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Existence and stability results for second order neutral stochastic Partial Functional differential equations driven by a fractional Brownian motion

P.Selvanayaki [1] and Dr.C.Loganathan[2]

[1]*Department of Mathematics,Sri Ramakrishna College of Arts and Science for Women, Coimbatore, India*
[2]*Department of Mathematics, Maharaja Arts and Science College, Coimbatore, India*
Corresponding Author:P.Selvanayaki

Abstract: *In this article, we study the existence and stability results for second order neutral stochastic functional differential equations driven by fractional Brownian motion. Our method of investigating the stability of solutions is based on successive approximation approach and Lipschitz conditions being imposed.*

Keywords*: Second – order system, non Lipschitzian conditions, fractional Brownian motion, mild solution.*

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I. Introduction

The neutral stochastic differential equations can play an important role in describing many sophisticated dynamical systems in physical, biological, medical, chemical engineering, aero elasticity and social sciences [4],[13],[14],[15] and[22] . More over many dynamical systems not only depend on present and past states but also involve derivatives with delays. Deterministic neutral functional differential equations, which was originally introduced by Hale and Lunel [9] are of great interest in theoretical and practical applications. Moreover, one of the simplest stochastic processes that is Gaussian, self – similar, and has stationary increments is FBm [2]. In particular, FBm is a generalization of the classical Brownian motion, which depends on a parameter $H \in (0,1)$ called the Hurst index [8]. It should be mentioned that when $H = 1/2$, the stochastic process is a standard Brownian motion; when $H \neq 1/2$, it behaves completely in a different way than the standard Brownian motion, In particular neither is a semi martingale nor a Markov process. It is a self-similar process with stationary increments and has a long- memory when $H \neq 1/2$. These significant properties make FBm a natural candidate as a model for noise in a wide variety of physical phenomena such as telecommunications networks, finance markets, biology and so on [10].

The existence and uniqueness of mild solutions for a class of stochastic differential equations in a Hilbert space with a standard, cylindrical FBm with the Hurst parameter in the interval (1/2,1) has been studied [6]. Dung studied the existence and uniqueness of impulsive stochastic volterraintegro-differential equations driven by FBm in [7]. LI [17] investigated the the existence of mild solution to a class of stochastic delay fractional evolution equations driven by FBm. Caraballo et al[5], and Boufousssi and Hajji [3] have discussed the existence, uniqueness and exponential asymptotic behaviour of mild solutions by using the wiener integral.

Even though there are many valuable results about neutral stochastic partial differential equations, they are mainly concerned with first order case. In many cases it is advantageous to treat the second order stochastic differential equations rather than to convert them to first-order systems. The second –order stochastic differential equations are the right model in continuous time to account for integrated processes than can be made stationary. For instance, it is useful for engineers to model mechanical vibrations or charge on a capacitor or condenser subjected to white noise excitation through a second- order evolution equation governed by the generator of a strongly continuous cosine family was proposed in [8,22] Ren and sakthivel [19] investigated the existence, uniqueness and stability of second order neutral stochastic evolution equations with infinite delay, Liang and Guo [16] probed the behaviour for second-order stochastic evolution equations with memory; Arthi et al [1] discussed the exponential stability for second-order neutral stochastic differential equations with impulses .Inspired by this consideration, in this paper we consider the second – order neutral stochastic functional differential equation driven by FBm with hurst parameter 1/2< h <1.

$$
\begin{aligned}
& d\left[x'(t) - g\left(t, x\big(t - r(t)\big)\right)\right] \\
&\qquad = \left[Ax(t) + f\left(t, x\big(t-\rho(t)\big)\right)\right]dt + h\left(t, x\big(t-\delta(t)\big)\right)dw(t) + \sigma(t)dB_Q^H(t), \quad t \geq 0,
\end{aligned}
$$

$x_0(t) = \phi(t)\theta \in BC_{F_0}^b[-r, 0\,; H],$

$T \in [-r, 0\,], x'(0) = \emptyset_1$

Where A : D(A)$\subset H \to H$ is the infinitesimal generator of a strongly continuous cosine family on H, B_Q^H is a fractional Bronian motion with Hurst parameter H ∈ (1/2, 1) and w is a standard wiener process on a real and separable Hilbert space K: [0, +∞) ×B→ $H(i = 0,1)$, h: [0,+∞)×B→ L_2^0 g,f and **σ**: [0,+∞) → $L_2(K, H)$] are some appropriate mappings specified later, Now Let us recall some basic concepts and facts on cosine families of operators (see [11].

