)} + T_{\left(\frac{11+3}{2}\right)}\right] \\ &= \frac{n}{2}[T_5 + T_7] \\ &= \frac{11}{2}[15 + 21] \\ &= 198 \end{aligned}$$

$\boldsymbol{Case. IV}: When\ p = n, S_n = \frac{n}{2}\left[T_p + T_{(n-p)+1}\right]\ will\ be,$

$$= \frac{n}{2}\left[T_n + T_{(n-n)+1}\right]$$

$$S_n = \frac{n}{2}[T_n + T_1]$$

$This\ resembles\ case. I.$

$Similarly, same\ results\ are\ obtainable\ using\ (8), i.e., S_n = \frac{n}{2}\left[2T_p + (n + 1 - 2p)d\right]$

IV. Conjugate Pair And Terms

In (7), it is noticeable that the sum of the terms $T_P\ and\ T_{(n-p)+1}$ is a constant to first n terms of an A.P. Conjugate pair or conjugate terms or pair of equidistant terms. This can be written as $\left(T_p, T_{(n-p)+1}\right)$.

$And\ the\ sum\ of\ conjugate\ pair\ is\ a\ constant, i.e, \boldsymbol{S_{c.p}^{p} = \left(T_p + T_{(n-p)+1}\right) = k - - - - - (10)}$

$where\ k\ is\ a\ constant\ for\ given\ n\ of\ A.P\ and\ in\ S_{c.p}^{p}\ sum\ of\ conjugate\ terms\ of\ p^{th}\ term.$

$For\ instance, consider\ the\ sequance\ 1,3,5, \ldots, 23. In\ this\ sequance\ n = 12.$

$From\ (9)\ we\ have\ S_{c.p}^{p} = \left(T_p + T_{(n-p)+1}\right)$

$For\ p = 1, (9)\ will\ be\ ;\ \ S_{c.p}^{1} = \left(T_1 + T_{(12-1)+1}\right) = (T_1 + T_{12}) = (1 + 23) = 24$

$p = 2, ;\ \ S_{c.p}^{2} = \left(T_2 + T_{(12-2)+1}\right) = (T_1 + T_{11}) = (3 + 21) = 24$

$p = 3, ;\ \ S_{c.p}^{3} = \left(T_3 + T_{(12-3)+1}\right) = (T_3 + T_{10}) = (5 + 19) = 24$

$\vdots$

$For\ p = 6, (9)\ will\ be\ ;\ \ S_{c.p}^{6} = \left(T_6 + T_{(12-6)+1}\right) = (T_6 + T_7) = (1 + 23) = 24$

$Here\ maximum\ value\ of\ p = \frac{n}{2} when\ n\ is\ even\ number\ and\ p = \frac{n-1}{2}\ when\ n\ is\ an\ odd\ number.$

$And\ a\ middle\ term\ exists\ when\ n\ is\ an\ odd\ number.$

$In\ this\ case, number\ of\ conjugate\ pairs\ is\ n_{c.p} = \frac{12}{2} = 6;\ where\ n_{c.p}\ denotes\ number\ of\ conjugate\ pairs.$

$If\ A.P\ is\ 1, 3, 5, \ldots, 25.\ Here\ conjugates\ are\ as\ following,$

$For\ p = 1, (9)\ will\ be\ ;\ \ S_{c.p}^{1} = \left(T_1 + T_{(13-1)+1}\right) = (T_1 + T_{13}) = (1 + 25) = 26$

$For\ p = 2, (9)\ will\ be\ ;\ \ S_{c.p}^{2} = \left(T_2 + T_{(13-1)+1}\right) = (T_2 + T_{12}) = (3 + 23) = 26$

$\vdots$

$For\ p = 6, (9)\ will\ be\ ;\ \ S_{c.p}^{6} = \left(T_6 + T_{(13-6)+1}\right) = (T_6 + T_8) = (11 + 15) = 26$

$In\ this\ case\ mid\ term\ exists, hence\ number\ of\ conjugate\ terms\ are\ \frac{n-1}{2}, when$

$n\ is\ odd. Here\ n_{c.p} = \frac{13-1}{2} = 6\ pairs\ of\ conjugates\ and\ next\ term\ is\ middle\ term\ i.e., T_7 = 13.$

Properties of conjugate pair and terms.

