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The Derivation of Implicit Second Derivative Method for Solving Second-Order Stiff Ordinary Differential Equations Odes.

Y. Skwame,
Department of Mathematic, Adamawa State University, Mubi-Nigeria.
Corresponding Author: Y. Skwame

Abstract: *A single-step hybrid block method for initial value problems of general second order Ordinary Differential Equations has been studied in this paper. In the derivation of the method, power series is adopted as basis function to obtain the main continuous scheme through collocation and interpolations approach. Taylor method is also used together with new method to generate the non-overlapping numerical results. The newly constructed method is then applied to solve the system of second-order stiff ordinary differential equations and the accuracy is better when compared with the existing methods in terms of error.*

Keywords: *Power Series, Collocation and Interpolation Method, Hybrid Block Method, Stiff ODEs, System of Second Order ODEs.*

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I. Introduction

Most real life problems that arise in various fields of study be it engineering or science are modeled as mathematical models before they are solved. These models often lead to differential equations. Numerous problems such as chemical kinetics, orbital dynamics, circuit and control theory and Newton's second law applications involve second-order ODEs [1-2]. Ordinary differential equations (ODEs) are commonly used for mathematical modeling in many diverse fields such as engineering, operation research, industrial mathematics, behavioral sciences, artificial intelligence, management and sociology. This mathematical modeling is the art of translatingproblem from an application area into tractable mathematical formulations whose theoretical and numerical analysis provide insight, answers and guidance useful for the originating application [3]. This type of problem can be formulated either in terms of first-order or higher order ODEs. In this article, the system of second-order ODEs of the following form is considered.

$$\begin{aligned} &{}^{1}y''\,{}^{1}f\left(x^{1}, y^{1}, {}^{2}y'\right), \quad {}^{1}y(x_0)=a_0,\ {}^{1}y'(x_0)=b_0 \\ &{}^{2}y''\,{}^{2}f\left(x^{2}, y^{2}, {}^{2}y'\right), \quad {}^{2}y(x_0)=a_1,\ {}^{2}y'(x_0)=b_1 \\ &\qquad\vdots \\ &{}^{m}y''\,{}^{m}f\left(x^{m}, y^{m}, {}^{m}y'\right), \quad {}^{m}y(x_0)=a_0,\ {}^{m}y'(x_0)=b_m \end{aligned} \tag{1}$$

The method of solving higher-order ODEs by reducing them to a system of first-order approach involves more functions to evaluate them and then leads to a computational burden as mentioned in [4]-[5]. The multistep methods for solving higher-order ODEs directly have been developed by many scholars such as [6]-[10]. The aim of this paper is to develop a new numerical method for solving systems of second-order stiff ODEs.

II. Derivation Of The Method

In this section, a one-step hybrid block method with two off-step points, $x_{n+\frac{1}{7}}$ *and* $x_{n+\frac{7}{8}}$ for solving Equation (1) is derived. Let the power series of the form

$$ {}^{j}y(x)=\sum_{i=0}^{v+m-1} a_i\left(\frac{x-x_n}{h}\right)^{i}, \quad j=1,\ldots,m. \tag{2}$$

be the approximate solution to Equation (1) for $x \in [x_n, x_{n+1}]$ where $n = 0, 1, 2, \cdots N-1$, $a's$ are the real coefficients to be determined, v is the number of collocation points, m is the number of interpolation points and $h = x_n - x_{n-1}$ is a constant step size of the partition of interval $[a, b]$, which is given by $a = x_0 < x_1 < \cdots < x_N = b$.
Differentiating Equation (2) once and twice yields:

$$ {}^j y'(x) = {}^j f\left(x^j, y^j, {}^j y\right) = \sum_{i=1}^{v+m-1} \frac{i a_i}{h}\left(\frac{x - x_n}{h}\right)^{i-1}, \quad j = 1, \cdots, m. \tag{3}$$

$$ {}^j y''(x) = {}^j f\left(x^j, y^j, {}^j y'\right) = \sum_{i=2}^{v+m-1} \frac{i(i-1) a_i}{h^2}\left(\frac{x - x_n}{h}\right)^{i-2}, \quad j = 1, \cdots, m. \tag{4}$$