II. Preliminaries

In this section, we briefly give some basic definitions and results for stochastic equations in infinite dimensions and cosine families of operators.We refer to [5],[12],[21] and the references therein).Let ($\mathbb{H}, \|\cdot\|_{\mathbb{k}}, \langle\cdot,\cdot\mathbb{k}$) are two real separable Hilbert spaces. The notation $L2\mathbb{P},\mathbb{H}$stands for the space of all $\mathbb{H}$ - valued random variables x such that$\mathrm{E}\|x\|^2 = \int_\Omega \|x\|^2 d\mathbb{P} < \infty$. For$x \in L^2(\mathbb{P}, \mathbb{H})$, let $\|x\|_2 = \int_\Omega (\|x\|^2 d\mathbb{P})^{\frac{1}{2}}$. It is easy to check that $L^2(\mathbb{P}, \mathbb{H})$ is a Hilbert space equipped with the norm
$\|\cdot\|_2$.Let $\mathcal{L}(\mathbb{k}, \mathbb{H})$ denotes the space of all bounded linear operators from$\mathbb{k} to \mathbb{H}$, and Q∈ $L^2(\mathrm{K}, \mathrm{K})$represents a non- negative self-adjoint operator.Let{W(t),t∈ $\mathbb{R}$} be a standard cylindrical Wiener process with values in $\mathbb{k}$ and defined on $(\Omega, \mathcal{F}, \mathbb{P})$ [18].
Let $\mathbb{L}_2^0 = \mathbb{L}^2(\mathbb{K}_{0,}\mathbb{H})$ beaseparable Hilbert space with respect to the Hilbert-Schmidt norm $\|\cdot\|_{\mathbb{L}_2^0}$. Let $\mathbb{L}_Q^0(\mathbb{k}, \mathbb{H})$ be the space of all $\psi \in \mathcal{L}(\mathbb{k}, \mathbb{H})$ such that $\psi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator from $\mathbb{k} to \mathbb{H}$. Let $\mathbb{L}_2^0(\Omega, \mathbb{H})$ denote the space of $\mathcal{F}_0 -$ measurable, $\mathbb{H}$ −valued and square integrable stochastic processes.
Let (Ω, $\mathcal{F}, \mathbb{P}$)be a complete probability space. Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{\beta^H(t),\ t \in [0, T]\}$ be the one-dimensional fractional Brownian motion with Hurst parameter H ∈ $\left(\frac{1}{2}, 1\right)$.This means by definition that $\beta^H$is a centered Gaussian process with covariance function:

$$R_H(t, s) = \mathbb{E}(\beta_t^H \beta_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

More over β^Hhas the following Wiener integral representation:

$$\beta^H(t) = \int_0^t \mathrm{K}_H(t, s) d\beta(s)$$

where$\beta = \{\beta^H(t), t \in [0, T]\}$ is a Wiener process and $\mathrm{K}_H(t, s)$ is the kernel given by

$$\mathrm{K}_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{1}{2}} u^{H-\frac{1}{2}} du$$

for t $> s$, where $c_H = \sqrt{H(2H - 1)/\beta\left(2 - 2H, H - \frac{1}{2}\right)}$ and $\beta(.,.)$ denotes the Beta function.We put $\mathrm{K}_H(t, s) = 0$ if t $\leq s$.
We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of theFBm. In fact$\mathcal{H}$isthe closure of set of indicator functions $\{\mathrm{I}_{[0,t]}, t \in [0, T]\}$with respect to the scalar product

$$\langle \mathrm{I}_{[0,t]}, \mathrm{I}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

The mapping$\mathrm{I}_{[0,t]} \to \beta^H(t)$ can be extended to an isometry between$\mathcal{H}$ andthe first Wiener chaos and we will denote by $\beta^H(\varphi)$the image ofφ by the previous isometry.
We recall that for $\psi, \Phi \in \mathcal{H}$ their scalar product in $\mathcal{H}$is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H - 1) \int_0^T \int_0^T \psi(s)\varphi(t)|t - s|^{2H-2} ds dt.$$

Let us consider the operator $K_H^*$from $\mathcal{H}$ to $L^2([0, T])$[19] defined by

$$(K_H^*)(s) = \int_s^T \varphi(r) \frac{\partial \mathrm{K}}{\partial r}(r, s) dr$$

Then K_H^* is an isometry between $\mathcal{H}$ and$L^2([0, T])$.
Moreover for any∈ $\mathcal{H}$, we have
$\beta^H(\varphi) = \int_0^t (K_H^*)(t) d\beta(t)$. Let $\{\beta_n^H(t)\}_{n \in \mathbb{N}}$ be a sequence of two-sided one- dimensional standard FBm mutually independent on (Ω, $\mathcal{F}, \mathbb{P}$).
Consider the following series
$\sum_{n=1}^{\infty} \beta_n^H(t) e_n,$ t≥ 0,
Where $\{e_n\}_{n \in \mathbb{N}}$ is a complete orthonormal basis in $\mathbb{k}$, the series does not necessarily converge in the space$\mathbb{k}$. Therefore,we consider a $\mathbb{k}$- valued stochastic process $\beta_Q^H(t)$given by the following series:

$\beta_Q^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) Q^{\frac{1}{2}} e_n,$ t≥ 0,

Moreover, if Q ia s non-negative self-adjoint trace class operator,then this series converges in the space K, that is, it holds that $\beta_Q^H(t) \in L^2(\mathbb{P}, \mathbb{k})$. Also, the above $\beta_Q^H(t)$ is a$\mathbb{k}-$valuedQ-cylindrical FBm with covariance operator Q. For example, if $\{\sigma_n\}_{n\in\mathbb{N}}$is abounded sequence of non-negative real numbers such that Qe_n=$\sigma_n e_n$, assuming that Q is a nuclear operator in$\mathbb{k}$, then the stochastic process

$\beta_Q^H(t)$ =$\sum_{n=1}^{\infty} \beta_n^H(t) Q^{\frac{1}{2}} e_n$, $= \sum_{n=1}^{\infty} \sqrt{\sigma_n} \beta_n^H(t) Q^{\frac{1}{2}} e_n,$ t ≥ 0,

Is well- defined as a $\mathbb{k}-$ valued Q-cylindrical FBm [5]

Definition 2.1[5] Let $\varphi: [0, T] \to \mathcal{L}_Q^0(\mathbb{k}, \mathbb{H})$ such that

$$\sum_{n=1}^{\infty} \left\| K_H^*(\varphi Q^{\frac{1}{2}} e_n) \right\|_{L^2([0,T];\mathbb{H})} < \infty$$

