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(M, N)-hesitant fuzzy implicative filters of hoops

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Abstract: *As a generalization of hesitant fuzzy implicative filters of hoops, we introduce the concept of (M,N)-hesitant fuzzy implicative filters. The relationships between (M,N)-hesitant fuzzy implicative filters and implicative filters are discussed by using the notion of (M, N)-τ -level sets of hesitant fuzzy sets. Several conditions for a (M,N)-hesitant fuzzy filter to be a (M,N)-hesitant fuzzy implicative filter are presented. Moreover, some characterizations of (M,N)-hesitant fuzzy implicative filters in regular hoops are derived.*
Keywords : *hoop, (M,N)-hesitant fuzzy implicative filter,hesitant fuzzy set*

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I. Introduction

Hoop-algebras or briefly hoops are ordered commutative residuated integral monoids satisfying a further conditions, as introduced by Bosbach [1] and have residuated lattices as a subclass. Hoops have long been considered of interest by algebraists, starting from the classical example of the lattice-ordered monoid. As an application of properties of pseudo valuations, [2-3] studied the concept of (implicative) pseudo valuations on hoops. Deductive systems are subsets closed with respect to Modus Ponens and they are sometimes called (implicative) filters. Filter theory plays an important role in studying these algebras. Kondo [4] considered that fundamental properties of filters in hoops, and then pointed out that any positive filter of a hoop is implicative and fantastic. To extend the research on filter theory of hoops in [4], [5] introduced the notions of n-fold (positive) implicative filters.

There are many complicated problems in real life that involve uncertain data. Therefore, to deal with uncertain data, some theories, such as the probability theory, the rough set theory, the fuzzy set theory, the interval valued fuzzy set theory, the intuitionistic fuzzy set theory, soft set theory etc. are proposed. The fuzzy set has been found to be a useful tool to model a collection of objects whose boundary is vague, however, the fuzzy set only involves the membership degree, and it can not manage those situations where the membership degree of an element has possible values. To address this drawback, Torra [6] first introduced the idea of hesitant fuzzy set theory which is a very useful technology in dealing with the situation that decision makers are hesitant among several values when asked to give evaluation information for alternatives. Hesitant fuzzy set theory has been applied to algebraic structures, such as BCK/BCI-algebras [7], residuated lattice[8], hoop-algebras [9] and BE-algebras [10].

In the paper, we put forward the notion of (M,N)-hesitant fuzzy implicative filters of hoops, which is a generalization of fuzzy implicative filters. By the notion of (M, N)-τ -level sets of hesitant fuzzy sets, the relationships between (M,N)-hesitant fuzzy implicative filters and implicative filters are discussed. Several characterizations of (M,N)-hesitant fuzzy implicative filters of hoops are present, moreover the characterizations of (M,N)-hesitant fuzzy implicative filters in regular hoops are also listed.

II. Preliminaries

An algebra structure $(A,\otimes,\to,1)$ of type $(2,2,0)$ is called a hoop if the following conditions are valid: for any $x,y,z\in A$,

(HP-1) $(A,\otimes,1)$ is a commutative monoid;

(HP-2) $x\to x=1$;

(HP-3) $x\otimes(x\to y)=y\otimes(y\to x)$;

(HP-4) $x\to(y\to z)=(x\otimes y)\to z$.

The order relation "$\le$" on a hoop A is defined as: $x\le y$ if and only if $x\to y=1$ for any $x,y\in A$. It is easy to see that $(A,\le)$ is a meet semilattice with $x\wedge y=x\otimes(x\to y)$ and 1 as the maximal element. We denote $x^n = \underbrace{x \otimes x \otimes \cdots \otimes x}_{n\text{ times}}$ if $n > 0$ and $x^0 = 1$ for any $n \in N$. A hoop is bounded if there is an element $0 \in A$ such that $0 \le x$ for all $x \in A$. Moreover, regular hoop can be considered as a bounded hoop with the axiom $x'' = x$, where $x' = x \to 0$.

In the following, unless mentioned otherwise, any hoop $(A, \otimes, \to, 1)$ will often be referred to by its support set A. Now we recall several properties of hoops which are useful for subsequent discussions.