1) $In\ \ conjugate\ pair, each\ term\ is\ conjugate\ to\ another. i.e., T_p\ is\ conjugate\ to\ \ T_{(n-p)+1}\ and\ vice\ versa.$

2) $When\ n\ is\ an\ odd\ number, Then\ value\ of\ mid-term\ is\ always\ equal\ to, T_{\left(\frac{n+1}{2}\right)} = \frac{\left[T_p + T_{(n-p)+1}\right]}{2}$

$Example. 4.1. Find\ the\ middle\ term\ of\ A.P. 10,7,4, \ldots, -62\ (AI\ CBSE\ 209\ C). [24]$

Method.1. Solution furnished in the book. [24]

$Solution$: $Here\ a = 10; d = 7 - 10 = -3;\ T_n = -62$

$\therefore using\ T_n = a + (n-1)d, we\ have$

$$-62 = 10 + (n-1) \times (-3)$$

$$\Longrightarrow n - 1 = \frac{-62 - 10}{-3} = \frac{-72}{-3} = 24$$

$$\Longrightarrow \quad n = 25$$

$$\therefore Middle\ term = \left(\frac{n+1}{2}\right) th\ term$$

$$= \frac{25+1}{2}$$

$$= 13^{th}\ term$$

$$Now\ T_{13} = 10 + 12d$$

$$= 10 + 12(-3)$$

$$= -26$$

$Thus\ the\ middle\ term = -26$

Method. 2. Solution using middle term equation , $T_{\left(\frac{n+1}{2}\right)} = \frac{\left[T_p + T_{(n-p)+1}\right]}{2}$

$Solution$: $Let\ we\ choose\ a\ conjugate\ terms, a = T_p = 10\ and\ l = T_{(n-p)+1} = -62$

$When\ n\ is\ considered\ as\ an\ odd\ number\ then\ middle\ term\ exists\ and\ putting\ the\ values\ in\ the\ middle\ term\ equation, we\ have\ T_{\left(\frac{n+1}{2}\right)} = \frac{[10 - 62]}{2}$

$$\Longrightarrow \quad Middle\ term\ T_{\left(\frac{n+1}{2}\right)} = -26$$

Hence the answer is also guessable by this method.

$\boldsymbol{Proof\ of\ mid\ term}$: $Consider\ the\ expression\ of\ mid\ term, T_{\left(\frac{n+1}{2}\right)} = \frac{\left[T_p + T_{(n-p)+1}\right]}{2}$ $whence\ n\ is\ odd\ number.$

$Let\ p = \frac{n+1}{2}$ $, here\ n\ is\ odd\ number.$

$Then, T_{\left(\frac{n+1}{2}\right)} = \frac{\left[T_p + T_{(n-p)+1}\right]}{2}$ $can\ be\ written\ as$

$$T_{\left(\frac{n+1}{2}\right)} = \frac{\left[T_{\left(\frac{n+1}{2}\right)} + T_{\left(n-\left(\frac{n+1}{2}\right)\right)+1}\right]}{2}$$

$$= \frac{\left[T_{\left(\frac{n+1}{2}\right)} + T_{\left(\frac{n+1}{2}\right)}\right]}{2}$$

$$= \frac{\left[2T_{\left(\frac{n+1}{2}\right)}\right]}{2}$$

$$T_{\left(\frac{n+1}{2}\right)} = T_{\left(\frac{n+1}{2}\right)}$$

$$LHS = RHS$$

Thus proved.

$Example. 4.2$: $let\ 3, 6, 9, \dots, 21\ is\ an\ A.P\ of\ 7\ terms, find\ mid\ term.$

$Here, \quad n = 7, then\ \frac{n+1}{2} = \frac{7+1}{2} = 4, T_4\ is\ the\ mid\ term. and\ value\ of\ this\ term\ is$

$$T_{\left(\frac{n+1}{2}\right)} = \frac{\left[T_p + T_{(n-p)+1}\right]}{2}$$

$Let\ value\ of\ p\ be\ arbitrarily\ choosen, i.e, let\ p = 3, then$

$Putting\ the\ values\ in\ the\ above\ equation, T_4 = \frac{\left[T_3 + T_{(7-3)+1}\right]}{2}$

$$T_4 = \frac{[T_3 + T_5]}{2}$$

$Middle\ term\ is;$ $\quad T_4 = \frac{9 + 15}{2} = 12$

3) $Sum\ of\ first\ 'n'\ terms\ is,\qquad S_n = \frac{n}{2}[k];.$

$Above\ equation\ can\ be\ written\ as\ S_n = \frac{n}{2}[\boldsymbol{S^p_{c.p}}]$

$Consequantly, \boldsymbol{S_n} = \frac{\boldsymbol{n}}{\boldsymbol{2}}\left[\left(\boldsymbol{T_p} + \boldsymbol{T_{(n-p)+1}}\right)\right] - - - - - (\mathbf{11})$