Then, its stochastic integral with respect to the FBm$B_Q^H(t) is$ defined, for t≥ 0, as follows:

$\int_0^t \varphi(s) dB_Q^H(s) := \sum_{n=1}^{\infty} \varphi(s)\, Q^{\frac{1}{2}} e_n \mathrm{d}\beta_n^H = \sum_{n=1}^{\infty} \int_0^t \left(K_H^* \left(\varphi Q^{\frac{1}{2}} e_n \right) \right)(s) d\,\beta(s).$

Lemma 2.2 [5]For any $\varphi: [0, T] \to \mathcal{L}_Q^0(\mathbb{k}, \mathbb{H})$ such that

$$\sum_{n=1}^{\infty} \left\| \varphi Q^{\frac{1}{2}} e_n \right\|_{L^{1/H}([0,T];\mathbb{H})} < \infty$$

Holds, and for any $\alpha, \beta \in [0, T]$with$> \beta$, we have

E$\left\| \int_\alpha^\beta \varphi(s) dB_Q^H(s) \right\|^2 \leq cH(2H-1)(\alpha-\beta)^{2H-1} \sum_{n=1}^{\infty} \int_\alpha^\beta \left\| \varphi Q^{\frac{1}{2}} e_n \right\|^2 ds,$

Where c = c(H). In addition, if $\sum_{n=1}^{\infty} \left\| \varphi(t) Q^{\frac{1}{2}} e_n \right\|$is uniformly convergent for t ∈ $[0, T]$, $then$

E$\left\| \int_\alpha^\beta \varphi(s) dB_Q^H(s) \right\|^2 \leq cH(2H-1)(\alpha-\beta)^{2H-1} \int_\alpha^\beta \|\varphi\|_{\mathcal{L}_Q^0}^2 ds.$

In this work,we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced in [12]

Definition 2.3$\mathcal{B}$is a linear space of family of $\mathcal{F}_0$-measurable functions from (-∞, 0] into $\mathbb{H}$ endowed with a norm $\|\cdot\|_{\mathcal{B}}$, which satisfies the following axioms:

(i) If x: (--∞,T) → $\mathbb{H}$, $T > 0$, is such that $x_0 \in \mathcal{B}$, then, for every t ∈ $[0, T]$, the following conditions hold:

(a) $x_t \in \mathcal{B}$;

(b)$\|x(t)\| \leq m\|x_t\|_{\mathcal{B}}$;

(c) $\|x_t\|_{\mathcal{B}} \leq \mathfrak{K}(t) n \sup_{0\leq s\leq t} \|x(s)\| + N(t)\, \|x_0\|_{\mathcal{B}}$,

Where m > 0 is a constant, $\mathfrak{K}, N: [0, +\infty) \to [1, +\infty)$, $\mathfrak{K}$ is continuous, N is locally bounded, $\mathfrak{K}, N$ are independent of x (·).

(ii) The space $\mathcal{B} is$complete.

Lemma 2.4[20] Let x:(--∞,T] → $\mathbb{H}$ be an $\mathcal{F}_{t-}$ adapted measurable process such that the $\mathcal{F}_0 -$ adapted processx_0 = $\varphi \in \mathcal{L}_2^0(\Omega, \mathcal{B})$, then

E$\|x_s\|_{\mathcal{B}} \leq \mathfrak{K}\, \sup \mathrm{E}_{0\leq s\leq t} \|x(s)\| + NE\, \|\varphi\|_{\mathcal{B}}$,

Where N = $sup_{t\in I}$ $\{N(t)\}$ and $\mathfrak{K} = \sup_{t\in I} \{\, \mathfrak{K}(t)\}$

Definition 2.5 [20] Denote by $\mathcal{M}^2(-\infty, T], \mathbb{H})$ be the space of all $\mathbb{H}-$ valued continuous $\mathcal{F}_{t-}$ adapted process x= {x(t)} $_{-\infty<t\leq T}$ such that

(i) $x_0 = \varphi \in \mathcal{B}$andx(t) is continuous on [0,T];

(ii) Define the norm $\|\cdot\|_{\mathcal{M}}$ in $\mathcal{M}^2(-\infty, T], \mathbb{H})$ by

$\|x\|_{\mathcal{M}}^2 = E\|\varphi\|_{\mathcal{B}}{}^2 + E\int_0^T \|x(s)\|^2 dt < \infty.$ (2.1)

Then, $\mathcal{M}^2(-\infty, T], \mathbb{H})$ with the norm (2.1) is a Banach space.

III. Existence and Uniqueness results

Definition 3.1 One parameter family$\mathrm{C}(t)_{t\geq 0}$ is called a strongly continuous cosine family if the following conditions hold:

(i) $\mathrm{C}(t)$ = I;

(ii) $\mathrm{C}(t)$x is continuous in t on R for all x∈H;

(iii) $\mathrm{C}(t+s) + \mathrm{C}(t-s)$ = 2$\mathrm{C}(t)$ $\mathrm{C}(s)$ for all t, s ∈ R.

