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On the Incidence Chromatic Number of Sierpiński Graphs

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Abstract : *In this paper we consider the incidence coloring of Sierpiński graphs $S(n,k)$ and prove that the incidence chromatic number of Sierpiński graphs $S(n,3)$ is 4 when $n > 1$. Moreover, an alternative proof for the incidence chromatic number of the complete bipartite graph $K_{m,n}$ is given. Algorithms for coloring incidence graphs of Sierpiński graphs $S(n,3)$ and complete bipartite graph $K_{m,n}$ are presented explicitly.*
Keywords: *Sierpiński graphs, incidence graph, incidence chromatic number*

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I. Introduction

The incidence coloring of a graph was first introduced by Brualdi and Massey (see [1]). An incidence in a graph $G = (V, E)$ is an ordered pair (v, e) with $v \in V$ and $e \in E$, such that the vertex v and the edge e are incident. The set of all incidences in G usually denoted by $I(G)$. For every vertex v, we denote by I_v the set of all incidences of the form $(v, \{v, w\})$ and by A_v the set of all incidences of the form $(w, \{w, v\})$.
The incidences $(v, e) \in I(G)$ and $(w, f) \in I(G)$ are called adjacent if one of the following holds:

i. $v = w$,
ii. $e = f$,
iii. The edge $\{v, w\}$ equals e or f.

A k-incidence coloring of a graph G is a mapping σ from $I(G)$ to a set X of k different colors such that adjacent incidences are assigned different colors. The incidence chromatic number $\chi_i(G)$ of G is the smallest number k such that G admits a k-incidence coloring.
Certainly, it is not easy to find the incidence chromatic number of an arbitrary graph, but many results on incidence coloring were obtained by several researchers, including especially Brualdi and Massey (see [1]). Brualdi and Massey proved that the following results hold for a graph G with maximum degree $\Delta(G)$:

$\chi_i(G) \geq \Delta(G) + 1,$ (1)
$\chi_i(G) \leq 2\Delta(G),$ (2)
$\chi_i(K_n) = n,\ (n \geq 2)$ (3)
$\chi_i(K_{m,n}) = m + 2,\ (m \geq n \geq 2)$ (4)
$\chi_i(T) = \Delta(T) + 1\ \text{(foreverytree}T\text{oforder}n \geq 2).$ (5)

Brualdi and Massey also conjectured that $\chi_i(G) \leq \Delta(G) + 2$, but in [2] Guiduli showed that $\chi_i(G) \geq \Delta(G) + \Omega(\log\Delta(G))$, which means that this conjecture is invalid. Guiduli also proved that $\chi_i(G) \leq \Delta(G) + O(\log\Delta(G))$ and showed that the problem of incidence coloring is a special case of directed star arboricity which was introduced by Algor and Alon (see [3]). The problem of determining incidence chromatic number of a graph has been extensively studied by many authors, and is still a fruitful area of research in graph theory (see [1, 2, 4—8] and references there in)Although Brualdi and Massey's result, given by equation (4), on complete bipartite graph $K_{m,n}$ is correct, their proof is sadly inaccurate (see [7]). In [7] a correction for the proof of equation (4) is proposed without any interpretation. In the following theorem, we provide an alternate proof for equality (4).

Theorem 1.1*For all $m \geq n \geq 2$, $\chi_i(K_{m,n}) = m + 2$.*

Proof. It is clear that $m+1$ colors are not sufficient to color $I(K_{m,n})$ (see [1]). So we have $m+2 \leq \chi_i(K_{m,n})$ and to complete the proof, it remains to be shown that $K_{m,n}$ has an incidence coloring with $m+2$ colors.Since for each $m \geq n \geq 2$, $K_{m,n}$ is a subgraph of $K_{m,m}$, the inequality $\chi_i(K_{m,n}) \leq \chi_i(K_{m,m})$ holds. Hence, it is enough to show that $\chi_i(K_{m,m}) \leq m+2$.