$Proof\ of\ (11) is\ very\ simple, i.e.,\ by\ putting\ p = 1\ we\ get\ the\ already\ known\ equation,$

$$S_n = \frac{n}{2}[T_1 + T_n]$$

4) $Sum\ of\ terms\ of\ each\ conjugate\ pair\ is\ constant\ to\ the\ given\ A.P\ to\ first\ n\ terms.$

$i.e., [T_1 + T_n] = \left[T_2 + T_{(n-1)}\right] = \cdots = \left[T_p + T_{(n-p)+1}\right] = \cdots = \left[T_n + [T_1]\right] = k$

$And\ k = 2a + (n-1)d$

Proof,

$We\ know\ that\ sum\ to\ first\ 'n'\ terms\ of\ an\ A.P\ is$

$S_n = T_1 + T_2 + T_3 + \cdots + T_p + \cdots + T_n - - - - - (1)$

$S_n = T_n + T_{(n-1)} + T_{(n-2)} + \cdots + T_{(n-p)+1} + \cdots + T_1 - - - - - (2)$ [Written in the reverse order]

Adding (1) and (2) we have,

$2S_n = [T_1 + T_n] + \left[T_2 + T_{(n-1)}\right] + \left[T_3 + T_{(n-2)}\right] + \cdots + \left[T_p + T_{(n-p)+1}\right] + \cdots + [T_n + T_1] - - - - - (3)$

Similarly, it is known that,

$2S_n = [2a + (n-1)d] + \cdots + [2a + (n-1)d] + \cdots + [2a + (n-1)d]\ upto\ n\ terms - - - (4)$

Comparing (3) and (4) we have

$[T_1 + T_n] = \left[T_2 + T_{(n-1)}\right] = \cdots = \left[T_p + T_{(n-p)+1}\right] = \cdots = \left[T_n + [T_1]\right] = [2a + (n-1)d]$

$\therefore k = [2a + (n-1)d]$

Thus proved.

Further, conjugate pairs that are made in a particular A.P, number of pairs will be equal to integral number (i.e., natural number), only when number of terms 'n' is an even number. Also, when n is odd number mid- term exists.

$i.e., When\ 'n'\ is\ even\ number, number\ of\ conjugate\ pairs = \frac{n}{2}$

$And, when\ 'n'\ is\ an\ odd\ number, then\ number\ of\ conjugate\ pairs = \frac{n-1}{2}\ and\ one$

$mid - term\ exists. Also\ the\ mid-\ term\ T_{\left(\frac{n+1}{2}\right)} is\ always\ equl\ to\ \frac{\left[T_p + T_{(n-p)+1}\right]}{2}.$

$Example. 4.3: In\ an\ A.P, if\ sum\ of\ conjugate\ terms\ 18^{th}\ and\ 52^{nd}\ terms\ is\ 176, find$
$middle\ term. And\ find\ the\ sum\ of\ first\ n\ terms.$ Also find the family of A.P.

$Solution: Data;\ p = 18;\ \left[T_p + T_{(n-p)+1}\right] = 176;\ T_{\left(\frac{n}{2}\right)} = ?, S_n = ?, and\ family\ of\ A.P = ?.$

$T_p = T_{18};\ \therefore p = 18$

$Similarly, T_{(n-p)+1} = T_{52}$

$\therefore Equating\ the\ subscripts\ of\ T\ on\ both\ sides, we\ have,\qquad (n-p) + 1 = 52$

$Putting\ the\ value\ of\ p\ in\ the\ above\ equation, we\ have, (n-18) + 1 = 52$

$Consequantly,\qquad n = 69$

$Since\ n = 69\ is\ an\ odd\ number, mid\ term\ exists,$

$$Hence, T_{\frac{n+1}{2}} = \frac{\left[T_p + T_{(n-p)+1}\right]}{2}$$

$$= \frac{176}{2} = 88$$

$$Also, 88 = \frac{k}{2}$$

$Further\ to\ find\ sum\ of\ first\ 69\ terms, S_n = n\left(\frac{k}{2}\right); since\ n\ is\ an\ odd\ number.$

$i.e.,\qquad S_{69} = 69 \times 88$

$\Rightarrow\qquad S_{69} = 6072$

Moreover to find the family of A.P, we have,

$\left[T_p + T_{(n-p)+1}\right] = 176$

$i.e., 2a + (n-1)d = 176$

$2a + (69-1)d = 176$

$2a + 68d = 176$

$a + 34d = 88$

This is the family of A.P in this case.