Proposition 2.1. ([1-3]) In any hoop A, the following statements are true: for any $x, y, z \in A$,

(1) $x \otimes y \le z$ if and only if $x \le y \to z$,

(2) $x \otimes y \le x \wedge y \le x \to y$, $x \le y \to x$, $1 \to x = x$,

(3) $x \to y \le (y \to z) \to (x \to z)$, $x \to y \le (z \to x) \to (z \to y)$,

(4) if $x \le y$, then $x \otimes z \le y \otimes z$, $y \to z \le x \to z$ and $z \to x \le z \to y$,

(5) $x \to (y \to z) = (x \otimes y) \to z = y \to (x \to z)$,

(6) $(x \to y) \to (x \to z) \le x \to (y \to z)$,

(7) if A is a bounded hoop, then $0' = 1$, $1' = 0$, $x \otimes x' = 0$ and $x \to y = y' \to x'$.

A nonempty subset F of A is called a filter if it satisfies: for any $x, y \in H$, (1) $x, y \in F$ implies $x \otimes y \in F$, (2) if $x \le y$ and $x \in F$, then $y \in F$. Georgescu et al. [1] pointed out that a nonempty subset F of A is a filter if and only if for any $x, y \in A$, (1) $1 \in F$; (2) $x \in F$ and $x \to y \in F$ imply $y \in F$. A nonempty subset F of A is called an implicative filter if it satisfies: (1) $1 \in F$, (2) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$ for any $x, y, z \in H$.

Definition 2.2. ([6]) Let E be a reference set. A hesitant fuzzy set H on E is defined in terms of a function H that when applied to E returns a subset of $[0,1]$, i. e.,

$$\mathsf{H} = \{(e, H(e)) \mid e \in E\},$$

where $H(e)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $e \in E$ to H. For convenience, the hesitant fuzzy set H will often be referred to by its function H.

In what follows, we take a hoop A as a reference set and $\Phi \subseteq M \subset N \subseteq [0,1]$.

To facilitate our discussion, we use the following notations: for a hesitant fuzzy set H of A, and for any $x, y \in A$,

(1) $H(x) \subseteq_N^M H(y)$ to mean $H(x) \cap N \subseteq H(y) \cup M$;

(2) $H(x) =_N^M H(y)$ to mean $(H(x) \cap N) \cup M = (H(y) \cap N) \cup M$.

Lemma 2.3. ([9]) Let H be a hesitant fuzzy set on A. Then for any $x, y, z \in A$,

(1) if $H(x) \subseteq_N^M H(y)$ and $H(y) \subseteq_N^M H(z)$, then $H(x) \subseteq_N^M H(z)$,

(2) $H(x) \subseteq_N^M H(y)$ if and only if $(H(x) \cap N) \cup M \subseteq (H(y) \cap N) \cup M$,

(3) $H(x) =_N^M H(y)$ if and only if $H(x) \subseteq_N^M H(y)$ and $H(y) \subseteq_N^M H(x)$,

(4) if $H(x) \subseteq_N^M H(y)$, then $H(x) =_N^M H(x) \cap H(y)$,

(5) if $H(x) \subseteq_N^M H(y)$, then $H(x) \cap H(z) \subseteq_N^M H(y) \cap H(z)$,

(6) if $H(x) \subseteq_N^M H(y)$ and $H(x) \subseteq_N^M H(z)$, then $H(x) \subseteq_N^M H(y) \cap H(z)$.

Definition 2.4. ([9]) A hesitant fuzzy set H on A is called a (M, N)-hesitant fuzzy filter if it satisfies: for any $x, y \in A$,

(1) $H(x) \cap H(y) \subseteq_N^M H(x \otimes y)$,

(2) if $x \le y$, then $H(x) \subseteq_N^M H(y)$.

Theorem 2.5. ([9]) A hesitant fuzzy set H on A is a (M, N)-hesitant fuzzy filter if and only if for for any $x, y \in A$,

(1) $H(x) \subseteq_N^M H(1)$,

(2) $H(x) \cap H(x \to y) \subseteq_N^M H(y)$.

Let H be a hesitant fuzzy set on A and τ be a subset of $[0,1]$. Then the set

$$L(H;\tau) := \{x \in A \mid \tau \subseteq_N^M H(x)\}$$

is called the (M, N)- τ -level set of H.

III. (M, N)-hesitant fuzzy implicative filters of hoops

Definition 3.1. A hesitant fuzzy set H on A is called a (M, N)-hesitant fuzzy implicative filter if it satisfies: for any $x, y \in A$,

(1) $H(x) \subseteq_N^M H(1)$;

(2) $H(x \to y) \cap H(x \to (y \to z)) \subseteq_N^M H(x \to z)$.

