IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 14, Issue 5 Ver. I (Sep - Oct 2018), PP 36-40
www.iosrjournals.org

Application of Residue Inversion Formula for Solving System of Initial Value Problems of Linear Ordinary Differential Equations with Constant Coefficients

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Abstract: *The Laplace Transformation is one of the most widely and frequently used transformation in sciences and Engineering. Its application in solving initial value Problems (IVP) of ordinary differential equations (ODE's) is well known to scholars. In this paper we reviewed the traditional algebraic method (i.e. the Laplace Transformation Method) of solving system of linear Ordinary Differential Equations with constant coefficients and now show how the newly established Residue Inversion Formula can best be applied directly in obtaining the Inverse Laplace Transform when solving system of linear ode's with constant coefficients hence, simplify the traditional method. This new Residue approached eliminates computational stress and resultant time wastage by circumventing the rigor of resolving into partial fractions and the use of Table of Laplace that is not always readily available. Numerical results experimented by applying the Residue Inversion approach in solving system of initial value problems of linear ordinary differential equations with constant coefficients are proven to be elegant, efficient, valid and reliable.*

Keywords:*Residue Inversion Formula, Initial Value Problems of Simultaneous Ordinary Differential equations, Table of Laplace Transform, Partial fractions,*

Date of Submission: 10-09-2018 Date of acceptance: 28-09-2018

I. Introduction

Laplace transforms is one of the most widely and frequently use transformation in solving ordinary differential equations [1]. It is mostly applied when solving differential equations with Initial Value Problems, which intricacies are well known to scholars. When it's applied in solving initial valued linear ode's or system of initial valued linear ode's, one of the major challenge encountered is, finding the Inverse Laplace transform of some unknown functions that always arises[2]. The traditional method of finding the Inverse Laplace Transformation is by resolving this unknown function into partial fractions before retracting their Inverse from the table of Laplace which is mostly intractable, more so, the table of Laplace is not always readily available [3]. This traditional method always subjects the researcher into serious computational stress, cramming of the Table of Laplace and resultant time wastage [4]. Over the years little or nothing has been done to address this challenge, in most scenarios, researchers shy away to other analytical methods. The work of [4] is truly a giant stride in addressing this challenge headlock. In their work [4], the established a new method of finding the Inverse Laplace Transform christen ***"Residue Inversion Formula (RIF)".*** This RIF employ the famous Cauchy Residue Theorem to find the Residues of $e^{-st}\,F(s)$ at the poles of $F(s)$, where $F(s)$ is that unknown function whose Inverse Laplace Transform is being sought after. The Inverse Laplace Transform of the unknown function is now the sum of these Residues of $e^{-st}\,F(s)$ at the poles of $F(s)$ [4 & 5]. This RIF approached truly circumvent any sort of resolving into partial fraction and also discard the reliance on table of Laplace for retracting the inverse Laplace Transformation that is not always readily available. The Work of [4] however, was limited to initial problems of ordinary differential equations. In this article, we seek to extends the application of this RIF from solving initial problems of ordinary differential equations to solving system of initial value problems of linear ordinary differential equations with constant coefficients and to also investigate its reliability, effectiveness, robustness, accuracy and doggedness.

II. Methodology

1.1 Review of the Laplace Transform Method of Solving System of Linear Ordinary Differential Equations

The traditional method (i.e. Laplace Transform Method) of solving system of ordinary differential equations is well known to scholars. However, we will review the method briefly and highlight some loopholes;

Without loss in generality say; given $n-$ system of initial value problems of linear ordinary differential equations with constant coefficients as

$$x_1' + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = g_1(t)$$
$$x_2' + a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = g_2(t)$$
$$x_3' + a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = g_3(t)$$
$$\vdots$$
$$x_n' + a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = g_n(t)$$

$$with\ x_1(0) = \alpha_1,\ x_2(0) = \alpha_2,\ x_3(0) = \alpha_3 \ldots\ x_n(0) = \alpha_n \qquad (1)$$

Where $x_1,\ x_2,\ x_3, \ldots, x_n$ are unknown functions of t

$a_{ij}, i, j = 1,2,3, \ldots, n$ are the constant coefficients and

$g_i(t), i = 1,2,3, \ldots, n$ are the right hand vectors

In solving (1) using the traditional algebraic method (Laplace Transform Method), we have the following steps;

