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Statistical Bourbaki-Cauchy Double Sequence And Its Relation To Bourbaki Completeness For Double Sequence

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Abstract: *In this paper the concept of statistical Bourbaki-Cauchy double sequence is introduced. Some inclusion relations between Bourbaki-Cauchy double sequence and statistical Bourbaki-Cauchy double sequence is also explored. We shall also give some essential analogous definitions of Bourbaki completeness and Bourbaki boundedness for double sequence in metric spaces which are characterized in terms of functions and preserved statistical Bourbaki-Cauchy double sequence.*

Keywords and phrases: *Bourbaki-Cauchy double sequence, Statistical Bourbaki-Cauchy double sequence and Statistical Bourbaki-Cauchy bounded.*
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I. Introduction

The concept of statistical convergence for real-valued sequences was introduced by Fast (1951) and Steinhaus (1951) independently. However, the idea of statistical convergence goes back to the work of Zygmund (1979) (first edition published in Warsaw 1935). The statistical convergence was generalized to sequences in some other spaces and studied on these spaces. It has been considered in metric spaces Kücükaslan, Değer and Dovgoshey (2014), cone metric spaces Li, Lin and Ge (2015), topological and uniform spaces Di Maio and Koćinac (2008) and topological group Cakalli (2009). Shoenberg (1959) gave some basic properties of statistical convergence and also studied the concept as a summability method. Later on, it was further investigated and linked with summability theory by Fridy (1985), Fridy and Orhan (1993), Mursaleen and Edely (2003). Quite recently Ilkhan and Kara (2018) introduced new a concept of statistical convergence of single sequence known as statistical Bourbaki-Cauchy sequence and its relation to Bourbaki completeness. In this paper, we shall extend this concept to double sequence and obtain some inclusion relations between Bourbaki-Cauchy double and statistical Bourbaki-Cauchy double sequences.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(n, m)$ be the numbers and (i, j) in K such that $i \leq n$ and $j \leq m$. Then the two dimensional analogue of natural density is defined as follows:
The lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as:

$$\underline{\delta_2}(K) = \lim_{n,m} \inf \frac{K(n,m)}{nm}$$

In case the sequence $\left(\frac{K(n,m)}{nm}\right)$ has a limit in Pringsheim's sense, then we say that K has a double natural density and is defined by

$$\lim_{n,m} \inf \frac{K(n,m)}{nm} = \delta_2(K)$$

For example, let $K = \{(i^2, j^2): i, j \in \mathbb{N}\}$.

$$\delta_2(K) = \lim_{n,m} \inf \frac{K(n,m)}{nm} = \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0,$$

i.e, the set K has double natural density zero, while the set $K = \{(i, 2j): i, j \in \mathbb{N}\}$ has double natural density ½.
Note that if $n = m$, we have a two-dimensional natural density considered by Christopher (1956).
Statistical analogue of double sequences $x = (x_{jk})$ was defined as follows:
Definition 1.1 (Mursaleen and Edely [2003]): A real double sequence $x = (x_{jk})$ is statistically convergent to a number L if for each $\varepsilon > 0$, the set

$$\{(j, k), j \leq n, k \leq m: |x_{jk} - L| \geq \varepsilon\}$$

has double natural density zero. In this case, we write $st_2 - \lim_{jk} x_{jk} = L$ and we denote the set of all statistically convergent double sequences by st_2.
In line with the paper, when $\delta_2(A)$ appears, we mean it is well defined. Also note that the following statements are true for any subsets A and B in $\mathbb{N} \times \mathbb{N}$.

1) $\delta_2(A)$ exists $0 \leq \delta_2(A) \leq 1$ and $\delta_2(\mathbb{N} \times \mathbb{N} \setminus A)$ also exists with $\delta_2(\mathbb{N} \times \mathbb{N} \setminus A) = 1 - \delta_2(A)$
2) If $\delta_2(A) = 1$ and $A \subset B$, then $\delta_2(B) = 1$
3) $\delta_2(A) = 0$ and $\delta_2(B) = 0$, then $\delta_2(A \cup B) = 0$
4) If $\delta_2(A) = 1$ and $\delta_2(B) = 1$, then $\delta_2(A \cap B) = 1$.