Example 3.2. Let $A = \{0, a, b, 1\}$ where $0 \le a \le b \le 1$. Define the operations $\otimes$ and $\to$ on A as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(A, \otimes, \to, 1)$ is a hoop. Let $M = \{0.2, 0.3\}$ and $N = \{0.2, 0.6\}$. Define a hesitant fuzzy set H on A by $H(0) = \{0.3, 0.5\}$, $H(a) = \{0.1, 0.3, 0.5\}$, $H(b) = \{0.2, 0.5\}$, $H(1) = \{0.4, 0.8\}$. It is easy to check that H is a (M, N)-hesitant fuzzy implicative filter.

Proposition 3.3. Every (M, N)-hesitant fuzzy implicative filter of a hoop is a (M, N)-hesitant fuzzy filter.

Proof: Let H be a (M, N)-hesitant fuzzy implicative filter of A. Then

$$H(x) \cap H(x \to y) = H(1 \to x) \cap H(1 \to (x \to y)) \subseteq_N^M H(1 \to y) = H(y),$$

Hence H is a (M, N)-hesitant fuzzy filter of A.

Proposition 3.4. Let H be a hesitant fuzzy set on A. Then H is a (M, N)-hesitant fuzzy implicative filter of A if and only if and only if $L(H;\tau)$ is an implicative filter of A for any subset τ of $[0,1]$ with $L(H;\tau) \neq \phi$.

Proof: Assume that H is a (M, N)-hesitant fuzzy implicative filter on A. Let $x, y \in A$ and $\tau \subseteq [0,1]$ be such that $x \to y \in L(H;\tau)$ and $x \to (y \to z) \in L(H;\tau)$. It means that $\tau \subseteq_N^M H(x \to y)$ and $\tau \subseteq_N^M H(x \to (y \to z))$, and so $\tau \subseteq_N^M H(x \to y) \cap H(x \to (y \to z)) \subseteq_N^M H(x \to z) \subseteq_N^M H(1)$. Thus $x \to z \in L(H;\tau)$ and $1 \in L(H;\tau)$, so $L(H;\tau)$ is an implicative of A.

Conversely, suppose that $L(H;\tau)$ is an implicative of A for any subset τ of $[0,1]$ with $L(H;\tau) \neq \phi$. For any $x \in A$, let $H(x) = \sigma$. It follows that $H(x) =_N^M \sigma$, and so $\sigma \subseteq_N^M H(x)$, therefore $x \in L(H;\sigma)$. By hypothesis, we get that $L(H;\sigma)$ is an implicative of A, hence $1 \in L(H;\sigma)$, and so $H(x) =_N^M \sigma \subseteq_N^M H(1)$ which implies $H(x) \subseteq_N^M H(1)$. For any $x, y \in A$, let $H(x \to y) = \sigma_{x \to y}$ and $H(x \to (y \to z)) = \sigma_{x \to (y \to z)}$. Taking $\sigma = \sigma_{x \to y} \cap \sigma_{x \to (y \to z)}$, we get $\sigma \subseteq_N^M H(x \to (y \to z))$ and $\sigma \subseteq_N^M H(x \to y)$, that is, $x \to (y \to z) \in L(H;\sigma)$ and $x \to y \in L(H;\sigma)$, then $x \to z \in L(H;\sigma)$, that is, $H(x \to y) \cap H(x \to (y \to z)) = \sigma \subseteq_N^M H(x \to z)$. Hence H is a (M, N)-hesitant fuzzy implicative filter on A.

Theorem 3.5. Let H be a (M, N)-hesitant fuzzy filter of A. Then the following conditions are equivalent:
(1) H is a (M, N)-hesitant fuzzy implicative filter of A,
(2) $H(x^2 \to y) \subseteq_N^M H(x \to y)$ for any $x, y \in A$,
(3) $H(x \to x^2) =_N^M H(1)$ for any $x \in A$,
(4) $H(x \to (y \to z)) \subseteq_N^M H((x \to y) \to (x \to z))$ for any $x, y, z \in A$.