The corresponding strongly continuous sine family $S(t)_{t\geq 0}$; which is defined as S(t)x = $\int_0^t C(s) x ds$, t ∈ R, X ∈ H. As for the infinitesimal generator A : D(A)⊂ $H \to H$ of a cosine family of operators $\mathrm{C}(t)_{t\geq 0}$, define Ax =

$\frac{d^2}{dt^2}C(t)x/t = 0$. A is also a closed and densely defined operator on H. Throughout this paper, we impose the following assumptions:

(H1)The cosine familyof operators $(C(t))_{t\geq 0}$ and its corresponding sine family $(S(t))_{t\geq 0}$ satisfy the following conditions for all t ≥ 0$\|C(t)\|^2 \leq M, \|S(t)\|^2 \leq M$, t ≥0 for a positive constant M

(H2) The function:σ: [0, +∞) → $L_2^0(K,H)$ satisfies the following conditions:

(i) there exists a positive constant L such that $\|\sigma(t)\|^2_{\mathcal{L}_Q^2} \leq L$ uniformly in [0, , ∞]

(ii) for thecomplete orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ in $\mathbb{K}$, we have $\sum_{n=1}^{\infty}\left\|\sigma Q^{\frac{1}{2}}e_n\right\|$<∞, $\sum_{n=1}^{\infty}\left\|\sigma Q^{\frac{1}{2}}e_n\right\|$ is uniformly convergent for t ∈ [0, , ∞],which imply that $\int_0^t \|\sigma(t)\|^2_{\mathcal{L}_Q^2} ds < \infty$ for every t ∈ [0, , ∞].

(H3) For all t∈ $[0,\infty]$, there exists a positive constant ∧, $\|f(t,0)\|^2 \vee \|g(t,0)\|^2 \vee \|h(t,0)\|^2 \leq$∧

(H4) The functions f,g : [0,+∞)× $\mathfrak{B}$ → $H$and h: I ×$\mathfrak{B}$ → $L_2 0, H)$ satisfy for all t ∈ $\varphi, \psi \in \mathfrak{B}$ and t ≥ 0

$$\|f(t,\varphi) - f(t,\psi)\|^2 \vee \|h(t,\varphi) - h(t,\psi)\|^2 \leq k\left(\|\varphi - \psi\|_\beta^2\right)$$

and

$$\|g(t,\varphi) - g(t,\psi)\|^2 \vee \|h(t,\varphi) - h(t,\psi)\|^2 \leq k\left(\|\varphi - \psi\|_\beta^2\right)$$

Where k(.) is concave. Non decreasing, continuousfunction from R_+ to R_+ such that k(0) = 0 ,k (y) > 0 for y> 0 and $\int_{0^+}\frac{du}{k(y)} \;= \infty$

Definition 3.2

A continuous stochastic process x : (-∞,T) → H is said to be a mild solution of (1.1) if

(i) x(t) is $\mathcal{F}_t$ – adapted and [x_t : t ∈ $[0,T]$ is B – valued

(ii) $\int_0^T \|\sigma(s)\|^2_{L_2^0} ds < \infty$,

(iii) x(t) satisfies the following integral equations

$$\begin{aligned} x(t) = c(t)\varphi(0) &+ s(t)\left[\varphi_1 - G(0, x(0 - r(0))\right) \\ &+ \int_0^t c(t-s)g\left(s, x\left(s - r(s)\right)\right)ds + \int_0^t S(t-s)f\left(s, x\left(s - \rho(s)\right)\right)ds \\ &+ \int_0^t S(t-s)h\left(s, x\left(s - \delta(s)\right)\right)dw(s) + \int_0^t S(t-s)\sigma(s)dB^H(s) \end{aligned}$$

(iv)x_0=$\varphi \in \mathcal{B}$

Note 3.2 Let us give some concrete functions k(·) . Let C > 0 and $\delta \in (0,1)$ be sufficiently small. Define

$$k_1(y) = Cy\,, y \geq 0.$$

$$k_2(y) = \begin{cases} ylog(y^{-1}), & 0 \leq y \leq \delta \\ \delta\log(y^{-1}) + k_2'(\delta -)(y-\delta), & y > \delta. \end{cases}$$

$$k_3(y) = \begin{cases} ylog(y^{-1})\log\log(y^{-1}), & 0 \leq y \leq \delta \\ \delta\log(y^{-1})\log log(\delta^{-1}) + k_3'(\delta -)(y-\delta), & y > \delta \end{cases}$$

Here $k_2'$and $k_3'$are the left derivative of k_2 and k_3 at the point δ.All the functions are concave nondecreasing and satisfy $\int_{0^+} du/\kappa(u) = +\infty$(I = 1,2,3). Also, it can be seen that the Lipschitz condition is a special case of the proposed conditions.

In order to obtain the uniqueness of solutions, we present the Bihari in- equality,

Lemma 3.1[Bihari inequality] : Let T > 0 and c> 0. Let k: $\mathbb{R}^+$ to $\mathbb{R}^+$ be a continuous nondecreasing function such that κ(t) > 0 for all t >0. Let u(.) be a Borel measurable bounded nonnegative function n $[0,\mathrm{T}]$. If

$$u(t) \leq c + \int_0^t v(s)k\left(u(s)\right)ds \text{for all } 0\leq t \leq \mathrm{T}.$$

$$\mathfrak{u}(t) \leq J^{-1}\left(J(c) + \int_0^t \mathfrak{v}(s)ds\right),$$

holds for all such t∈ $[0,\mathrm{T}]$ that

$$J(c) + \int_0^t \mathfrak{v}(s)ds \in \mathrm{Dom}\,(J^{-1}),$$

where$J(r) = \int_0^r ds/k(s)$,on r > 0 , and J^{-1} is the inverse function of J . In Particular,if, c = 0 and $\int_{0^+}^r ds/\kappa(s) = \infty$,then $\mathfrak{u}(t) = 0$ for all t∈ $[0,\mathrm{T}]$.