Definition 1.2: A sequence (x_k) in a metric space (X, ρ) is said to be Bourbaki-Cauchy if for every $\varepsilon > 0$ there exist $m, n_0 \in \mathbb{N}$ and $x \in X$ such that $x_k \in \beta^m(x, \varepsilon)$ for $n \geq n_0$ where $\beta^m(x, \varepsilon)$ consists of points $y \in X$ satisfying $\rho(x, a_1) < \varepsilon, \rho(a_1, a_2) < \varepsilon, \dots \rho(a_{m-1}, y) < \varepsilon$ for some $a_1, a_2, \dots, a_{m-1} \in X$.

Definition 1.3: A subset A of a metric space (X, ρ) is said to be Bourbaki bounded if for every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and finitely many points $x_1, x_2, \dots x_n \in X$ such that $A \subset \bigcup_{i=1}^{n} \beta^m(x_i, \varepsilon)$.

Definition 1.4.: A sequence (x_k) is a statistical Cauchy sequence in X if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\Delta\left(A_{N(\varepsilon)}\right) = 1$, where $A_{N(\varepsilon)} = \{k \in \mathbb{N}: \rho(x_N, x_k) < \varepsilon\}$.

Definition 1.5(Ilkhan and Kara [2018]): A sequence (x_k) is said to be statistically bounded in X if there exist $x \in X$ and $M > 0$ such that $\Delta\{k \in \mathbb{N}: \rho(x_k, x) \leq M\} = 1$.

Definition 1.6(Ilkhan and Kara [2018]): A sequence (x_k) in a metric space (X, ρ) is said to be statistical Bourbaki-Cauchy if for every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and $x \in X$ such that $\Delta\{k \in \mathbb{N}: x_k \in \beta^m(x, \varepsilon)\} = 1$

II. Main Results

We now give our main definitions and results as follows:

Definition 2.1: A double sequence (x_{jk}) in a metric space (X, ρ) is said to be statistical Bourbaki-Cauchy if for every $\varepsilon > 0$ there exist $m, n \in \mathbb{N}$ and $x \in X$ such that $\delta_2\{(j,k) \in \mathbb{N} \times \mathbb{N}: x_{jk} \in \beta^{m,n}(x, \varepsilon)\} = 1$.

Definition 2.2: A double sequence (x_{jk}) in a metric X is statistical Bourbaki-Cauchy bounded if given any $\varepsilon > 0$, we have $\delta_2\{(j,k) \in \mathbb{N} \times \mathbb{N}: x_{jk} \in \beta(x, mn\varepsilon)\} \geq \delta_2\{(j,k) \in \mathbb{N} \times \mathbb{N}: x_{jk} \in \beta^{m,n}(x, \varepsilon)\} = 1$.
For some $x \in X$ and $m, n \in \mathbb{N}$ which shows that (x_{jk}) is statistically bounded.

Example 2.1: Consider $\mathbb{R}$ with the usual metric. The double sequence (x_{jk}) defined in the following way

$$x_{jk} = \begin{cases} jk, j \text{ and } k \text{ are primes} \\ 1, \quad \text{otherwise} \end{cases}$$

Is indeed statistically Cauchy and so statistically Bourbaki-Cauchy due to the fact that the natural density of the set of all primes equals zero.

Theorem 2.1: For a double sequence (x_{jk}) in a metric space (X, ρ), the following statements are equivalent.

i. (x_{jk}) is a statistical Bourbaki-Cauchy double sequence in X.
ii. There exists a Bourbaki-Cauchy subsequence $(x_{j_r k_s})$ of (x_{jk}) such that $\delta_2\{(j_r, k_s) \in \mathbb{N} \times \mathbb{N}: (j,k) \in \mathbb{N} \times$ *N=1.*
iii. There exists a statistical Bourbaki-Cauchy subsequence $(x_{j_r k_s})$ of (x_{jk}) such that $\delta_2\{(j_r, k_s) \in \mathbb{N} \times$ *N:(j,k)∈N× N=1.*

