On Fuzzy Round Digraphs

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Abstract: In this paper we introduce Fuzzy Round Digraph by means of labelling. Also, some basic results of Fuzzy Round Digraph are discussed.

Keywords: Fuzzy digraph, Round Digraph, Locally semicomplete Digraph, Fuzzy Round Digraph.

I. Introduction

The concept of fuzzy graph was introduced by Rosenfeld [1] in 1975. Fuzzy graph theory has a vast area of applications. It is used in evaluation of human cardiac function, fuzzy neural networks, etc. Fuzzy graphs can be used to solve traffic light problem, time table scheduling, etc. In fuzzy set theory, there are different types of fuzzy graphs which may be a graph with crisp vertex set and fuzzy edge set or fuzzy vertex set and crisp edge set or fuzzy vertex set and fuzzy edge set or crisp vertices and edges with fuzzy connectivity, etc. A lot of works have been done on fuzzy graphs [3], [4], [5]. The Fuzzy Directed Graph, Fuzzy competition digraphs are well known topic. In this article Fuzzy Round digraphs are defined and their properties are investigated. Locally semicomplete fuzzy digraph, ordinary arc in a fuzzy round digraph are discussed

II. Preliminaries

Definition 2.1:
Fuzzy digraph $\mathcal{G} = (V, \sigma, \mu)$ is a non-empty set $V$ together with a pair of functions $\sigma : V \rightarrow [0,1]$ and $\mu : V \times V \rightarrow [0,1]$ such that for all $x, y \in V$, $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$. Since $\mu$ is well defined, a fuzzy digraph has at most two directed edges (which must have opposite directions) between any two vertices. Here $\mu(u,v)$ is denoted by the membership value of the edge $(u,v)$. The loop at a vertex $x$ is represented by $\mu(x,x) \neq 0$. Here $\mu$ need not be symmetric as $\mu(x,y)$ and $\mu(y,x)$ may have different values. The underlying crisp graph of directed fuzzy graph is the graph similarly obtained except the directed arcs are replaced by undirected edges.

Definition 2.2:
Fuzzy out-neighbourhood of a vertex $v$ of a directed fuzzy graph $\mathcal{G} = (V, \sigma, \mu)$ is the fuzzy set $N^+(v) = (X_v^+, m_v^+)$ where $X_v^+ = \{ u | \mu(v,u) > 0 \}$ and $m_v^+ : X_v^+ \rightarrow [0,1]$ defined by $m_v^+(u) = \mu(v,u)$. Similarly, fuzzy in-neighbourhood of a vertex $v$ of a directed fuzzy graph $\mathcal{G} = (V, \sigma, \mu)$ is the fuzzy set $N^-(v) = (X_v^-, m_v^-)$ where $X_v^- = \{ u | \mu(u,v) > 0 \}$ and $m_v^- : X_v^- \rightarrow [0,1]$ defined by $m_v^-(u) = \mu(u,v)$.
Example 2.2.1: Let \( \bar{G} \) be a directed fuzzy graph. Let the vertex set be \{a, b, c\} with membership values \( \sigma(a) = 0.4, \sigma(b) = 0.5, \sigma(c) = 0.6 \). The membership values of arcs are \( \mu(a,b) = 0.3, \mu(b,c) = 0.5 \). So \( N^+(a) = \{(b,0.3)\} \) and \( N^-(c) = \{(a,0.3),(b,0.5)\} \). (Note that \( (a,0.3) \) represents the vertex a with membership value \( \sigma(a) \)). It is shown in Figure 1.

Definition 2.3:

A Fuzzy Digraph on ‘n’ vertices is round if its vertices are labelled as \( v_1, v_2, \ldots, v_n \) so that or each
\[
\Gamma N^+(v_i) = (X_{v_{i+1}}, m_{v_{i+1}}^+) \ldots \ldots \ldots (X_{v_{i+d^+}(v_i)}, m_{v_{i+d^+}(v_i)}^+) \] and
\[
N^-(v_i) = (X_{v_{i-d^-}(v_i)}, m_{v_{i-d^-}(v_i)}^-) \ldots \ldots (X_{v_{i-1}}, m_{v_{i-1}}^-)
\]

Definition 2.4: An arc \( \langle x\sigma(x), y\sigma(y) \rangle \) of a fuzzy digraph \( D \) is ordinary iff \( y\sigma(y), x\sigma(x) \) is not in \( D \). A cycle or path \( Q \) of a fuzzy digraph \( D \) is ordinary if all arcs of \( Q \) are ordinary.

III. Fuzzy Round Digraph

Proposition 3.1: Every Fuzzy round digraph is locally semicomplete.