Further in order to prove the existence and uniqueness result, we construct the sequence of successive approximations as follows

$$x^0(t) = C(t)\varphi(0) + S(t)[\varphi_1 - f(0,\varphi)],\ t\in [0,\ T].$$

$$y^n(t) = C(t)\varphi(0) + S(t)[\varphi_1 - f(0,\varphi)] + \int_0^t C(t-s) f\left(s, x^{n-1}(s - r(s))\right) ds$$
$$+ \int_0^t S(t-s) g\left(s, x^{n-1}(s-\rho(s))\right) ds$$
$$+ \int_0^t S(t-s) h\left(s, x^{n-1}(s-\delta(s))\right) dw(s) + \int_0^t S(t-s)\sigma(s) dB^H(s)$$
$$x^n(t) = \varphi(t),$$

The following theorem establish the existence anduniqueness of mild solution to equation (1.1)

Theorem 3.1

Then Assume that the condition (H1) – (H3) hold. Then there exists a unique mild solution of (1.1) in $\mathcal{M}^2(-\infty,T],H$) proof of this theorem is long and technical, therefore it is convenient to divide it into three steps.

Step 1:

For all t∈ $[0,\mathrm{T}]$and n≥ 0 it holds that $y^n(t) \in \mathcal{M}^2(-\infty,T],H]$ ie there exists a +ve constantC_1, which is independent of n such that $\|y_n(t)\|^2 \le C_1$. It is obvious that $y^0(t) \in \mathcal{M}^2(-\infty,T],H]$by induction. We prove that $y^n(t) \in \mathcal{M}^2(-\infty,T],H]$. It follows from (3.2) Lemma 2.2, Holders and Doob's Inequality

$$\mathbb{E}\left(\sup_{0\le t\le T}\|y^n(t)\|^2\right) \le 6M\mathbb{E}\|\varphi(0)\|^2 + 12ME\|\xi\|^2 + 12\,ME\|f(0,\varphi\|^2 +$$
$$6MT\mathbb{E}\left\|\int_0^t f\left(s,x^{n-1}(s-r(s))\right)ds\right\|^2 +$$
$$6MT\mathbb{E}\left\|\int_0^t g\left(s,x^{n-1}(s-\rho(s))\right)ds\right\|^2 +$$
$$6MT\mathbb{E}\left\|\int_0^t h\left(s,x^{n-1}(s-\delta(s))\right)dw(s)\right\|^2 +$$
$$6\mathrm{MC}H(2H-1)T^{2H-1}\int_0^t\|\sigma(s)\|^2_{\mathcal{L}_Q^0}ds$$
$$\le 6M\mathbb{E}\|\varphi(0)\|^2 + 12ME\|\xi\|^2 + 24\,ME_k\left(\|\varphi\|^2_{B)}\right) + 24M \wedge +$$
$$12\,MT\mathbb{E}\int_0^t k\left(\left\|\int_0^t k(\|x^{n-1}(s-r(s)\|^2_B)ds\right\|^2_B\right) + 12MT^2 \wedge +$$
$$12MT\mathbb{E}\int_0^t k\left(\left\|\int_0^t k(\|x^{n-1}(s-r(s)\|^2_B)ds\right\|^2_B\right) + 12\,\mathrm{M}T^2 \wedge +$$
$$12MT\mathbb{E}\int_0^t k\left(\left\|\int_0^t k(\|x^{n-1}(s-r(s)\|^2_B)ds\right\|^2_B\right) + 12\,\mathrm{M}T^2 \wedge +$$
$$6\mathrm{MC}H(2H-1)T^{2H}L$$

Where $K_1 = 6\,ME\|\varphi\|^2_B + 12\,ME\|\xi\|^2 + 24\,ME_K(\|\varphi\|^2_B) + 24M \wedge + 12MT(2T+1)\wedge + 6\,McH(2H-$ *1T2HL.* Given that k(·) is concave and k(0) = 0, we can find positive constants a and b such that k(y) ≤a + by , for all y ≥ 0. Also, by using the Lemma 2.4 in the above inequality, we obtain

$$\mathrm{E}\left(\sup_{0\le t\le T}\|y^n(t)\|^2\right)$$
$$\le K_1 + 24\,MaT^2 + 24MTbEt\int_0^t\|y^n_s\|^2_B\,ds + 12\,MaT + 12\,MbE\int_0^t\|y^n_s\|^2_B\,ds$$
$$\le K_1 + 12\,MaT(2T+1) + 12BMM(2T+1)E\int_0^t\left(\sup_{0\le r\le s}\|y^{n-1}(r)\| + N\|y^n_0\|_B\right)^2 ds$$
$$\le K_3 + 24\,bM(2T+1)\int_0^t \sup_{0\le r\le s}\|y^{n-1}(r)\|ds$$

Where $K_3 = K_1 + 12\,MaT(2T+1) + 12BMM(2T+1)N^2E\|\varphi\|_B{}^2$. From any k≥ 1,it follows from (3.3) that

$$\mathrm{MaxE}\left(\sup_{0\le t\le T}\|y^n(t)\|^2\right)$$
$$\le K_3 + 24\,bM(2T+1)\int_0^t\left[E\|y^0(s)\|^2 + Emax\left(\sup_{0\le r\le s}\|y^{n-1}(r)\|\right)^2\right]ds$$
$$\le K_3 + 24\,bM(2T+1)\int_0^t\left[M\mathbb{E}\|\varphi\|^2_B + 2ME\|\xi\|^2 + 4\,ME_k\left(\|\varphi\|^2_{B)}\right) + 4M\wedge\right]ds$$