Then let us consider the complete bipartite graph $K_{m,m}$ with partition sets $V_1 = \{w_1, w_2, \dots, w_m\}$ and $V_2 = \{u_1, u_2, \dots, u_m\}$, and color it using the following method:

If $(w_i, \{w_i, u_j\}) \in I_{w_i} \subset I(K_{m,m})$ then

$$\sigma_1((w_i, \{w_i, u_j\})) = \begin{cases} m & \text{if} j = m-i+1, \\ m+1 & \text{if} j = m \text{and} i \neq 1, \\ j & \text{otherwise.} \end{cases}$$

If $(u_j, \{u_j, w_i\}) \in A_{w_i} \subset I(K_{m,m})$ then

$$\sigma_2((u_j, \{u_j, w_i\})) = \begin{cases} m+2 & if j = m-i+1, \\ m+1 & if j \neq m \text{and} i = 1, \\ m-i+1 & \text{otherwise,} \end{cases}$$

where $i, j \in \{1,2,\dots,m\}$.

By means of σ_1 and σ_2 we define a new function $\sigma: I(K_{m,m}) \to \{1,2,\dots,m+2\}$ by

$$\sigma((v,e)) = \begin{cases} \sigma_1((v,e)) & \text{if} v \in V_1, \\ \sigma_2((v,e)) & \text{if} v \in V_2. \end{cases}$$

We shall now show that adjacent incidences are assigned different colors by σ. Let $(w_i, \{w_i, u_j\}) \in I_{w_i}$ be any incidence element then it has three type of neighbors:

1. $(w_i, \{w_i, u_r\})$ $(1 \leq r \leq m, r \neq j)$, using the definition of σ_1, we have $\sigma((w_i, \{w_i, u_j\})) \neq \sigma((w_i, \{w_i, u_r\}))$.
2. $(u_k, \{u_k, w_i\})$ $(1 \leq k \leq m)$, by virtue of the definition of σ_1 and σ_2, we get $\sigma((w_i, \{w_i, u_j\})) \neq \sigma((u_k, \{u_k, w_i\}))$.
3. $(u_j, \{u_j, w_p\})$ $(1 \leq p \leq m)$, in the same way, because of the definition of σ_1 and σ_2, we obtain $\sigma((w_i, \{w_i, u_j\})) \neq \sigma((u_j, \{u_j, w_p\}))$.

Therefore, all neighbors of $(w_i, \{w_i, u_j\}) \in I_{w_i}$ are colored different from itself.Similarly, let $(u_j, \{u_j, w_i\}) \in A_{w_i}$ be any incidence element. If we interchange the partition sets V_1 and V_2, then one can easily conclude that $(u_j, \{u_j, w_i\})$ and its all neighbors are assigned different colors. This completes the proof.In the following example, we demonstrate how to color incidence graph of a complete bipartite graph by the method given in the proof of Theorem 1.1.

Example 1.2 We can give an incidence coloring of the complete bipartite graph $K_{4,4}$ using the method presented in the proof of Theorem 1.1 as follo*ws:*