Interpolating Equation (2) at the selected intervals, i.e., x_n, and collocating Equation (3) and (4) at all points in the selected interval, i.e., $x_n, x_{n+\frac{1}{7}}, x_{n+\frac{7}{8}}$ *and* x_{n+1}, gives the following equations which can be written in matrix form:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{h} & \frac{1}{4h} & \frac{3}{64h} & \frac{1}{121h} & \frac{5}{4096h} & \frac{3}{16384h} & \frac{7}{262144h} & \frac{1}{32768h} \\
0 & \frac{1}{h} & \frac{7}{8h} & \frac{14}{64h} & \frac{343}{128h} & \frac{12005}{4096h} & \frac{50421}{16384h} & \frac{823543}{262144h} & \frac{8234543}{262144h} \\
0 & \frac{1}{h} & \frac{2}{h} & \frac{3}{h} & \frac{4}{h} & \frac{5}{h} & \frac{6}{h} & \frac{7}{h} & \frac{8}{h} \\
0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{h^2} & \frac{3}{4h^2} & \frac{31}{16h^2} & \frac{5}{128h^2} & \frac{15}{2048h^2} & \frac{21}{16384h^2} & \frac{14}{65536h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{21}{4h^2} & \frac{147}{16h^2} & \frac{1715}{128h^2} & \frac{36015}{2048h^2} & \frac{352947}{16384h^2} & \frac{823543}{32768h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2}
\end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix}
=
\begin{pmatrix} {}^j y_n \\ {}^j y_{n+\frac{1}{8}} \\ {}^j y_{n+\frac{7}{8}} \\ {}^j y_{n+1} \\ {}^j f_n \\ {}^j f_{n+\frac{1}{8}} \\ {}^j f_{n+\frac{7}{8}} \\ {}^j f_{n+1} \end{pmatrix}$$

$$j = 1, \cdots, m. \tag{5}$$

Applying the Gaussian elimination method on Equation (5) gives the coefficient $a_i's$, *for* $i = 0\,(1)\,10$.
These values are then substituted into Equation (2) to give the implicit continuous hybrid method of the form:

$$ {}^j y(x) = \sum_{i=\frac{1}{4}, \frac{1}{2}, \frac{3}{4}} {}^j \beta_i(x)^j f_{n+i} + \sum_{i=0}^{1} {}^j \beta_i(x)^j f_{n+i}, \quad j = 1, \cdots, m. \tag{6}$$

Differentiating Equation (6) once yields:

$$ {}^{j}y^{'}(x)=\sum_{i=\frac{1}{4},\frac{1}{2},\frac{3}{4}}\frac{d}{dx}{}^{j}\beta_i(x)^{j}f_{n+i}+\sum_{i=0}^{1}\frac{d}{dx}{}^{j}\beta_i(x)^{j}f_{n+i},\quad j=1,\cdots,m. \qquad (7)$$

Where the continuous schemes are

$$ {}^{j}\alpha_0=0 $$

$$ {}^{j}\beta_0= \quad x-x_n-\frac{13331}{147}\frac{(x-x_n)^3}{h^2}+\frac{418391}{686}\frac{(x-x_n)^4}{h^3} $$
$$ -\frac{505984}{343}\frac{(x-x_n)^5}{h^4}+\frac{1751168}{1029}\frac{(x-x_n)^6}{h^5}-\frac{2297856}{2401}\frac{(x-x_n)^7}{h^6}+\frac{72704}{343}\frac{(x-x_n)^8}{h^7} $$

$$ {}^{j}\beta_{\frac{1}{8}}= \quad \frac{51200}{567}\frac{(x-x_n)^3}{h^2}-\frac{799744}{1323}\frac{(x-x_n)^4}{h^3}+\frac{13428736}{9261}\frac{(x-x_n)^5}{h^4}-\frac{46174208}{27783}\frac{(x-x_n)^6}{h^5} $$
$$ +\frac{60293120}{64827}\frac{(x-x_n)^7}{h^6}-\frac{1900544}{9261}\frac{(x-x_n)^8}{h^7} $$