Proof: $(i) \Rightarrow (ii)$ Let (x_{jk}) be a statistical Bourbaki-Cauchy double sequence in X. Then there exist $m_1, n_1 \in \mathbb{N}$ and $j_1, k_1 \in \mathbb{N}$ and such that $\delta_2(A_1) = 1$, where $A_1 = \{(j,k) \in \mathbb{N} \times \mathbb{N}: x_{jk} \in \beta^{r_1 s_1}(x_{j_1 k_1})\}$. Similarly, there exist $r_2, s_2 \in \mathbb{N}$ and $j_2, k_2 \in \mathbb{N}$ such that $\delta_2(B_1) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N}: x_{jk} \in \beta^{r_2 s_2}(x_{j_2 k_2}, \frac{1}{2^2})\right\}$. Put $A_2 = A_1 \cap B_1$. Then, we have $\delta_2(A_2) = 1, A_2 \subset A_1$ and $x_{j_2 k_2} \in \beta^{2 r_2 s_2}(x_{j_1 k_1}, \frac{1}{2^2})$ for all $j_1, k_1, j_2 k_2 \in A_2$. By continuing this process, we obtain a decreasing sequence $A_{1,1} \supset A_{2,2} \supset \cdots \supset A_{i,j} \supset \cdots$ of subset of $\mathbb{N} \times \mathbb{N}$ with $\delta_2\left(A_{ij}\right) = 1$ and $x_{j_2 k_2} \in \beta^{2 r_i s_j}\left(x_{j_1 k_1}, \frac{1}{2^{ij}}\right)$ for all $j_1, k_1, j_2 k_2 \in A_{i,j}$. Let $r_1, s_1 \in A_{1,1}$ and $r_2, s_2 \in A_{2,2}$ with $r_2 > r_1$ and $s_2 > s_1$ such that

$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\chi_{A_{2,2}}(k)=1-\frac{1}{2}$ for all $r,s\geq r_2,s_2$. In this manner, we construct an increasing sequence (n_{ij}) in $\mathbb{N}$ such that $\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\chi_{A_{2,2}}(k)=1-\frac{1}{ij}$ for all $n\geq n_{ij}$, where $n_{ij}\in A_{ij}$ for each $i,j\in\mathbb{N}$. Set

$$A=\{j,k:1\leq j\leq r_1,1\leq k\leq s_1\}\cup\left[\bigcup_{j,k\in\mathbb{N}}\{j,k:r_i\leq j\leq r_{i+1},s_j\leq k\leq s_{j+1}\}\cap A_{ij}\right].$$

For any $i,j\in\mathbb{N}$ and $r_i\leq j\leq r_{i+1},s_j\leq k\leq s_{j+1}$, we have

$$\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}\chi_A(K)\geq\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}\chi_{A_{ij}}(K)>1-\frac{1}{ij}$$

Which implies that $\delta_2(A)=1$. Now, given any $\varepsilon>0$, we can find a natural numbers $i_o,j_0\in\mathbb{N}$ satisfying $\frac{1}{2^{i_0j_0}}<\varepsilon$. Choose fixed $j,k\in A$ and arbitrary $l\in A$ with $l>k>n_{i_0j_0}$. Then there exist $r,s,p,q\in\mathbb{N}$ with $s\geq r\geq i_0$ and $q\geq p\geq j_0$ such that $u\in A_{r,s}$ and $v\in B_{p,q}$ and $m_{r,s}<u\leq m_{r+1,s+1},n_{p,q}<v\leq n_{p+1,q+1}$ and $\alpha\in A_{t,w}$, $\gamma\in B_{c,d}$ and $f_{t,w}<\alpha\leq f_{t+1,w+1},e_{c,d}<\gamma\leq e_{c+1,d+1}$. Hence, we have $u,\alpha\in A_{r,s}$ and so $x_o\in\beta^{2m_{r,s}}(x_{u,\alpha},\frac{1}{2^{r,s}})\subset\beta^{2m_{r,s}}(x_{u,\alpha},\varepsilon)$ which means that $(x_{u,\alpha})_{u,\alpha\in A}$ is the required Bourbaki-Cauchy subsequence.

$(ii)\Rightarrow(iii)$. The implication follows from the fact that a Bourbaki-Cauchy double sequence is a statistical Bourbaki-Cauchy double sequence.

$(iii)\Rightarrow(i)$. Let $\left(x_{j_rk_s}\right)$ be a statistical Bourbaki-Cauchy subsequence of $\left(x_{jk}\right)$, where $\delta_2\left(\{n_{ij}\in\mathbb{N}\times\mathbb{N}:(i,j)\in \mathbb{N}\times\mathbb{N}=1.\right.$ Then, given any $\varepsilon>0$ there exist $m,n\in\mathbb{N}$ and $x\in X$ such that

$$\lim_{j,k\to\infty}\frac{1}{jk}\sum_{p=1}^{j}\sum_{q=1}^{k}\chi_A(pq)\geq\lim_{j,k\to\infty}\frac{1}{jk}\sum_{p=1}^{j}\sum_{q=1}^{k}\chi_{\tilde{A}}(n_{pq})=1$$

Where $A=\{(i,j)\in\mathbb{N}\times\mathbb{N}:x_{ij}\in\beta^{m,n}(x,\varepsilon)\}$ and $\tilde{A}=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:x_{n_{ij}}\in\beta^{m,n}(x,\varepsilon)\right\}$, respectively. We conclude that $\delta_2(A)=1$ which proves that the double sequence (x_{jk}) is statistical Bourbaki-Cauchy in X.