$\sigma((w_1, \{w_1, u_1\})) = 1$	$\sigma((w_2, \{w_2, u_1\})) = 1$	$\sigma((w_3, \{w_3, u_1\})) = 1$	$\sigma((w_4, \{w_4, u_1\})) = 4,$
$\sigma((w_1, \{w_1, u_2\})) = 2$	$\sigma((w_2, \{w_2, u_2\})) = 2$	$\sigma((w_3, \{w_3, u_2\})) = 4$	$\sigma((w_4, \{w_4, u_2\})) = 2,$
$\sigma((w_1, \{w_1, u_3\})) = 3$	$\sigma((w_2, \{w_2, u_3\})) = 4$	$\sigma((w_3, \{w_3, u_3\})) = 3$	$\sigma((w_4, \{w_4, u_3\})) = 3,$
$\sigma((w_1, \{w_1, u_4\})) = 4$	$\sigma((w_2, \{w_2, u_4\})) = 5$	$\sigma((w_3, \{w_3, u_4\})) = 5$	$\sigma((w_4, \{w_4, u_4\})) = 5,$
$\sigma((u_1, \{u_1, w_1\})) = 5$	$\sigma((u_2, \{u_2, w_1\})) = 5$	$\sigma((u_3, \{u_3, w_1\})) = 5$	$\sigma((u_4, \{u_4, w_1\})) = 6,$
$\sigma((u_1, \{u_1, w_2\})) = 3$	$\sigma((u_2, \{u_2, w_2\})) = 3$	$\sigma((u_3, \{u_3, w_2\})) = 6$	$\sigma((u_4, \{u_4, w_2\})) = 3,$
$\sigma((u_1, \{u_1, w_3\})) = 2$	$\sigma((u_2, \{u_2, w_3\})) = 6$	$\sigma((u_3, \{u_3, w_3\})) = 2$	$\sigma((u_4, \{u_4, w_3\})) = 2,$
$\sigma((u_1, \{u_1, w_4\})) = 6$	$\sigma((u_2, \{u_2, w_4\})) = 1$	$\sigma((u_3, \{u_3, w_4\})) = 1$	$\sigma((u_4, \{u_4, w_4\})) = 1.$

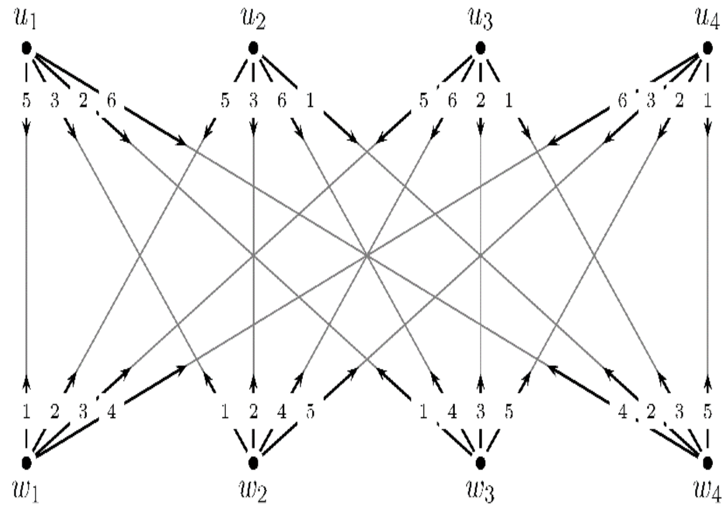

Figure1: A proper coloring of $I(K_{4,4})$ with 6 colors

In Fig. 1, $w_1 \bullet\!-\!\xrightarrow{1}\!-\!\bullet\, u_1$ means that the incidence element $(w_1, \{w_1, u_1\})$ is colored by color 1.
The Sierpiński graphs $S(n,k)$ $(n,k \geq 1)$ is defined on the vertex set $\{0,1,\dots,k-1\}^n$, in which two different vertices $u = (u_1,\dots,u_n)$ and $v = (v_1,\dots,v_n)$ are adjacent if and only if there exists an $h \in \{1,\dots,n\}$ such that

i. $u_t = v_t$, for $t = 1,\dots,h-1$,
ii. $u_h \neq v_h$, and
iii. $u_t = v_h$ and $v_t = u_h$ for $t = h+1,\dots,n$.