$For\ instance, let\ a = 10d, then\ by\ putting\ this\ relation\ in\ the\ above\ equation.$

$we\ have\ a + 34d = 88$

$10d + 34d = 88$

$\Rightarrow d = 2$

$Then,\quad a = 10d$

$= 10 \times 2$

$a = 20$

$Also, 69^{th}\ term\ is, T_{69} = 20 + (69 - 1)2$

$T_{69} = 156$

$Consequantly, member\ of\ family\ of\ \ A.P. is\ 10d, 11d, 12d, \dots, 78d$

$i.e., 20,22,24, \dots, 156$

This fulfils all the conditions that an A.P should have to exhibit, They are,

1) $Common\ differance\ is\ constant. i.e., d = 22 - 20 = 2\ and\ d = 24 - 22 = 2\ and\ so\ on.$

2)$sum\ of\ terms\ of\ each\ conjugate\ pairs\ is\ constant. for\ example, 20 + 156 = 22 + 154 = \cdots = 176$

3) $Since\ n = 69\ is\ an\ odd\ number, this\ A.P\ has\ a\ middle\ term.$

$and\ etc.$

$\therefore 20,22,24, \dots, 156\ is\ a\ member\ of\ family\ of\ A.P, a + 34d = 88$

$To\ find\ another\ one, let\ a = -23d$

$Then, a + 34d = 88\ \Longrightarrow -23d + 34d = 88$

$i.e., 11d = 88$

$hence, d = 8$

$Further\ to\ find\ the\ last\ term, T_n = a + (n - 1)d$

$= -23 \times 8 + (69 - 1)8$

$= -184 + 544$

$T_n = 360$

$Now\ another\ member\ of\ this\ family\ is, -184, -176, -168, \dots, 360$

$Similarly, this\ A.P, also\ follows\ all\ conditions.$

$i.e, 1)\ Common\ differance\ is\ constant.$

$i.e., d = -176 - (-184) = 8\ and\ d = -168 - (-176) = 8\ and\ so\ on.$

2)$sum\ of\ terms\ of\ each\ conjugate\ pairs\ is\ constant.$

$i.e., -184 + 360 = -176 + 352 = \cdots = 176.$

3) $Since\ n = 69\ which\ is\ an\ odd\ number, this\ A.P\ has\ a\ middle\ term.$

$And\ etc.$

$Example. 4.4: The\ sum\ of\ first\ and\ fifth\ terms\ of\ an\ A.P\ is\ equal\ to\ 26, and\ product\ of\ second\ and\ fourth\ is\ 160, find\ the\ sum\ of\ first\ six\ terms.$

$Solution: Given, T_1 + T_5 = 26\ \&\ T_2 \times T_4 = 160$

$From\ 1^{st}\ to\ 5^{th}\ term, number\ of\ terms\ is\ 5\ terms, then\ mid\ term\ is\ 3^{rd}\ term.$

$Since\ T_1 and\ T_5\ are\ conjugate\ terms\ \ in\ A.P\ of\ 1\ to\ 5\ terms$

$Hence, T_3 = \frac{T_1 + T_5}{2} = \frac{26}{2} = 13$

$Using\ value\ of\ T_3\ ,'d'\ can\ be\ found\ as\ following.$

$It\ is\ given\ that\ T_2 \times T_4 = 160$

$i.e.,\quad (T_3 - d)(T_3 + d) = 160$

$(13 - d)(13 + d) = 160$

$13^2 - d^2 = 160$

$\Rightarrow d = 3$

$Then, S_n = \frac{n}{2}\left[2T_p + \{(n + 1) - 2p\}d\right]$

$S_6 = \frac{6}{2}[2 \times 13 + \{(6 + 1) - 2 \times 3\}3]$

$S_6 = 87$

$Example. 4.5: Given\ a_3 = 15, S_{10} = 125, find\ d\ and\ a_{10}.$ [15]

This problem is solved at section III, example.3.1, using current method and remodelled equation. Here same problem is solved using (11) which is one of conjugate property.

$Solution: consider, S_n = \frac{n}{2}\left[\left(T_p + T_{(n-p)+1}\right)\right]$

$Here\ T_3 = 15,\qquad n = 10,$

$$Then\ ,125 = \frac{10}{2}[15 + T_{10-3+1}]$$
$$\Rightarrow \quad 25 = 15 + T_8$$
$$\Rightarrow \quad T_8 = 10$$
$$Then, d = \frac{T_8 - T_3}{8-3}$$
$$= \frac{10-5}{5}$$
$$d = 1$$
$$Then, T_{10} = T_8 + 2d$$
$$= 10 + 2$$
$$T_{10} = 12$$

V. Conlusion

Generalisation of equations and shortcuts to results will be helpful to students. This article is aimed at both perspectives.

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