Corollary 2.1: Every statistical Bourbaki-Cauchy double sequence has a Bourbaki-Cauchy subsequence in a metric space.

Theorem 2.2: A metric space is Bourbaki complete if and only if every statistical Bourbaki-Cauchy double sequence has a statistical convergent subsequence.

Proof: Suppose that a metric space X is Bourbaki complete. Let (x_{jk}) be a statistical Bourbaki-Cauchy double sequence in X. By the previous theorem, it has a Bourbaki-Cauchy subsequence. Then, Bourbaki completeness of X implies that it has a usual convergent and so statistical convergent subsequence.

Conversely, take a Bourbaki-Cauchy double sequence in X. Since it is also statistical Bourbaki-Cauchy, by hypothesis there exists a statistical convergent subsequence of (x_{jk}). Since every statistical convergent double sequence has a convergent subsequence, it follows that X is a Bourbaki complete metric space.

Lemma 2.1: Each Bourbaki-Cauchy regular function is statistical Bourbaki-Cauchy regular.

Proof: Let $f:(X,\rho)\to(X',\rho')$ be a Bourbaki-Cauchy regular function and (x_{jk}) be a statistical Bourbaki-Cauchy double sequence in X. Then, by theorem 2.1, it has a Bourbaki-Cauchy subsequence $(x_{j_rk_s})$ such that $\delta_2\left(\{n_{ij}\in\mathbb{N}:(i,j)\mathbb{N}\times\mathbb{N}\}\right)=1$. Hence, our assumption implies that that the double sequence $\left(f(n_{ij})\right)$ is also Bourbaki-Cauchy. It follows again from theorem 2.1 that the double sequence $\left(f(x_{jk})\right)$ is statistical Bourbaki-Cauchy. Thus, we conclude that the function f is statistical Bourbaki-Cauchy regular.

Theorem 2.3: The following statements are equivalent for a metric space (X,ρ).

i. (X,ρ) is a Bourbaki complete.
ii. Every continuous function from (X,ρ) into chainable metric space (X',ρ') is Bourbaki-Cauchy regular.
iii. Every continuous function from (X,ρ) into chainable metric space (X',ρ') is statistical Bourbaki-Cauchy regular.
iv. Every continuous function from (X,ρ) into $\mathbb{R}$ is statistical Bourbaki-Cauchy regular.

Proof: $(i) \Rightarrow (ii)$ It is proved in [(iii), theorem 2.1].

$(ii) \Rightarrow (iii)$ The proof comes from the fact that every Bourbaki-Cauchy regular function is statistical Bourbaki-Cauchy regular which is proved in lemma 2.1.

$(iii) \Rightarrow (iv)$ It is clear since $\mathbb{R}$ os chainable with respect to the usual metric.

$(iv) \Rightarrow (i)$ Let (x_{jk}) be a Bourbaki-Cauchy double sequence in X. We can say that (x_{jk}) has a Bourbaki-

Cauchy subsequence whose terms are distinct, otherwise, there is nothing to prove. Now, we suppose that a Bourbaki-Cauchy double sequence (x_{jk}) with distinct terms has no convergent subsequence. It is follows that $Y = \{x_{jk} : (j,k) \in \mathbb{N} \times \mathbb{N}\}$ is a closed subset of X. Also the subspace topology on the set Y is discrete topology since it consists of only isolated points. Define a real-valued function g on Y with $g(x_{jk}) = jk, j, k(x_{jk})$. Then g is a continuous function since every function defined on a discrete topological space is continuous. Accordingly, Tietze extension theorem implies that there is a continuous function $f: (X, \rho) \to \mathbb{R}$ with $f(x_{jk}) = g(x_{jk})$ for all $j, k \in \mathbb{N}$. But this function cannot be statistical Bourbaki-Cauchy regular since $\left(f(x_{jk})\right) = (jk)$ is not a statistical Bourbaki-Caucjy double sequence in $\mathbb{R}$ whereas (x_{jk}) is statistical Bourbaki-Cauchy in X. Therefore, every Bourbali-Cauchy double sequence in X must have a convergent subsequence which means that X is Bourbaki complete.