In the sequel we will use abbreviation $u_1u_2\dots u_n$ for the vertex $(u_1,\dots,u_n)$. From the definition of $S(n,k)$, it is clear that $|V(S(n,k))| = k^n$, $|E(S(n,k))| = \frac{k(k^n-1)}{2}$, $S(1,k) \cong K_k$ and $S(2,k) \cong P_{2^k}$. For $i \in \{0,1,\dots,k-1\}$ we call vertices of the form $ii\dots i$ extreme vertices of $S(n,k)$. Obviously, only k of k^n vertices are extreme vertices of $S(n,k)$. In Fig. 2, the Sierpiński graphs $S(2,5)$ and $S(3,4)$ with their vertex labelings are illustrated. Let $n \geq 2$, then for $i \in \{0,1,\dots,k-1\}$ we define $iS(n-1,k)$ be the subgraph of $S(n,k)$ induced by the vertices of the form $iv_2v_3\dots v_n$. The edge $\{ijj\dots j, jii\dots i\}$ is the unique edge between $iS(n-1,k)$ and $jS(n-1,k)$ and it is denoted by $e(n)_{ij}$ or $e(n)_{ji}$ (see Fig. 3). For further information on Sierpiński graphs $S(n,k)$, please see [9].

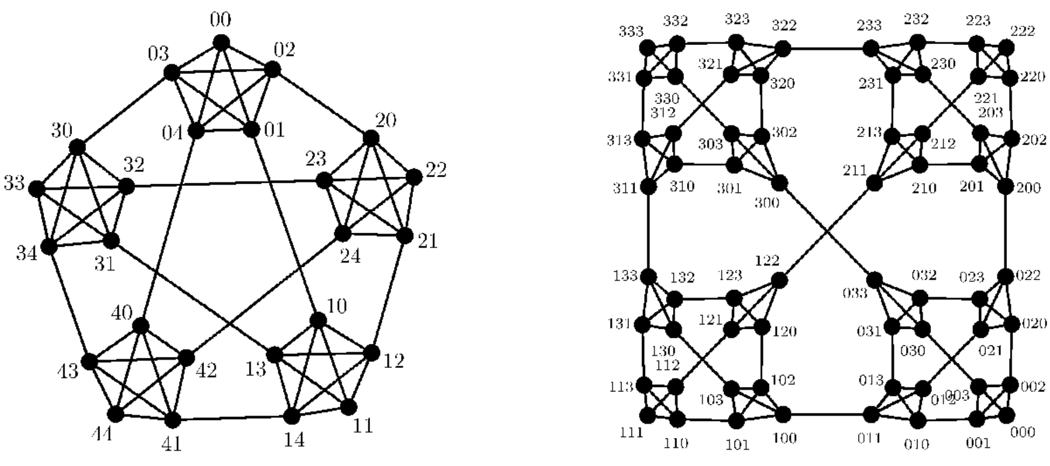

Figure2: Sierpiński graphs $S(2,5)$ and $S(3,4)$

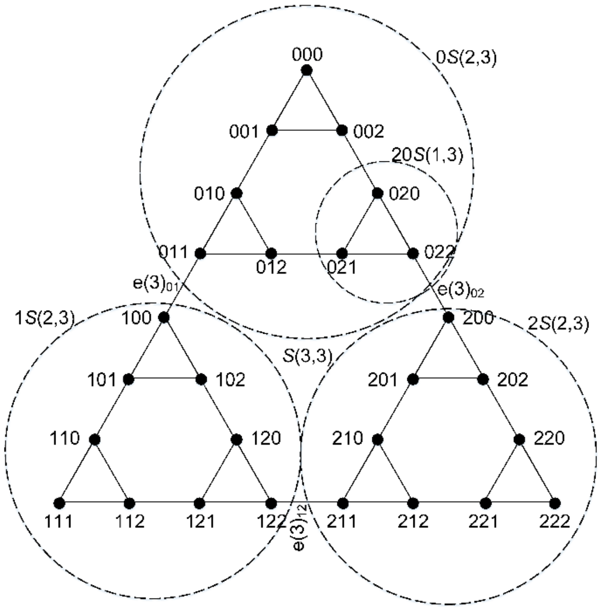

Figure3: Sierpiński graph $S(3,3)$ and its subgraphs $0S(2,3)$, $1S(2,3)$, $2S(2,3)$ and $20S(1,3)$