$$ {}^{j}\beta_{\frac{7}{8}}= \quad -\frac{26624}{3969}\frac{(x-x_n)^3}{h^2}+\frac{898048}{9261}\frac{(x-x_n)^4}{h^3}-\frac{4470784}{9261}\frac{(x-x_n)^5}{h^4}+\frac{24940544}{27783}\frac{(x-x_n)^6}{h^5} $$
$$ -\frac{46137344}{64827}\frac{(x-x_n)^7}{h^6}+\frac{1900544}{9261}\frac{(x-x_n)^8}{h^7} $$

$$ {}^{j}\beta_1= \quad \frac{149}{21}\frac{(x-x_n)^3}{h^2}-\frac{10033}{98}\frac{(x-x_n)^4}{h^3}+\frac{174208}{343}\frac{(x-x_n)^5}{h^4}-\frac{964736}{1029}\frac{(x-x_n)^6}{h^5} $$
$$ +\frac{1773568}{2401}\frac{(x-x_n)^7}{h^6}-\frac{72704}{343}\frac{(x-x_n)^8}{h^7} $$

$$ {}^{j}\gamma_0= \quad -\frac{1}{2}x_n\left(-x_n+2x\right)-\frac{142}{21}\frac{(x-x_n)^3}{h}+\frac{6833}{196}\frac{(x-x_n)^4}{h^2}-\frac{19072}{245}\frac{(x-x_n)^5}{h^3} $$
$$ +\frac{12736}{147}\frac{(x-x_n)^6}{h^4}-\frac{16384}{343}\frac{(x-x_n)^7}{h^5}+\frac{512}{49}\frac{(x-x_n)^8}{h^6} $$

$$ {}^{j}\gamma_{\frac{1}{8}}= \quad -\frac{128}{27}\frac{(x-x_n)^3}{h}+\frac{2752}{63}\frac{(x-x_n)^4}{h^2}-\frac{258176}{2205}\frac{(x-x_n)^5}{h^3}+\frac{187904}{1323}\frac{(x-x_n)^6}{h^4} $$
$$ -\frac{253952}{3087}\frac{(x-x_n)^7}{h^5}+\frac{8192}{441}\frac{(x-x_n)^8}{h^6} $$

$$ {}^{j}\gamma_{\frac{7}{8}}= \quad -\frac{128}{189}\frac{(x-x_n)^3}{h}+\frac{4288}{441}\frac{(x-x_n)^4}{h^2}-\frac{105344}{2205}\frac{(x-x_n)^5}{h^3}+\frac{114176}{1323}\frac{(x-x_n)^6}{h^4} $$
$$ -\frac{204800}{3087}\frac{(x-x_n)^7}{h^5}+\frac{8192}{441}\frac{(x-x_n)^8}{h^6} $$

$$ {}^{j}\gamma_1= \quad -\frac{1}{3}\frac{(x-x_n)^3}{h}+\frac{135}{28}\frac{(x-x_n)^4}{h^2}-\frac{5888}{245}\frac{(x-x_n)^5}{h^3}+\frac{6592}{147}\frac{(x-x_n)^6}{h^4} $$
$$ -\frac{12288}{343}\frac{(x-x_n)^7}{h^5}+\frac{512}{49}\frac{(x-x_n)^8}{h^6} $$

III. Convergenceanalysis

3.1 Order and error Constants of the Methods

According to [11] the order of the new method in Equation (5) is obtained by using the Taylor series and it is found that the developed method has an uniformly order Ten, with an error constants vector of:

$$C_8 = \left[1.4703\times 10^{-11},\quad 1.0490\times 10^{-8},\quad 1.0505\times 10^{-13},\quad 9.7083\times 10^{-8}\right]^T$$

3.2 Consistency

Definition 3.1: The hybrid block method (5) is said to be consistent if it has an order more than or equal to one i.e. $P \geq 1$. Therefore, the method is consistent.