$+24\, bM(2T+1)\int_0^t E \underset{0\le r\le s}{max}\left(\underset{0\le r\le s}{sup}\|y^n(r)\|\right)^2 ds$

$$\le K_3 + 24\, bM(2T+1)\int_0^t\int_0^t E \underset{0\le r\le s}{max}\left(\underset{0\le r\le s}{sup}\|y^n(r)\|\right)^2 ds,$$

Where $K_4 = K_3 + 48\, bM^2(2T+1)T\left[\mathbb{E}\|\varphi\|_B^2 + 2ME\|\xi\|^2 + 4a + 4bE\left(\|\varphi\|_{B)}^2\right) + 4\wedge\right]$.
Using the Gronwall inequality in the above inequality, we get

$$\underset{1\le n\le k}{max}\left(\underset{0\le r\le s}{sup}\|y^n(r)\|\right)^2 \le K_4 e^{24Mb(2T+1)T}.$$

Since k is arbitrary, we have

$$E\left(\underset{0\le r\le s}{sup}\|y^n(r)\|\right)^2 \le K_4 e^{24Mb(2T+1)T}.forall\ 0\le t\le T, n\ge 1.$$

Hence by the result, we obtain
$\|y^n\|^2 = \mathrm{E}\|u_0^n\|_B^2 + E\int_0^T \|\varphi\|_B^{2\,2} ds \le C_1 < \infty$
Where $C_1 = \mathrm{E}\|\varphi\|_B^2 + \mathrm{T} \le K_4 e^{24Mb(2T+1)T}$.

Step 2. Next we show that there exists a positive constant $\bar{C}$which independent of n such that such that, for all $0\le t\le T$ and $\mathfrak{n},\mathfrak{m}\ge 0$

$$\mathbb{E}\underset{0\le t\le T}{sup}\|x^{\mathfrak{n}+\mathfrak{m}}(s) - x^{\mathfrak{n}}(s)\|^2 \le \bar{\mathrm{C}}\int_0^t \kappa\left(\mathbb{E}\underset{0\le u\le s}{sup}\|x^{\mathfrak{n}+\mathfrak{m}-1}(u) - x^{\mathfrak{n}-1}(u)\|^2\right) ds.$$

For $\mathfrak{n},\mathfrak{m}\ge 1$, from 3.2, we obtain

$$\mathbb{E}\underset{0\le t\le T}{sup}\|x^{\mathfrak{n}+\mathfrak{m}}(s) - x^{\mathfrak{n}}(s)\|^2 \le 3\, M(2T+1)E\int_0^t \kappa(\|x^{\mathfrak{n}+\mathfrak{m}-1}(u) - x^{\mathfrak{n}-1}(u)\|^2) ds.$$

$$\le 3\, M(2T+1)E\int_0^t \kappa\left(\underset{0\le r\le s}{sup}\|x^{\mathfrak{n}+\mathfrak{m}-1}(u) - x^{\mathfrak{n}-1}(u)\|^2\right) ds.$$

Further, it follows from the Jensen's inequality that

$$\mathbb{E}\underset{0\le t\le T}{sup}\|x^{\mathfrak{n}+\mathfrak{m}}(s) - x^{\mathfrak{n}}(s)\|^2 \le C_2\int_0^t \kappa\left(\underset{0\le r\le s}{sup}\|x^{\mathfrak{n}+\mathfrak{m}-1}(u) - x^{\mathfrak{n}-1}(u)\|^2\right) ds.$$

Where$C_2 = 3\, M(2T+1)$. The proof of Step 2 is complete.

Step 3. By using a similar procedure as in the proof of (Lemma. 15, [19]), we can prove that there a positive constant C_3 such that E $(sup_{0\le r\le t}\|x^{\mathfrak{n}+\mathfrak{m}}(s) - x^{\mathfrak{n}}(s)\|^2) \le C_3 t$ for all $0\le t\le T$, n, m ≥ 1. Next, we define
$\varphi_1(t) = C_3(\mathrm{t})$,
$\varphi_{n+1}(t) = C_2\int_0^t \kappa(\varphi_n(s))ds,\ \mathfrak{n}\ge 1$,
$\varphi_{n,m}(t) = E\left(\underset{0\le t\le T}{sup}\|x^{\mathfrak{n}+\mathfrak{m}}(r) - x^{\mathfrak{n}}(r)\|^2\right),\ \mathfrak{n}, m\ge 1.$
Choose $T_1 \in$[0,T) such that C_{2k} $(C_3\mathrm{t}) \le C_3$,for all $0\le t\le T_1$. Next, by following the similar procedure as in (Lemma 16, [19] , by induction one can show that there exists a positive $0\le T_1\le T$. Such that for all n,m $\ge 1, 0\le \varphi_{n,m}(t)\le\varphi_n(t)\le\varphi_{n-1}(t)\le\cdots ...\le\varphi_1(t)$ for all $0\le t\le T_1$.
Next we prove the existence and uniqueness results:

Uniqueness: Let x(t) and y(t) be any solutions of (1.1). By employing the similar procedure as in Step 2, on the interval $(-\infty, 0]$ and for $0\le t\le T$, we have

$$\mathbb{E}\underset{0\le t\le T}{sup}\|x(s) - y(s)\|^2 \le 3\, M(2T+1)\int_0^t \kappa\left(\mathbb{E}\underset{0\le t\le T}{sup}\|x(s) - y(s)\|^2\right) ds$$

Further, it follows from the Bihari inequality that
$\mathbb{E}\underset{0\le t\le T}{sup}\|x(s) - y(s)\|^2 = 0$, t$\in[0,T]$.
Consequently x=y which implies the uniqueness.The proof of theorem is complete.