Sierpiński graphs and their coloring properties have been extensively studied in the literature by some authors. In [10], Parisse showed that $\chi(S(n,k)) = k$, for $n, k \in \mathbb{N}$. In [11], Klavžar showed that $\chi'(S(n,3)) = 3$ and this coloring is unique. This result was extended by Jakovac and Klavžar who showed that for any k, $\chi'(S(n,k)) = k$. They also proved that for any $n \geq 2$ and any odd $k \geq 3$, $\chi''(S(n,k)) = k+1$ (see [12]). The exact value of the total chromatic number of $S(n,k)$ was given by Hinz and Parisse and they obtained for any $k, n \geq 2$, $\chi''(S(n,k)) = k+1$ (see [13]).In [14], authors concentrated on Sierpiński graphs and presented certain results on their metric aspects,domination-type invariants with an emphasis on perfect codes, different colorings, and embeddingsinto other graphs.We previously studied the game chromatic number and the game coloring number of Sierpiński graphs, and obtain certain results (see [15]). Namely, the game chromatic number of a graph G is defined via a two-person finite game. Two players, generally called Alice and Bob, with Alice going first, alternatively color the uncolored vertices of G with a color from a color set X, such that no two adjacent vertices have the same color. Bob wins if at any stage of the game before the G is completely colored, one of the players has no legal move; otherwise, that is, if all the vertices of G are colored properly, Alice wins. The game chromatic number $\chi_g(G)$ of G is the least number of colors in the color set X for which Alice has a winning strategy.Accordingly, the game coloring number of a graph G is defined by modifying the rules of the above-mentioned coloring game as follows. The players fix a positive integer k, and instead of coloring vertices, only mark an unmarked vertex each turn. Bob wins if at some stage, some unmarked vertex has k marked neighbors, while Alice wins if this never happens. The game coloring number of a graph G is defined as the least number k for which Alice has a winning strategy on graph G, and it is denoted by the symbol $\mathrm{col}_g(G)$.
Although game chromatic number and game coloring number of the graphs are extensively studied, and many significant results are obtained, there are still many open problems in this subject (please see [15—29] and references therein).In the following theorem, we obtain the game coloring number of Sierpiński graphs $S(n,k)$, and particularly the game chromatic number of Sierpiński graphs $S(n,3)$.

Theorem 1.3 ([15])*For any* $n \geq 1$ *and* $k \geq 2$

$$\mathrm{col}_g(S(n,k)) = \begin{cases} k & \text{if} n = 1, \\ k+1 & \text{if} n > 1, \end{cases}$$

and

$$\chi_g(S(n,3)) = \begin{cases} 3 & \text{if} n = 1, \\ 4 & \text{if} n > 1. \end{cases}$$

We could not prove, but we conjecture that for each $k \geq 1$

$$\chi_g(S(n,k)) = \begin{cases} k & \text{if} n = 1, \\ k+1 & \text{if} n > 1. \end{cases}$$

1.2 Main Theorem

In this section, we state a theorem which gives the exact value of the incidence chromatic number of Sierpiński graphs $S(n,3)$.

Theorem 2.1 $\chi_i(S(n,3)) = \begin{cases} 3 & \text{if } n = 1, \\ 4 & \text{if } n > 1. \end{cases}$

Proof. If $n = 1$ then $S(1,3) \cong K_3$, and since for each $k \geq 1$, $\chi_i(K_k) = k$, we get $\chi_i(S(1,3)) = 3$.
Now let $n \geq 2$. Clearly, $\Delta(S(n,3)) = 3$ and from (1) we obtain

$$4 \leq \chi_i(S(n,3)).$$

Let $X = \{0,1,2,3\}$ be the set of colors. If we prove there is a proper coloring σ of $I(S(n,3))$ with colors from X, then we complete the proof for $n \geq 2$. We prove it by induction on n.
Consider the base case when $n = 2$, then we have a proper coloring σ_0 of $I(0S(1,3))$ using only 3 colors, since $\chi_i(0S(1,3)) = 3$. We choose σ_0 as follows:

$$\sigma_0: I(0S(1,3)) \to \{0,1,2\} \subset X,\ \ \sigma_0((0i,\{0i,0j\})) = j\ \ (i,j = 0,1,2).$$