First, take the Laplace Transform of (1)

$$L\{x_1' + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = g_1(t)\}$$
$$L\{x_2' + a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = g_2(t)\}$$
$$=> \quad L\{x_3' + a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = g_3(t)\}$$
$$\vdots$$
$$L\{x_n' + a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = g_n(t)\}$$

Secondly, apply the initial conditions $x_1(0) = \alpha_1,\ x_2(0) = \alpha_2,\ x_3(0) = \alpha_3 \ldots\ x_n(0) = \alpha_n$ and let say $L\{x_1\} = \bar{x}_1,\ L\{x_2\} = \bar{x}_2,\ L\{x_3\} = \bar{x}_3, \ldots, L\{x_n\} = \bar{x}_n$

Thirdly, solve for $\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3, \ldots, \bar{x}_n$ simultaneously

Finally, find the Inverse Laplace Transform of $\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3, \ldots, \bar{x}_n$

i.e. $x_1 = L^{-1}\{\bar{x}_1\},\ x_2 = L^{-1}\{\bar{x}_2\},\ x_3 = L^{-1}\{\bar{x}_3\}, \ldots, x_n = L^{-1}\{\bar{x}_n\}$ to get solution $x_1, x_2,\ x_3, \ldots, x_n$

Now,

The challenge of finding this Inverse Laplace Transforms in using the traditional method is

i) Resolving the unknown functions $\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3, \ldots, \bar{x}_n$ into partial fractions. This is always very tedious resulting into computational stress and resultant time wastage

ii) Retracting the corresponding Inverse Laplace Transform from the Table of Laplace. This is mostly intractable and more so, the Table of Laplace is not always readily available.

2.2. Residue Inversion Approach of Finding Inverse Laplace Transform

The Challenges outline in the traditional method of finding the Inverse Laplace Transform can now be circumvent from the established Residue Inversion Formula (RIF) in the following ways;

i. Let $\bar{x}_i = F_i(s)$, $i = 1,2,3, \ldots, n$

ii. Find the poles of each of the $F_i(s) = \bar{x}_i$

iii. Find the Residues of $e^{st}F_i(s)$ at each of its poles

iv. Finally, from the Residue Inversion Formula $x_i(t) = L^{-1}\{\bar{x}_i\} = L^{-1}\{F_i(s)\} = sum\ of\ residues\ of\ e^{st}F_i(s)\ at\ the\ poles\ of\ F_i(s)$

III. Numerical Experiment

Solve the below system of initial value Problems (IVP) of ordinary differential Equations

1. Solve the system of linear ordinary differential equation

$$\frac{dx}{dt} + 5x - 2y = t$$
$$\frac{dy}{dt} + 2x + y = 0, \qquad y(0) = x(0) = 1$$

2. Solve the Stiff differential equation

$$x_1{}' = x_1 - x_2$$
$$x_2{}' = -x_1 + x_2 \qquad x_1(0) = 6, x_2(0) = 4$$

Solution 1:

Given $\frac{dx}{dt} + 5x - 2y = t$

$\frac{dy}{dt} + 2x + y = 0,$

Take the Laplace transform of the equations

$$=> L\left\{\frac{dy}{dx}\right\} + 5L\{x\} - 2L\{y\} = L\{t\} \;=>\; SL\{x\} - S(0) + 5L\{x\} - 2L\{y\} = \frac{1}{S^2}$$
$$L\left\{\frac{dy}{dt}\right\} + 2L\{x\} + L\{y\} = L\{0\} \quad => \; SL\{y\} - y(0) + 2L\{x\} + L\{y\} = 0$$

Applying the initial conditions and letting $L\{x\} = \bar{x}$ and $L\{y\} = \bar{y}$

$$=> \quad (s+5)\bar{x} - 2\bar{y} = \frac{1}{S^2} \ldots\ldots\ldots *$$
$$2\bar{x} + (s+1)\bar{y} = 0 \ldots\ldots\ldots **$$