Theorem 2.4: The following statements are equivalent for a metric space (X, ρ)**.**

i. Every double sequence in X has a statistical Bourbaki-Cauchy subsequence.
ii. If $f: (X, \rho) \to (X', \rho')$ is statistical Bourbaki-Cauchy regular function, where (X', ρ') is any metric space, then f is bounded.
iii. If $f: (X, \rho) \to (\tilde{X}, \tilde{\rho})$ is statistical Bourbaki-Cauchy regular function, where $(\tilde{X}, \tilde{\rho})$ is an unbounded chainable metric space, then f is unbounded
iv. (X, ρ) is Bourbaki bounded

Proof: $(i) \Rightarrow (ii)$ Suppose that $f: (X, \rho) \to (X', \rho')$ is statistical Bourbaki-Cauchy regular function but not a bounded function. Then for all $m, n \in \mathbb{N}$, we can construct a double sequence (x_{jk}) in X satisfying $\rho'\left(f(x_{j+1,k+1}), f(x_{u,v})\right) > jk(j, k = 1 \dots, n)$ since $f(X)$ is not bounded. By hypothesis, the double sequence (x_{jk}) has a statistical Bourbaki-Cauchy subsequence, say $(x_{j_r k_s})$. However, $\left(f(x_{j_r k_s})\right)$ is not a statistical Bourbaki-Cauchy double sequence. Indeed, given any $m, n \in \mathbb{N}$ and $x' \in X'$, the set $\{(j,k) \in \mathbb{N} \times \mathbb{N}: f(x_{j_r k_s}) \in$ *βm,n(x′,1)* is finite. Otherwise, for fixed numbers *p,q∈A*, the inclusion

$$\beta^{m,n}(x', 1) \subset \beta^{m,n}\left(f\left(x_{j_p k_q}\right), 1\right) \subset \beta\left(f\left(x_{j_p k_q}\right), 2mn\right)$$

Implies that $\rho'\left(f(x_{j_r k_s}), f\left(x_{j_p k_q}\right)\right) < 2mn$ for finitely many $j, k \in \mathbb{N}$. Hence $\left(f(x_{j_r k_s})\right)$ is not a statistical Bourbaki-Cauchy double sequence which contradicts the fact that f is statistical Bourbaki-Cauchy regular. Thus, f must be a bounded function.

$(ii) \Rightarrow (iii)$ This is obvious.

$(iii) \Rightarrow (iv)$ Suppose that (X, ρ) is not Bourbaki bounded. Then, there exists an $\varepsilon_0 > 0$ such that for all $m, n \in \mathbb{N}$, X cannot be covered by a union of finitely many sets $\beta^{m,n}(x, \varepsilon_0)$ $(x \in X)$. Fixed $x_{0,0} \in X$. Then, we can choose $x_{1,1} \in X$ such that $x_{1,1} \notin \beta^{1,1}(x_{0,0}, \varepsilon_0)$. In some manner, we can choose $x_{1,1} \in X$ such that $x_{2,2} \notin \beta^{2,2}(x_{0,0}, \varepsilon_0) \cup \beta^{2,2}(x_{1,1}, \varepsilon_0)$. By continuing this process, we obtain a sequence (x_{uv}) in X such that $x_{uv} \notin \beta^{u,v}(x_{u,v}, \varepsilon_0)$ for every $u, v \in \mathbb{N}$ and $u, v = 0, \dots u-1, v-1$. Let $\tilde{x}_0 \in \tilde{X}$. Since $(\tilde{X}, \tilde{\rho})$ is an unbounded metric space, there is a point $\tilde{x}_{m,n} \in \tilde{X}$ such that $\tilde{\rho}(\tilde{x}_{m,n} \tilde{x}_{0,0}) > mn$ for all $m, n \in \mathbb{N}$. By virtue of this fact, we define an unbounded function $f: (X, \rho) \to (\tilde{X}, \tilde{\rho})$ as:

$$f(x) = \begin{cases} \tilde{x}_{uv}, if\ x = x_{uv}\ for\ some\ u, v \in \mathbb{N} \\ \tilde{x}_{0,0}, & else. \end{cases}$$