Existence: In order to prove the existence result, we claim that

$$\mathbb{E}\underset{0\le s\le t}{sup}\|y^{\mathfrak{n}+1}(s) - y^n(s)\|^2 \to 0$$

For al$\infty\le t\le T_1$, as n,m $\to\infty$. Notice that $\varphi_n$is continuous on $[0,T_1]$, and for each t, $\varphi_n is$a decreasing sequence. Now, we can define the function $\varphi(t)$ as $\varphi(t) = lim_{n\to\infty}\varphi_n(t) = lim_{n\to\infty}C_2\int_0^t k(\varphi_{n-1}(s))ds = C_2\int_0^t k(\varphi\ (s))ds\, dsforall 0\le t\le T_1$. Also, from Step 3, we have $\varphi_{n,n}(t)\le sup_{0\le t\le T_1}\varphi_n\le\varphi_n(T_1)\to 0$ as n$\to\infty$.That is $y^n(t)$ is a Cauchy sequence in L^2 on $(-\infty, T_1]$.

Also from Step 1, we can obtain that $\|y(t)\|^2 \leq C$, where C is a positive constant. More over, for all $0 \leq t \leq T_1$, by using the property of the function k(·), we can obtain

E$\left\|\int_0^t C(t-s)\,[\,f\big(s, y^n(t-r(t))\big) - f(s, y(t-r(t)]ds\right\|^2 \to 0, as n \to \infty,$

E$\left\|\int_0^t S(t-s)\,[\,g\big(s, y^n\big(t-\rho(t)\big) - g(s, (y-\rho(t)]ds\right\|^2 \to 0, as n \to \infty,$

and

E $\left\|\int_0^t S(t-s)\,[\;\, h\big(s, y^n\big(t-\delta(t)\big) - h\big(s, (y, \big(s-\delta(t)\big)(t)\big]d(s)\right\|^2 \to 0, as n \to \infty,$

For all $0 \leq t \leq T_1$ by taking limit on both sides of (3.2), we obtain

$$\begin{aligned} y^n(t) = \quad & C(t)\varphi(0) + S(t)[\varphi_1 - f(0,\varphi)] + \int_0^t C(t-s) f\left(s, y^{n-1}\big(s-r(s)\big)\right) ds \\ & + \int_0^t S(t-s) g\big(s, y^{n-1}(s-\rho(s))\big) ds \\ & + \int_0^t S(t-s) h\left(s, y^{n-1}\big(s-\delta(s)\big)\right) dw(s) + \int_0^t S(t-s)\sigma(s) dB^H(s) \end{aligned}$$

The expression reveals that y (t) is one solution of (1.1) on [0, T_1] . By iteration technique, the existence of solutions of (1.1) on [0, T] can be obtained. Thus, the proof of this theorem is completed.

Next, we investigate the stability of mild solutions for the stochastic equations (1.1). In particular, we will provide the continuous dependence of solutions on the initial value by using the Bihari inequality. In order to establish the stability result, we impose the following condition on f:

(H5) For all t [0,T],$\varphi, \psi \in \mathcal{B}$, the function f satisfies $\|f(t,\varphi) - f(t,\psi)\|^2 \leq k\|\varphi - \psi\|_{\mathcal{B}}^2$

Where k > 0 is a constant.

Definition 3.3 [19] A mild solution $y^{\xi,y}(t)$ of (1.1) with initial value (ξ, y) is said to be stable in mean square if for all ϵ, when E$\|\xi - \eta\|_{\mathcal{B}}^2$ + E $\|y - z\|^2 \leq \delta$,

Where$z^{\eta,z}(t)$ is another solution of (1.1) with initial value (η, z).

To obtain the stability of solutions, we need the following lemmas

Lemma 3.2[14]: Let T $>$ 0 and c$>$ 0. Let k: $\mathbb{R}^+$ to $\mathbb{R}^+$ be a continuous nondecreasing function such that κ(t) > 0 for all t >0. Let u(.) be a Borel measurable bounded nonnegative function n $[0, \mathrm{T}]$. If

$u(t) \leq c + \int_0^t v(s) k\big(u(s)\big) ds$for all 0≤ $t \leq$ T.

$$\mathfrak{u}(t) \leq J^{-1}\left(J(c) + \int_0^t \mathfrak{v}(s) ds\right),$$

holds for all such t∈ $[0, \mathrm{T}]$that

$J(c) + \int_0^t \mathfrak{v}(s) ds \in$ Dom (J^{-1}),

where$J(r) = \int_0^r ds/k(s)$,on r $>$ 0 , and J^{-1} is the inverse function of J . In Particular, if, c = 0 and$\int_{0^+}^r ds/\kappa(s) = \infty$,then $\mathfrak{u}(t) = 0$ for all t∈ $[0, \mathrm{T}]$.

Lemma 3.3 Let the assumptions of Lemma 3.3 hold and v(t) ≥ 0 for t $\epsilon[0, T]$.If for all $\epsilon, > 0$, there exists $t_1 \geq 0, forall 0 \leq u_0 < \epsilon, \int_{t_1}^T v(s) ds \leq \int_{u_0}^{\epsilon} \frac{ds}{k(s)}$ holds. Then for every t ∈ $[t_1, T]$, the estimate u(t) ≤ ϵholds

Theorem 3.2

Assume that the conditions of Theorem 3.4 are satisfied and f satisfies (H4) instead of (H1), then the solution of (1.1) is stable in mean square.