Let us define a permutation p_1 from $I(0S(1,3))$ to $I(1S(1,3))$

$$p_1((00,\{00,01\})) = (11,\{11,10\}),\quad p_1((00,\{00,02\})) = (11,\{11,12\}),$$
$$p_1((01,\{01,00\})) = (10,\{10,11\}),\quad p_1((01,\{01,02\})) = (10,\{10,12\}),$$
$$p_1((02,\{02,00\})) = (12,\{12,11\}),\quad p_1((02,\{02,01\})) = (12,\{12,10\}),$$

and a permutation p_2 from $I(0S(1,3))$ to $I(2S(1,3))$

$$p_2((00,\{00,01\})) = (22,\{22,21\}),\quad p_2((00,\{00,02\})) = (22,\{22,20\}),$$
$$p_2((01,\{01,00\})) = (21,\{21,22\}),\quad p_2((01,\{01,02\})) = (21,\{21,20\}),$$
$$p_2((02,\{02,00\})) = (20,\{20,22\}),\quad p_2((02,\{02,01\})) = (20,\{20,21\}).$$

We also define two permutations $c_1, c_2: X \to X$,

$$c_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{pmatrix} \text{and} c_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{pmatrix}.$$

Now, composing the functions c_1, σ_0 and the inverse of p_1 we get a proper coloring of $I(1S(1,3))$ as follows:

$$\sigma_1: I(1S(1,3)) \to \{0,2,3\} \subset X,\ \ \sigma_1 = c_1 \circ \sigma_0 \circ p_1^{-1}.$$

Similarly, we obtain a proper coloring of $I(2S(1,3))$ as follows:

$$\sigma_2: I(2S(1,3)) \to \{0,1,3\} \subset X,\ \ \sigma_2 = c_2 \circ \sigma_0 \circ p_2^{-1}.$$

$$\begin{tikzcd} I(0S(1,3)) \arrow[r, "\sigma_0"] \arrow[d, "p_1"'] & X \arrow[d, "c_1"] \\ I(1S(1,3)) \arrow[r, "c_1 \circ \sigma_0 \circ p_1^{-1}"'] & X \end{tikzcd} \qquad \begin{tikzcd} I(0S(1,3)) \arrow[r, "\sigma_0"] \arrow[d, "p_2"'] & X \arrow[d, "c_2"] \\ I(2S(1,3)) \arrow[r, "c_2 \circ \sigma_0 \circ p_2^{-1}"'] & X \end{tikzcd}$$

Figure4: Proper colorings of $I(1S(1,3))$ and $I(2S(1,3))$ by virtue of proper coloring of $I(0S(1,3))$

Finally, we give a proper coloring of incidence elements $(ij, e(2)_{ij}) = (ij,\{ij,ji\}) i,j = 0,1,2,\ i \neq j$ which are not belong to $I(0(S(1,3)))$, $I(1(S(1,3)))$ and $I(2(S(1,3)))$. We color an incidence element $(ij,\{ij,ji\}) i,j = 0,1,2,\ i \neq j$ with a color not used in $I(iS(1,3))$. Then, since $3 \notin \mathcal{R}(\sigma_0)$, we can color incidence element $(0j,\{0j,j0\}))$ with color 3. Similarly, since $1 \notin \mathcal{R}(\sigma_1)$, we can color incidence element $(1j,\{1j,j1\}))$ with color 1, and finally since $2 \notin \mathcal{R}(\sigma_2)$, we can color incidence element $(2j,\{2j,j2\}))$ with color 2. Here, $\mathcal{R}(\sigma_0)$ denotes the range of σ_0. Hence, we get

$$\sigma^*((0j,\{0j,j0\})) = 3, \sigma^*((1j,\{1j,j1\})) = 1,\ \text{and} \sigma^*((2j,\{2j,j2\})) = 2.$$