Proof: Let \( D_R \) be a Fuzzy round digraph and let \( v_1, v_2, \ldots, v_n \) be a round labelling of \( D_R \). Consider an arbitrary vertex, say \( v_i \). Let \( \langle x\sigma(x), y\sigma(y) \rangle \) be a pair of out-neighbours of \( v_i \). We show that and \( x\sigma(y) \) are adjacent. Assume without loss of generality that \( y\sigma(y), x\sigma(x) \) appear in that circular order in the round labelling. Since \( v_i \sigma(v_i), y\sigma(y) \) and the in-neighbours of \( y\sigma(y) \) appear consecutively preceding \( y\sigma(y), \) i.e., \( x\sigma(x) \rightarrow y\sigma(y) \). Thus the out-neighbours of \( v_i \) are pairwise adjacent. Similarly, we can show that the in-neighbours of \( v_i \) are also pairwise adjacent. Therefore, \( D_R \) is locally semicomplete.

(a\sigma(a) \) represents the vertex a with membership value \( \sigma(a) \))

Result 3.2: Let \( D_R \) be a Fuzzy round digraph then the following is true:
(a) Every induced fuzzysubdigraph of \( D_R \) is round.
(b) For each \( x\sigma(x) \in V(D_R) \), the subdigraphs induced by \( N^+(x\sigma(x)) - N(x\sigma(x)) \) and \( N^-(x\sigma(x)) - N^+(x\sigma(x)) \) are transitive tournaments.

Lemma 3.3: Let \( D_R \) be a fuzzy round digraph. Then, for each vertex \( x\sigma(x) \) of \( D_R \), the fuzzysubdigraph induced by \( N^+(x\sigma(x)) - N(x\sigma(x)) \) contains no ordinary cycle.

Proof: Suppose the fuzzy subdigraph induced by some \( N^+(x\sigma(x)) - N(x\sigma(x)) \) contains an ordinary cycle \( C \). Let \( v_1, v_2, \ldots, v_n \) be a round labelling of \( D_R \). Without loss of generality, assume that \( x\sigma(x) = v_1, \sigma(v_1) \). Then \( C \) must contain an arc \( \langle v_i, \sigma(v_i), \sigma(v_i) \rangle \), such that \( v_i, \sigma(v_i), v_j, \sigma(v_j) \in E(D_R) \) and \( i > j \). We have \( v_i, \sigma(v_i) \in EN(v_i, \sigma(v_i)) \) but \( v_j, \sigma(v_j) \notin EN(v_i, \sigma(v_i)) \) contradicting the assumption that \( v_1, \sigma(v_1), v_2, \sigma(v_2), \ldots, v_n \) is a round labelling of \( D_R \) ut.
Theorem 3.4: A connected locally semicomplete fuzzy digraph $D$ is round if and only if the following holds for each vertex $\sigma(x)$ of $D$:
(a) $N^+(\sigma(x)) - N^-(\sigma(x))$ and $N^-(\sigma(x)) - N^+(\sigma(x))$ induce transitive tournaments and
(b) $N^+(\sigma(x)) \cap N^-(\sigma(x))$ induces a (semicomplete) fuzzy subdigraph containing no ordinary cycle.

Proof: The necessity follows from Lemmas 3.3 and result 3.2. To prove the sufficiency, we consider two cases.

**Case 1:** $D$ has an ordinary cycle. We start by proving that $D$ contains an ordinary Hamilton cycle. Let $C = x_1, \sigma(x_1), x_2, \sigma(x_2), \ldots, x_k, \sigma(x_k)$ be a longest ordinary cycle in $D$. Assume that $k \neq n$, the number of vertices in $D$. Since $D$ is connected there is a vertex $v \in V(D) - V(C)$ such that $v \sigma(v)$ is adjacent to some vertex of $C$. Suppose that there is an ordinary arc between $v \sigma(v)$ and some vertex, say $x_1 \sigma(x_1)$, of $C$. We may without loss of generality assume that the ordinary arc $(x_1, y_1) \sigma(y_1)$ and $\sigma(y_1)$ are adjacent since they are out-neighbours of $x_1 \sigma(x_1)$. The arc between $v \sigma(v)$ and $\sigma(y_1)$ must be ordinary since $D$ does not contain as an induced subdigraph. Since $C$ is a longest ordinary cycle, $\sigma(y_1)$ cannot dominate $x_2 \sigma(x_2)$. Thus, $x_2 \sigma(x_2) \rightarrow v \sigma(v)$ similarly, we can prove that $\sigma(x_1) \rightarrow v \sigma(v)$ for every $i = 3, 4, \ldots, k$. Hence, $N^+(\sigma(x_1)) - N^-(\sigma(x_1))$ contains all vertices of $C$, which contradicts the assumption that $N^+(\sigma(x_1)) - N^-(\sigma(x_1))$ induces a transitive tournament. Since there is no ordinary arc between $v$ and $C$, we may assume that $v \sigma(v) x_1 \sigma(x_1) v \sigma(v)$ is a $2$-cycle of $D$. Using the fact that $D$ is locally semicomplete, it is easy to derive that $V(C) \subseteq N^+(\sigma(x_1)) \cap N^-(\sigma(x_1))$. This contradicts the assumption that $N^+(\sigma(x_1)) \cap N^-(\sigma(x_1))$ contains no ordinary cycle. Thus, we have shown that $D$ contains an ordinary Hamilton cycle. It follows from the definition of the labelling that $\sigma(x) \neq \sigma(y)$ for every $x, y \in V(D)$.