Proof: (1) ⇒(2) Suppose that H is a (M, N)-hesitant fuzzy implicative filter, then for $x, y \in A$,

$$H(x \to x) \cap H(x \to (x \to y)) = H(1) \cap H(x \to (x \to y)) = H(x \to (x \to y)) \subseteq_N^M H(x \to y),$$

hence $H(x^2 \to y) \subseteq_N^M H(x \to y)$.
(2) ⇒(3) For any $x \in A$, we get that $H(1) = H(x^2 \to x^2) \subseteq_N^M H(x \to x^2)$. By Theorem 2.5, $H(x \to x^2) \subseteq_N^M H(1)$, and thus $H(x \to x^2) =_N^M H(1)$.
(3) ⇒(4) For any $x, y, z \in A$, we have

$$x \to (y \to z) \le x \to ((x \to y) \to (x \to z)) = x \to (x \to ((x \to y) \to z))$$
$$= x^2 \to ((x \to y) \to z)) \le (x \to x^2) \to (x \to ((x \to y) \to z))).$$

Notice that H is a (M, N)-hesitant fuzzy filter, we obtain that

$$\begin{aligned} H((x \to y) \to z) &\subseteq_N^M H((x \to x^2) \to (x \to ((x \to y) \to z)))) \\ &=_N^M H(1) \cap H((x \to x^2) \to (x \to ((x \to y) \to z)))) \\ &=_N^M H(x \to x^2) \cap H((x \to x^2) \to (x \to ((x \to y) \to z)))) \\ &\subseteq_N^M H(x \to ((x \to y) \to z)) \\ &= H((x \to y) \to (x \to z)), \end{aligned}$$

and hence (4) is valid.
(4) ⇒(1) Since H is a (M, N)-hesitant fuzzy filter of A, we have

$$H(x \to y) \cap H(x \to (y \to z)) \subseteq_N^M H(x \to y) \cap H((x \to y) \to (x \to z)) \subseteq_N^M H(x \to z),$$

therefore H is a (M, N)-hesitant fuzzy implicative filter of A.

Theorem 3.6. Let H be a (M, N)-hesitant fuzzy filter of A. Then H is a (M, N)-hesitant fuzzy implicative filter of A if and only if $H(z \to (x^2 \to y)) \cap H(z) \subseteq_N^M H(x \to y)$ for any $x, y, z \in A$.

Proof: Suppose that H is a (M, N)-hesitant fuzzy implicative filter of A, then for $x, y, z \in A$, we have $H(z \to (x^2 \to y)) \cap H(z) \subseteq_N^M H(x^2 \to y) \subseteq_N^M H(x \to y)$.
Conversely, taking $z = 1$, we get $H(1 \to (x^2 \to y)) \cap H(1) = H(x^2 \to y) \subseteq_N^M H(x \to y)$. According to Theorem 3.5, H is a (M, N)-hesitant fuzzy implicative filter of A.

Proposition 3.7. Let H_1, H_2 be two (M, N)-hesitant fuzzy filters of A such that $H_1(1) =_N^M H_2(1)$ with H_2 ô $_N^M$ H_1, that is, $H_2(x) \subseteq_N^M H_1(x)$ for any $x \in A$. If H_1 is a (M, N)-hesitant fuzzy implicative filter of A, then H_2 is also a (M, N)-hesitant fuzzy implicative filter of A.

Proof: Since H_1 is a (M, N)-hesitant fuzzy implicative filter of A, then for any $x \in A$, we have $H_2(1) =_N^M H_1(1) =_N^M H_1(x \to x^2) \subseteq_N^M H_2(x \to x^2)$. Notice that $H_2(x \to x^2) \subseteq_N^M H_2(1)$, we obtain that $H_2(x \to x^2) =_N^M H_2(1)$, hence H_2 is a (M, N)-hesitant fuzzy implicative filter of A.

Theorem 3.8. Let A be a regular hoop and H be a (M, N)-hesitant fuzzy filter of A. Then the following conditions are equivalent:
(1) H is a (M, N)-hesitant fuzzy implicative filter of A,
(2) $H(x \to (z' \to y)) \cap H(y \to z) \subseteq_N^M H(x \to z)$ for any $x, y, z \in A$,
(3) $H(x \to (z' \to z)) \subseteq_N^M H(x \to z)$ for any $x, z \in A$,
(4) $H(y \to (x \to (z' \to z))) \cap H(y) \subseteq_N^M H(x \to z)$ for any $x, y, z \in A$.