Proof. By assumption let $y^{\xi,y}(t)$ and $z^{\eta,z}(t)$ be two solutions of (1.1) with initial value $(\cdot)$ and (η, z), respectively. Then for all 0≤ $t \leq T$ and using the same arguments as lemma 14.

We get

$$\begin{aligned} \mathrm{E}\Big(\sup_{0 \leq t \leq T} |x(s) - y(s)|^2\Big) \leq\ & 5M(1+2K_1)E\|\xi - \eta\|^2 + 10\, ME|x - y|^2 \\ & + 5MT \int_0^t \big\|Ex\big(s-r(s)\big) - y\big(s-r(s)\big)\big\|^2 + \ 5M(1+T)\int_0^t \big\|k(x\big(s-\rho(s)\big)y\big(s-\rho(s)\big)\big\|^2 \\ & + 10C \int_0^t \big\|k(x\big(s-\delta(s)\big) - y\big(s-\delta(s)\big)\big\|^2 \end{aligned}$$

$$\leq\ 5M(1+2K_1)E\|\xi-\eta\|^2+10ME|x-y|^2$$
$$+5MT\int_0^t E\left(\sup_{0\leq t\leq T}|x(r)-y(r)|^2\right)ds$$
$$+5\,[M(1+)+2C]\int_0^t k\left(\sup_{0\leq t\leq T}|x(s)-y(s)|^2 ds\right)$$

Let $K_1(u)=5M(T+1)k(u)+5MTu$, for k is a concave increasing function from R to R, such that k(0) = 0, k(u) > 0 and $\int_{0^+}\frac{du}{k(u)}=\infty. k_1(0)=0, k(u)\geq k(1)u,$ for any $0\leq u\leq 1$ and $\int_{0^+}\frac{du}{k(u)}=\infty.$ So, for all$\varepsilon>0,$ letting $\epsilon_1=\frac{1}{2}\epsilon$, we have $\lim_{s\to 0}\int_\delta^{\epsilon_1}\frac{du}{k_1(u)}=\infty.$ So there exists a positive constant $\delta<\epsilon_1$such that $\int_s^{\epsilon_1}\frac{du}{k_1(u)}\geq T.$ Let $u_0=5M(1+2k_1)E\|\xi-\eta\|^2$+ 10 ME $|x-y|^2$, u(t) = E$\left(\sup_{0\leq t\leq T}|x(s)-y(s)|^2\right)$, v(t) =1. Therefore, when $u_0\leq t\leq\varepsilon_1,$

We have $\int_{u_0}^{\epsilon_1}\frac{du}{k_1(u)}\geq\int_\delta^{\epsilon_1}\frac{du}{k_1(u)}\geq T=\int_0^{\epsilon_1}v(s)\,ds. So, for any t\in[0,T]$, the estimate u(t) $\leq\varepsilon_1$, holds. This completes the proof of the theorem.

IV. An Example

Consider the following stochastic nonlinear wave equation with infinite delays driven by FBM in the following form

$$\partial\left[\frac{\partial(t,\psi)}{\partial t}-f_1\left(t,x(t-\tau,\psi)\right)\right]$$
$$=\frac{\partial^2(t,\psi)}{\partial\psi^2}\partial t+\ f_2\left(t,x(t-\tau,\psi)\right)\partial t+\varphi\left(t,x(t-\tau,\psi)\right)dw(t)+\Theta(t)dB_Q^H(t)$$

$0\leq\psi\leq\pi,0\leq t\leq T$

x(t,ψ) = $\varphi(t,\pi)$ -∞ $<t\leq 0,\quad 0<\psi<\pi$

x(t,0) = x(t,π) = 0

$$\frac{\partial x(t,\psi)}{\partial t}=\ \xi(\psi)0<\psi<\pi$$

Where $\xi\in L_0^2(\Omega,H)\varphi\in\mathcal{B},\ \ H=\ L^2([0,\pi])$w is an H-valued Winer Process and the phase $\mathcal{B}$ which denotes the family of bounded continuous H- valued fuctions φdefined]-∞, 0],

With norm $\|\varphi\|_{\mathcal{B}}=\ sup_{-\infty<\theta\leq 0}|\varphi(\theta)|$

The operation A is defined by

A(z) (ψ) $=\ \frac{d^2z(\psi)}{d\psi^2}$, with domain D(A) = { z $\epsilon H\colon D(A)=\{z\epsilon H\colon z(0)=z(\pi)\}$

The spectrum of A consists of eigenvalues -n^2 for n ϵN, with associated eigenvectors

$z_n(\psi)=\left(\frac{2}{\pi}\right)^{1/2}$sin(n$\psi$). Furthermore the set $\{z_n\colon n\epsilon N\}$ is an orthonormal basis of H.

In Particular

Ax = $\sum_{n=1}^\infty -n^2\langle x,z_n\rangle z_n,$ x $\epsilon D(A)$

And the operator C(t), defined by

C(t) x =$\sum_{n=1}^\infty$ cos$(nt)\,\langle x,z_n\rangle z_n,$ t ϵR

From a cosine function on H, with associated sine function

S(t) x = $\sum_{n=1}^\infty\frac{sn(nt)}{n}\langle x,z_n\rangle z_n$ t ϵR

From [21] for all xϵH,t $\epsilon R\|s(t)\|\leq 1$ and $\|C(t)\|\leq 1$

Therefore the above system can be rewritten in the form of (1) – (3). Assume that $f_i\colon[0,T]\times R\to R(i=1,2),\sigma\colon[0,T]\times R\to L_Q^{(H)}$ satisfy the conditions of Therorem3.1 Then the above system has a unique mild solution

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