As a result, by combining all these colorings together, we obtain the following proper coloring of $I(S(2,3))$:

$$\sigma: I(S(2,3)) \to X = \{0,1,2,3\},$$
$$\sigma((ij,\{ij,kl\})) = \begin{cases} \sigma_0((ij,\{ij,kl\})) & \text{if } (ij,\{ij,kl\}) \in I(0S(1,3)), \\ \sigma_1((ij,\{ij,kl\})) & \text{if } (ij,\{ij,kl\}) \in I(1S(1,3)), \\ \sigma_2((ij,\{ij,kl\})) & \text{if } (ij,\{ij,kl\}) \in I(2S(1,3)), \\ \sigma^*((ij,\{ij,kl\})) & \text{otherwise}, \end{cases}$$

where $i,j,k,l \in \{0,1,2\}$.

Now, suppose that the induction hypothesis holds for n where $n \geq 2$, that is, $\chi_i(S(n,3)) = 4$ for $n \geq 2$, and we want to prove that it also holds for $n+1$. By the induction hypothesis, we have a proper coloring $\sigma_0: I(0S(n,3)) \to X$ of $I(0S(n,3))$ using 4 colors.

Similar to the above strategy we define the permutation

$$p_1: I(0S(n,3)) \to I(1S(n,3)),$$
$$(0u_2u_3 \dots u_{n+1}, \{0u_2u_3 \dots u_{n+1}, 0v_2v_3 \dots v_{n+1}\}) \to$$
$$(1u_2'u_3' \dots u_{n+1}', \{1u_2'u_3' \dots u_{n+1}', 1v_2'v_3' \dots v_{n+1}'\})$$

such that for all $i = 2,3,\dots,n+1$

$$u_i' = \begin{cases} 1 & \text{if} u_i = 0, \\ 0 & \text{if} u_i = 1, \\ 2 & \text{if} u_i = 2, \end{cases} \text{and} v_i' = \begin{cases} 1 & \text{if} v_i = 0, \\ 0 & \text{if} v_i = 1, \\ 2 & \text{if} v_i = 2. \end{cases}$$

Likewise, we define the permutation

$$p_2: I(0S(n,3)) \to I(2S(n,3)),$$
$$(0u_2u_3 \dots u_{n+1}, \{0u_2u_3 \dots u_{n+1}, 0v_2v_3 \dots v_{n+1}\}) \to$$
$$(2u_2'u_3' \dots u_{n+1}', \{2u_2'u_3' \dots u_{n+1}', 2v_2'v_3' \dots v_{n+1}'\})$$

such that for all $i = 2,3,\dots,n+1$

$$u_i' = \begin{cases} 2 & \text{if} u_i = 0, \\ 1 & \text{if} u_i = 1, \\ 0 & \text{if} u_i = 2, \end{cases} \text{and} v_i' = \begin{cases} 2 & \text{if} v_i = 0, \\ 1 & \text{if} v_i = 1, \\ 0 & \text{if} v_i = 2. \end{cases}$$

We define two permutation functions $c_1, c_2: X \to X$,

$$c_1 = \begin{cases} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{pmatrix} & \text{if} n \text{isodd}, \\ \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 3 \end{pmatrix} & \text{if} n \text{iseven}; \end{cases}$$

and

$$c_2 = \begin{cases} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{pmatrix} & \text{if} n \text{isodd}, \\ \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 3 \end{pmatrix} & \text{if} n \text{iseven}. \end{cases}$$