However, this function is statistical Bourbaki-Cauchy regular. To observe this, take a statistical Bourbaki-Cauchy double sequence (y_{jk}) in X. Then for $\varepsilon_0 > 0$, then there exists natural numbers $m_0, n_0 \in \mathbb{N}$ and a point $x_{0,0} \in X$such that $\delta_2\left(\{(j,k) \in \mathbb{N} \times \mathbb{N}: y_{jk} \in \beta^{m_0 n_0}(x_{0,0}, \varepsilon_0)\}\right) = 1$, that is $\beta^{m_0 n_0}(x_{0,0}, \varepsilon_0)$ contains infinitely many terms of the double sequence (y_{jk}). On the other hand, for only finitely many $m, n \in \mathbb{N}, x_{jk} = \{y_{mn} : m, n \in A\}$, where $A = \{(m,n) \in \mathbb{N} \times \mathbb{N}: y_{jk} \in \beta^{m_0 n_0}(x_{0,0}, \varepsilon_0)\}$. Otherwise, since inclusion $\beta^{m_0 n_0}(x_{0,0}, \varepsilon_0) \subset \beta^{2 m_0 n_0}(x_{m_0 n_0})$ holds for infinitely many $m_0, n_0 \in \mathbb{N}$, we contradict with the construction of the double sequence (x_{mn}). Hence, $\{f(y_{jk}): j,k \in A\}$ is a finite subset of $\tilde{X}$. It follows that given any $\varepsilon > 0, f(y_{jk}) \in \beta^{MN}(\tilde{x}_{0,0}, \varepsilon)$, where $MN = max\{m_j n_k: j,k \in A\}$ and $m_j n_k$ is the length of the ε-chain from $\tilde{x}_{0,0}$ to $f(x_{jk})$ for every $j,k \in A$. Thus, we conclude that the subsequence $\left(f(y_{jk})\right)_{j,k \in A}$ is Bourbaki-Cauchy with $\delta_2(A) = 1$ which means the double sequence itself $\left(f(y_{jk})\right)$ is a statistical Bourbaki-Cauchy double sequence in $\tilde{X}$. Consequently, we obtain an unbounded statistical Bourbaki-Cauchy regular function from X into unbounded chainable metric space $\tilde{X}$ opposite to hypothesis and so X is Bourbaki bounded.

$(iv) \Rightarrow (i)$ It is proved that if X is Bourbaki bounded, then every double sequence in X has a Bourbaki-Cauchy subsequence and so it has a statistical Bourbaki-Cauchy subsequence.

References

[1]. J. Christopher, The asymptotic density of some k-dimensional sets, Amer. Math. Monthly ***63***(1956), 399 – 401.

[2]. 2.Cakalli, H. (2008). Sequential defiitions of compactness. *Appl. Math. Lett. 21*(6), 594-598.

[3]. Fast, H. (1951). Sur la convergence statistique. Colloquium Mathematicum, 2 (3–4), 241–244.

[4]. Fridy, J. A., & Orhan, C. (1993). Lacunary statistical summability. *Journal of Mathematical Analysis and Applications, 173*, 497–504.

[5]. Ilkhan, M. & Kara, E. E. (2018). A new type of statistical Cauchy sequence and its relation to Bourbaki completeness. Cogent Mathematics & Statistics, 5, 1-9.

[6]. J. A. Fridy and C. Orhan, Lacunary statistical summability, J. Math, Annal. Appl. 173(1993), 497-504.

[7]. J. Schoenberg, the integrability of certain functions and related summability methods. *American Mathematics. Monthly* ***66***(1959), 361-375.

[8]. J.A. Fridy, C. Orhan, Lacunary statistical convergence, Pacific J. Math. 160 (1993) 45–51.

[9]. M. Mursaleen, and O. H. H. Edely, "Statistical convergence of double sequences," *J. Math. Anal. Appl.,* ***288***(2003), 223–231.

[10]. Mursaleen, M. & Edely, O. H. H. (2003). "Statistical convergence of double sequences," *J. Math. Anal. Appl., 288*, 223–231.

[11]. Mursaleen, Osama H.H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl. 288 (2003) 223–231.

[12]. Steinhaus, H. (1951). Sur la convergence ordinate et al convergence asymptotique. *Colloquium Mathematicum,* 2, 73–84.

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