Solving $*$ and $**$simultenuoesely for $\bar{x}$ and $\bar{y}$

$$=> \bar{x} = \frac{S+1}{S^2(S+3)^2} \; and \; \bar{y} = -\frac{2}{S^2(S+3)^2}$$
$$=> x = L^{-1}\left\{\frac{S+1}{S^2(S+3)^2}\right\} \quad \text{and} \quad y = L^{-1}\left\{-\frac{2}{S^2(S+3)^2}\right\}$$

Now to find x and y we can circumvent the traditional method of resolving into partial fraction and the application of the table of Laplace, by applying the established Residue Inversion formula (RIF) which state that

$$x(t) = L^{-1}\{F(x)\} = \text{sum of residues at } e^{st}F(x) \text{ at the poles of } F(s)$$

Now,

For $x = L^{-1}\left\{\frac{S+1}{S^2(S+3)^2}\right\}$, clearly $\bar{x} = F_1(s) = \frac{S+1}{S^2(S+3)^2}$ have poles of order 2 at $s = 0$ and at $s = -3$

From definition of residue at pole z_0 of order n $\text{Res}_{s=z_0} F(s) = \lim_{s\to 0} \frac{d^{n-1}}{ds^{n-1}}[(s - z_0)^n F(s)]$

At $s = 0$ which is a pole of order 2

$$=> Res_{s=0}e^{st} F_1(s) = Res_{s=0}e^{st}\left\{\frac{S+1}{S^2(S+3)^2}\right\} = \lim_{s\to 0}\left\{\frac{d}{ds}\left[(s-0)^2 e^{st}\left\{\frac{S+1}{S^2(S+3)^2}\right\}\right]\right\}$$
$$= \lim_{s\to 0}\left\{\frac{d}{ds}\left[e^{st}\left\{\frac{S+1}{(S+3)^2}\right\}\right]\right\} = \lim_{s\to 0}\left\{\frac{(S+3)^2[e^{st} + te^{st}(s+1)] - 2e^{st}(s+1)}{(S+3)^2}\right\}$$
$$= \frac{3[1+t]-2}{27} = \frac{1}{27} + \frac{t}{9}$$

Similarly, at $s = -3$ which is also a pole of order 2

$$=> Res_{s=-3}e^{st}\left\{\frac{S+1}{S^2(S+3)^2}\right\} = \lim_{s\to -3}\left\{\frac{d}{ds}\left[(s+3)^2 e^{st}\left\{\frac{S+1}{S^2(S+3)^2}\right\}\right]\right\}$$
$$= \lim_{s\to -3}\left\{\frac{d}{ds}\left[e^{st}\left\{\frac{S+1}{s^2}\right\}\right]\right\} = \lim_{s\to -3}\left\{\frac{d}{ds}[s^{-2}e^{st} + s^{-1}e^{st}]\right\}$$
$$= \lim_{s\to -3}\{ts^{-2}e^{st} - 2s^{-3}e^{st} + ts^{-1}e^{st} - s^{-2}e^{st}\}$$
$$= -\frac{1}{27}e^{-3t} + \frac{2}{9}te^{-3t}$$

From RIF $x(t) = L^{-1}\left\{\frac{S+1}{S^2(S+3)^2}\right\}$ = sum of residues at $e^{st}\left\{\frac{S+1}{S^2(S+3)^2}\right\}$

$$=> x(t) = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} + \frac{2}{9}te^{-3t}$$

Similarly,

For $y = L^{-1}\left\{-\frac{2}{S^2(S+3)^2}\right\}$, clearly $\bar{y} = F_2(s) = -\frac{2}{S^2(S+3)^2}$ have both poles of order 2 at $s = 0$ and -3

At $s = 0$

$$=> Res_{s=0}e^{st} F_2(s) = Res_{s=0}e^{st}\left\{-\frac{2}{S^2(S+3)^2}\right\} = \lim_{s\to 0}\left\{\frac{d}{ds}\left[(s-0)^2 e^{st}\left\{-\frac{2}{S^2(S+3)^2}\right\}\right]\right\}$$
$$= \lim_{s\to 0}\left\{\frac{d}{ds}\left[\frac{-2e^{st}}{(S+3)^2}\right]\right\} = -\frac{2}{9}t + \frac{4}{27}$$