3.3 Zero Stability

Definition 3.2: The hybrid block method (5)said to be zero stable if the first characteristic polynomial $\pi(r)$ having roots such that $|r_z| \leq 1 \;\; and \;\; if \;\; |r_z| = 1$, then the multiplicity of r_z must not greater than two.

In order to find the zero-stability of hybrid block method (5), we only consider the first characteristic polynomial of the method according to definition (3.2) as follows

$$\Pi(r) = \left| r^{[2]_3} - \bar{B}_1^{[3]_3} \right| = \left| r\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| = r^2(z-1)$$

Which implies $r = 0,\, 0,\, 1$.Hence the method is zero-stable since $|r_z| \leq 1 \;\; and \;\; if \;\; |r_z| = 1$.

3.4 Convergence

Theorem (3.1): Consistency and zero stability are sufficient condition for linear multistep method to be convergent.Since the method (5) is consistent and zero stable, it implies the method is convergent for all point.

3.5 Regions of Absolute Stability (RAS)

Using the MATLAB package, we were able to plot the stability regions of the block method (see fig. below). This is done by reformulating the block method as general linear method to obtain the values of the matrices according to [11], [12]. The matrices are substituted into the stability matrix and using MATLAB software, the absolute stability regions of the new methods are plotted as shown in fig. below.

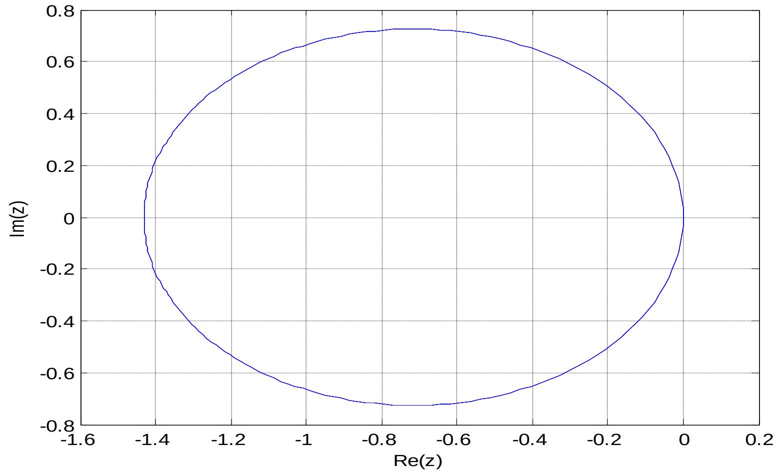

Figure: Region of Absolute Stability.

3.6 Numerical Implementation

To study the efficiency of the block hybrid method for $K=1$, we present some numerical examples by Skwame et-al 2017, [13]. In this section, the performance of the new single-step hybrid block method is examined using the following two systems of second-order initial value problems of ordinary differential equations with $K=2$ *and* $K=3$. Tables 1 and 2 show the comparison of the numerical results of the new method with the existing methods Skwame et-al 2017, [13]for solving Example 1 and 2.

Example 1

$$y_1' = 998y_1 + 1998y_2 \quad y_1(0)=1$$

$$y_2' = -999y_1 - 1999y_2 \quad y_2(0)=0, \quad h=0.1$$

With Exact Solution

$$y_1(x) = 2e^{-x} - e^{-1000x}$$

$$y_2(x) = -e^{-x} - e^{-1000x}$$

$$x \in [0,1]$$

(See Skwame et-al 2017, [13])

Table 1: Comparison of absolute errors for example 1.