**Case 2:** $D$ contains no ordinary cycle. If $D$ has no ordinary arc, $D$ is complete. Thus, any labelling of $V(D)$ is round. So assume that $D$ has an ordinary arc. Since $D$ has an ordinary arc, but has no ordinary cycle, we claim that there is a vertex $z_1 \sigma(z_1) \in N^+(z_1, \sigma(z_1)) \cap N^-(z_1, \sigma(z_1)) \neq \emptyset$ and $N^+(z_1, \sigma(z_1)) \cap N^-(z_1, \sigma(z_1)) \neq \emptyset$. Indeed, let $w_1 \sigma(w_1) w_2 \sigma(w_2)$ be an ordinary arc in $D$. We may set $z_1 \sigma(z_1) = w_1 \sigma(w_2)$ unless $N^+(w_1 \sigma(w_2)) \cap N^-(w_1 \sigma(w_2)) \neq \emptyset$. In the last case there is an ordinary arc whose head is $w_1 \sigma(w_2)$. Let $w_1 \sigma(w_1) w_2 \sigma(w_2)$ be such an arc. Again, either we may set $z_1 \sigma(z_1) = w_1 \sigma(w_2)$ or there is an ordinary arc $w_1 \sigma(w_1) w_2 \sigma(w_2)$. Since $D$ is finite and contains no ordinary cycle, the above process cannot repeat vertices.
and hence terminates at some vertex \( w_i \sigma(w_i) \) such that we may set \( z_1 \sigma(z_1) = w_i \sigma(w_i) \). We apply the following algorithm to find a path in \( \mathcal{D} \). Begin with \( z_1 \sigma(z_1) \) and, for each \( i = 1,2,\ldots \), let \( w_{i+1} \sigma(w_{i+1}) \) be the vertex of in-degree zero in the (transitive) tournament induced by \( N(z_i \sigma(z_i)) \cup N(z_{i+1} \sigma(z_{i+1})) \) unless this set is empty. Since \( \mathcal{D} \) has no ordinary cycle, this produces a path \( P = z_1 \sigma(z_1), z_2 \sigma(z_2), \ldots, z_k \sigma(z_k) \) with \( N(z_i \sigma(z_i)) \cup N(z_{i+1} \sigma(z_{i+1})) \neq \emptyset \). Applying an argument similar to that used above, we can show that \( z_1 \sigma(z_1), z_2 \sigma(z_2), \ldots, z_k \sigma(z_k) \) is a round labelling of the fuzzy subdigraph induced by \( V(P) \). Thus, if \( P \) contains all vertices of \( \mathcal{D} \), then a round labelling of \( \mathcal{D} \) is established. So assume that there is a vertex \( v \) not in \( P \), which is adjacent to some vertex of \( P \). It is easy to see that there is no ordinary arc between \( v \) and \( P \). This implies that \( v \) is a vertex in \( \mathcal{D} \) not in \( P \). Applying an argument similar to that used above, we can show that \( z_1 \sigma(z_1), z_2 \sigma(z_2), \ldots, z_k \sigma(z_k) \) is a round labelling of the fuzzy subdigraph induced by \( V(P) \). Therefore, if we apply the above algorithm starting from an appropriate vertex not in \( P \), we obtain a collection of vertex-disjoint ordinary paths \( P^k = z_1^k \sigma(z_1^k), z_2^k \sigma(z_2^k), \ldots, z_t^k \sigma(z_t^k) \) \( k = 1,2,\ldots \). Let \( z_1^t+1 \sigma(z_1^t+1), \ldots, z_m^t+1 \sigma(z_m^t+1) \) be the remaining vertices. It is easy to verify that labelling the vertices according to the ordering

\[
( z_1^t \sigma(z_1^t), z_2^t \sigma(z_2^t), \ldots, z_m^t \sigma(z_m^t), z_1^t+1 \sigma(z_1^t+1), \ldots, z_m^t+1 \sigma(z_m^t+1) )
\]

results in a round labelling of \( \mathcal{D} \).

IV. Conclusion

Finally, we have analyzed about the fuzzy round digraph and the conditions for the fuzzy locally semicomplete digraph being a fuzzy round digraph and their properties. This study will help fulfill in systems where fuzzy digraphs are applied.

References


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