Similarly, at $s = -3$

$$=> Res_{s=-3}e^{st}\left\{-\frac{2}{S^2(S+3)^2}\right\} = \lim_{s\to -3}\left\{\frac{d}{ds}\left[(s+3)^2 e^{st}\left\{-\frac{2}{S^2(S+3)^2}\right\}\right]\right\}$$

$$= \lim_{s\to -3}\left\{\frac{d}{ds}\left[\frac{-2e^{st}}{s^2}\right]\right\} = -\frac{2}{9}te^{-3t} + \frac{4}{27}e^{-3t}$$
$$= -\frac{1}{27}e^{-3t} + \frac{2}{9}te^{-3t}$$

From RIF $y = L^{-1}\left\{-\frac{2}{S^2(S+3)^2}\right\}$ = sum of residues at $e^{st}\left\{-\frac{2}{S^2(S+3)^2}\right\}$

$$=> y(t) = \frac{4}{27} - \frac{2}{9}t - \frac{1}{27}e^{-3t} + \frac{2}{9}te^{-3t}$$

$\therefore\ x(t) = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} + \frac{2}{9}te^{-3t}$ and $y(t) = \frac{4}{27} - \frac{2}{9}t - \frac{1}{27}e^{-3t} + \frac{2}{9}te^{-3t}$.

The rigor of expressing $\bar{x}$ and $\bar{y}$ into partial fraction and using Table of Laplace as obtainable in the traditional method has been circumvented. This would have plunge us into the rigor of expressing

$$x = L^{-1}\left\{\frac{S+1}{S^2(S+3)^2}\right\} = L^{-1}\left\{\frac{A}{s}\right\} + L^{-1}\left\{\frac{B}{s^2}\right\} + L^{-1}\left\{\frac{C}{s+3}\right\} + L^{-1}\left\{\frac{D}{(S+3)^2}\right\}$$

and

$$y = L^{-1}\left\{-\frac{2}{S^2(S+3)^2}\right\} = L^{-1}\left\{\frac{E}{s}\right\} + L^{-1}\left\{\frac{F}{s^2}\right\} + L^{-1}\left\{\frac{G}{s+3}\right\} + L^{-1}\left\{\frac{H}{(S+3)^2}\right\}$$

where A, B, C, D, E, F, G and H are constant to be found by partial fraction.

Solution 2:

Given $x_1{}' = x_1 - x_2$

$x_2{}' = -x_1 + x_2 \qquad x_1(0) = 6,\ x_2(0) = 4$

Take the Laplace transform of the equations

$$=> L\{x_1'\} = L\{x_1\} - L\{x_2\} \ => \ SL\{x_1\} - S(0) = L\{x_1\} - L\{x_2\} \ \ldots\ldots\ldots *$$
$$L\{x_2'\} = -L\{x_1\} + L\{x_2\} => SL\{x_2\} - S(0) = -L\{x_1\} + L\{x_2\} \ldots\ldots\ldots **$$

Applying the initial conditions and letting $L\{x_1\} = \bar{x}_1$ and $L\{x_2\} = \bar{x}_2$

$=> (S-1)\bar{x}_1 + \bar{x}_2 = 6\ \ldots\ldots\ldots *$

$\bar{x}_1 + (S-1)\bar{x}_2 = 4\ \ldots\ldots\ldots **$

Solving $*\ and\ **$simultenuoesely $for\ \bar{x}_1$ and $\bar{x}_2$

$=> \bar{x}_1 = \frac{6S-10}{S(S-2)}$ and $\bar{x}_2 = \frac{4S-10}{S(S-2)}$

$=> x_1 = L^{-1}\left\{\frac{6S-10}{S(S-2)}\right\}$ and $x_2 = L^{-1}\left\{\frac{4S-10}{S(S-2)}\right\}$

Now to find $x\ and\ y$ we can circumvent the traditional method of resolving into partial fraction and the application of the table of Laplace, by applying the established Residue Inversion formula which state that

$$x(t) = L^{-1}\{F(x)\} = \text{sum of residues at } e^{st}F(x) \text{ at the poles } of F(s)$$

Now,

For $x_1 = L^{-1}\left\{\frac{6S-10}{S(S-2)}\right\}$, Let $F_1(S) = L^{-1}\left\{\frac{6S-10}{S(S-2)}\right\}$, clearly $F_1(S) = \frac{6S-10}{S(S-2)}$ have both simple poles at $s = 0$ and 2