Proof: (1) ⇒(2) For any $x, y, z \in A$,

$$H(x \to (z' \to y)) \cap H(y \to z) = H(z' \to (x \to y)) \cap H(y \to z)$$
$$= H(z' \to (y' \to x')) \cap H(z' \to y')$$
$$\subseteq_N^M H(z' \to x') = H(x \to z),$$

thus (2) holds.
(2) ⇒(3) For any $x, z \in A$,

$$H(x \to (z' \to z)) =_N^M H(x \to (z' \to z)) \cap H(1) =_N^M H(x \to (z' \to z)) \cap H(z \to z) \subseteq_N^M H(x \to z),$$

and so (3) is valid.
(3) ⇒(4) Since H is a (M, N)-hesitant fuzzy filter of A, then

$$H(y \to (x \to (z' \to z))) \cap H(y) \subseteq_N^M H(x \to (z' \to z)) \subseteq_N^M H(x \to z).$$

(4) ⇒(3) Putting $y = 1$, $H(1 \to (x \to (z' \to z))) \cap H(1) = H(x \to (z' \to z)) \subseteq_N^M H(x \to z)$.
(3) ⇒(2) Since H is a (M, N)-hesitant fuzzy filter of A and for any $x, y, z \in A$,
$(x \to (z' \to y)) \otimes (y \to z) = (x \otimes z' \to y) \otimes (y \to z) \leq x \otimes z' \to z = x \to (z' \to z)$,
then $H((x \to (z' \to y)) \otimes (y \to z)) \subseteq_N^M H(x \to (z' \to z)) \subseteq_N^M H(x \to z)$. Thus (2) is valid.
(2) ⇒(1) For any $x, y, z \in A$,

$$H(x \to y) \cap H(x \to (y \to z)) = H(y' \to x') \cap H(x \to (z' \to y'))$$
$$= H(y' \to x') \cap H(z' \to (x \to y')) = H(y' \to x') \cap H(z' \to (x'' \to y'))$$
$$\subseteq_N^M H(z' \to x') = H(x \to z),$$

hence H is a (M, N)-hesitant fuzzy implicative filter of A.

Theorem 3.9. Let A be a regular hoop and H be a (M, N)-hesitant fuzzy filter of A. Then the following conditions are equivalent:
(1) H is a (M, N)-hesitant fuzzy implicative filter of A,
(2) $H(x' \to x) \subseteq_N^M H(x)$ for any $x \in A$,
(3) $H((x \to y) \to x) \subseteq_N^M H(x)$ for any $x, y \in A$,
(4) $H(z \to ((x \to y) \to x)) \cap H(z) \subseteq_N^M H(x)$ for any $x, y, z \in A$.

Proof: (1) ⇒(2) Using Theorem 3.8(3), we obtain that

$$H(x' \to x)) = H(1 \to (x' \to x)) \subseteq_N^M H(1 \to x) = H(x),$$

that is, $H(x' \to x) \subseteq_N^M H(x)$.
(2) ⇒(3) For any $x, y \in A$, $x' = x \to 0 \leq x \to y$, and so $(x \to y) \to x \leq x' \to x$. Notice that H is a (M, N)-hesitant fuzzy filter of A, we get that $H((x \to y) \to x) \subseteq_N^M H(x' \to x) \subseteq_N^M H(x)$.

(3) ⇒(4) Since H is a (M, N)-hesitant fuzzy filter of A, then

$$H(z \to ((x \to y) \to x)) \bigcap H(z) \subseteq_N^M H((x \to y) \to x) \subseteq_N^M H(x).$$

(4) ⇒(1) From $z \le x \to z$, it follows that $z' \to (x \to z) \le (x \to z)' \to (x \to z)$, and so $H(z' \to (x \to z)) \subseteq_N^M H((x \to z)' \to (x \to z))$. By hypothesis, we get that

$$\begin{aligned} & H(x \to (z' \to z)) = H(z' \to (x \to z)) \\ & \subseteq_N^M H((x \to z)' \to (x \to z)) \\ & =_N^M H(1 \to ((x \to z)' \to (x \to z))) \bigcap H(1) \\ & = H(1 \to (((x \to z) \to 0) \to (x \to z))) \bigcap H(1) \\ & \subseteq_N^M H(x \to z). \end{aligned}$$

Thus H is a (M, N)-hesitant fuzzy implicative filter of A.

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