Now, composing the functions c_1, σ_0 and the inverse of p_1 we get a proper coloring of $I(1S(n,3))$ as follows:

$$\sigma_1: I(1S(n,3)) \to X, \ \sigma_1 = c_1 \circ \sigma_0 \circ p_1^{-1}.$$

Analogously, we obtain a proper coloring of $I(2S(n,3))$ as follows:

$$\sigma_2: I(2S(n,3)) \to X, \ \sigma_2 = c_2 \circ \sigma_0 \circ p_2^{-1}.$$

Finally, we give a proper coloring of incidence elements

$$(ijj \dots j, e(n+1)_{ij}) = (ijj \dots j, \{ijj \dots j, jii \dots i\}), \ i,j = 0,1,2, \ i \neq j$$

which are not belong to $I(0(S(n,3)))$, $I(1(S(n,3)))$ and $I(2(S(n,3)))$. By the coloring strategy we always have a color from X which is not used in $I(i_1i_2 \dots i_nS(1,3))$ where $i_1, i_2, \dots, i_n \in \{0,1,2\}$. Thus, we can color an incidence element $(ijj \dots j, \{ijj \dots j, jii \dots i\}) i,j = 0,1,2, \ i \neq j$ with a color not used in $I(jj \dots jiS(1,3))$. Hence, we get

$$\sigma^*((0jj \dots j, \{0jj \dots j, j00 \dots 0\})) = 3,$$
$$\sigma^*((1jj \dots j, \{1jj \dots j, j11 \dots 1\})) = 1,$$
$$\sigma^*((2jj \dots j, \{2jj \dots j, j22 \dots 2\})) = 2$$

when n is odd, and

$$\sigma^*((i00 \dots 0, \{i00 \dots 0, 0ii \dots i\})) = 0,$$
$$\sigma^*((i11 \dots 1, \{i11 \dots 1, 1ii \dots i\})) = 1,$$

$$\sigma^*((i22\dots2,\{i22\dots2\,,2ii\dots i\}))=2$$

when n is even.

As a result, by combining all these colorings together, we obtain the following proper coloring of $I(S(n+1,3))$:

$$\sigma: I(S(n+1,3)) \to X=\{0,1,2,3\},$$

$$\sigma((u_1\dots u_{n+1},\{u_1\dots u_{n+1},v_1\dots v_{n+1}\}))=\begin{cases}\sigma_0((u_1\dots u_{n+1},\{u_1\dots u_{n+1},v_1\dots v_{n+1}\}))\ \text{if}\\ (u_1\dots u_{n+1},\{u_1\dots u_{n+1},v_1\dots v_{n+1}\})\in I(0S(n,3)),\\ \sigma_1((u_1\dots u_{n+1},\{u_1\dots u_{n+1},v_1\dots v_{n+1}\}))\ \text{if}\\ (u_1\dots u_{n+1},\{u_1\dots u_{n+1},v_1\dots v_{n+1}\})\in I(1S(n,3)),\\ \sigma_2((u_1\dots u_{n+1},\{u_1\dots u_{n+1},v_1\dots v_{n+1}\}))\ \text{if}\\ (u_1\dots u_{n+1},\{u_1\dots u_{n+1},v_1\dots v_{n+1}\})\in I(2S(n,3)),\\ \sigma^*((u_1\dots u_{n+1},\{u_1\dots u_{n+1},v_1\dots v_{n+1}\}))\ \text{otherwise.}\end{cases}$$

This completes the proof.

Although, for any n and k, it is not easy to use the technique, presented in the proof of Theorem 2.1, to determine the incidence chromatic number of Sierpiński graphs $S(n,k)$, we conjecture that

$$\chi_i(S(n,k))=\begin{cases}k & \text{if} n=1,\\ k+1 & \text{if} n>1.\end{cases}$$

II. Conclusion

In this study,we focus on Sierpińskigraphsandprove that the incidence chromatic number of Sierpiński graphs $S(n,3)$is 4 when$n>1$. We also presentaproof for the incidence chromatic number of the complete bipartite graph $K_{m,n}$. We have given algorithms for coloring incidence graphs of Sierpińskigraphs$S(n,3)$ and complete bipartite graph $K_{m,n}$. Further, we conjecture that the incidence chromatic number of Sierpński graphs $S(n,k)$ is $k+1$ when $n>1$.

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