X	Absolute error in Skwame et-al 2017, [13]				Absolute error in New method	
	$K=2$		$K=3$		$K=1$	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	2.43×10^{-2}	2.43×10^{-2}	5.82×10^{-2}	5.83×10^{-2}	2.98×10^{-1}	2.98×10^{-1}
0.2	3.87×10^{-2}	3.81×10^{-2}	4.02×10^{-3}	3.95×10^{-3}	8.90×10^{-2}	8.90×10^{-2}
0.3	9.31×10^{-4}	9.85×10^{-4}	9.17×10^{-3}	2.16×10^{-3}	2.65×10^{-2}	2.65×10^{-2}
0.4	1.51×10^{-5}	1.51×10^{-3}	7.13×10^{-4}	6.28×10^{-4}	7.91×10^{-3}	7.91×10^{-3}
0.5	2.32×10^{-5}	2.20×10^{-5}	1.20×10^{-4}	4.22×10^{-4}	2.35×10^{-3}	2.36×10^{-3}
0.6	6.99×10^{-5}	7.14×10^{-5}	2.28×10^{-4}	1.57×10^{-4}	6.97×10^{-4}	7.01×10^{-4}
0.7	2.15×10^{-5}	1.22×10^{-5}	1.60×10^{-4}	7.77×10^{-5}	2.03×10^{-4}	2.06×10^{-4}
0.8	2.34×10^{-5}	1.46×10^{-5}	1.52×10^{-4}	7.61×10^{-5}	5.52×10^{-5}	5.89×10^{-5}
0.9	2.17×10^{-5}	160×10^{-5}	1.36×10^{-4}	6.78×10^{-5}	1.11×10^{-5}	1.49×10^{-5}
1.0	1.97×10^{-5}	1.48×10^{-5}	1.52×10^{-4}	7.60×10^{-5}	2.10×10^{-6}	1.74×10^{-6}

Example 2

$$y_1' = 198y_1 + 199y_2 \quad y_1(0)=1$$

$$y_2' = -398y_1 - 399y_2 \quad y_2(0)=-1, \quad h=0.1$$

With Exact Solution

$$y_1(x) = e^{-x}$$

$$y_2(x) = -e^{-x}$$

$$x \in [0,1]$$

(See Skwame et-al 2017, [13])

Table 2: Comparison of absolute errors for example 2.

X	Absolute error in Skwame et-al 2017, [13]				Absolute error in New method	
	$K=2$		$K=3$		$K=1$	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	3.61×10^{-7}	3.60×10^{-7}	2.60×10^{-6}	2.60×10^{-6}	2.20×10^{-9}	3.17×10^{-8}
0.2	3.21×10^{-7}	3.30×10^{-7}	2.42×10^{-6}	2.42×10^{-6}	3.31×10^{-8}	1.00×10^{-9}
0.3	6.28×10^{-7}	3.27×10^{-7}	2.18×10^{-6}	2.18×10^{-6}	6.10×10^{-8}	3.27×10^{-8}
0.4	5.65×10^{-7}	5.65×10^{-7}	3.90×10^{-6}	3.90×10^{-6}	8.25×10^{-8}	5.63×10^{-8}
0.5	6.69×10^{-7}	6.68×10^{-7}	3.58×10^{-6}	3.58×10^{-6}	9.95×10^{-7}	7.55×10^{-8}
0.6	6.03×10^{-7}	6.02×10^{-7}	3.23×10^{-6}	3.23×10^{-6}	1.13×10^{-7}	9.12×10^{-8}
0.7	5.92×10^{-7}	5.92×10^{-7}	4.35×10^{-6}	4.35×10^{-6}	1.22×10^{-7}	1.04×10^{-7}
0.8	5.36×10^{-7}	5.37×10^{-7}	3.97×10^{-6}	3.97×10^{-6}	1.29×10^{-7}	1.11×10^{-7}
0.9	7.38×10^{-7}	7.38×10^{-7}	3.59×10^{-6}	3.59×10^{-6}	1.32×10^{-7}	1.17×10^{-7}
1.0	6.70×10^{-7}	6.70×10^{-7}	4.31×10^{-6}	4.30×10^{-6}	1.35×10^{-7}	1.21×10^{-7}

IV. Conclusions

In this article, an uniformly order eight implicit single-step block method with two off-step points is derived via the interpolation and collocation approach. The absolute errors arising from examples 1 and 2 using the new method were compared with the existing method [13]. Skwame et-al 2017 solves examples 1 and 2, it is evident from the tables presented above that the newly proposed method performs better than Skwame et-al 2017 [13]. The method is also desirable by virtue of possessing high order of accuracy. The developed method is consistent, $A-stable$, convergent, with a region of absolute stability and has uniformly order eight.

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