From definition of residue at simple pole z_0 $\text{Res}_{s=z_0} F(s) = \lim_{s\to 0}(s - z_0)F(s)$

At $s = 0$ which is a simple pole

$$=> Res_{s=z_0}\, e^{st}F_1(s) = \lim_{s\to 0}(s - z_0)e^{st}F_1(s)$$
$$=> Res_{s=0}e^{st}\left\{\frac{6S-10}{S(S-2)}\right\} = \lim_{s\to 0}(s-0)\left\{e^0\left\{\frac{6S-10}{S(S-2)}\right\}\right\}$$
$$= \lim_{s\to 0}\left\{e^0\left\{\frac{6S-10}{S-2}\right\}\right\} = 5$$

Similarly, at $s = 2$ which is also a simple pole

$$=> Res_{s=z_0}\, e^{st}F_1(s) = \lim_{s\to 2}(s - z_0)e^{st}F_1(s)$$
$$=> Res_{s=2}e^{st}\left\{\frac{6S-10}{S(S-2)}\right\} = \lim_{s\to 2}(s-2)\left\{e^{2t}\left\{\frac{6S-10}{S(S-2)}\right\}\right\}$$
$$= \lim_{s\to 2}\left\{e^{2t}\left\{\frac{6S-10}{S}\right\}\right\} = e^{2t}$$

From RIF $\qquad => x_1(t) = 5 + e^{2t}$

Similarly,

For $x_2 = L^{-1}\left\{\frac{4S-10}{S(S-2)}\right\}$, Let $F_2(S) = L^{-1}\left\{\frac{4S-10}{S(S-2)}\right\}$, clearly $F_2(S) = \frac{4S-10}{S(S-2)}$ has simple pole at $s = 0$ and 2

At $s = 0$ which is a simple pole

$$=> Res_{s=z_0}\, e^{st}F_2(s) = \lim_{s\to 0}(s - z_0)e^{st}F_2(s)$$

$$=> Res_{s=0} e^{st} \left\{\frac{4S-10}{S(S-2)}\right\} = \lim_{s\to 0}(s-0)\left\{e^{0}\left\{\frac{4S-10}{S(S-2)}\right\}\right\}$$
$$= \lim_{s\to 0}\left\{e^{0}\left\{\frac{4S-10}{S-2}\right\}\right\} = 5$$

Similarly, at $s = 2$ which is also a simple pole

$$=> Res_{s=2} e^{st} \left\{\frac{4S-10}{S(S-2)}\right\} = \lim_{s\to 2}(s-2)\left\{e^{2t}\left\{\frac{4S-10}{S(S-2)}\right\}\right\}$$
$$= \lim_{s\to 2}\left\{e^{2t}\left\{\frac{4S-10}{S}\right\}\right\} = -e^{2t}$$

From RIF $=> x_2(t) = 5 - e^{2t}$

$$\therefore\ x_1(t) = 5 + e^{2t} \text{ and } x_2(t) = 5 - e^{2t}.$$

This gives us the exact solution as can be obtained using the traditional method or any other known analytic methods.

The rigor of expressing $\bar{x}_1$ and $\bar{x}_2$ into partial fraction and using Table of Laplace for retracting the Inverse Laplace Transform as obtainable in the traditional method has been circumvent/eliminated.

IV. Conclusion

The results experimented from the Residue Inversion formula (RIF) in solving system of Initial Value Problem of Ordinary Differential Equations is more direct, valid, accurate and efficient. The Residue Inversion Formula approached has proved to be a major breakthrough in addressing the problem associated with the traditional method of finding Inverse Laplace Transform by circumventing and/or eliminating the untold computational stress and resultant time wastage incurred in resolving the unknown function into partial fractions before retracting their Inverse Laplace Transform from the Table of Laplace that is not always readily available. Finally, from this work and the previous reviewed literatures, it can be concluded that the Residue Inversion Formula can be use doggedly use on any type of initial value problems involving finding Inverse Laplace Transformation.

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Oko Nlia "Application of Residue Inversion Formula for Solving System of Initial Value Problems of Linear Ordinary Differential Equations with Constant Coefficients " IOSR Journal of Mathematics (IOSR-JM) 14.5(2018): 36-40.