Two simples proofs of Fermat 's last theorem and Beal conjecture

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Abstract: If after 374 years the famous theorem of Fermat-Wiles was demonstrated in 110 pages by A. Wiles [4], the purpose of this article is to give a simple demonstration and deduce a proof of the Beal conjecture.

Résumé : Si après 374 ans le célèbre théorème de Fermat-Wiles a été démontré en 110 pages par A. Wiles [4], le but de cet article est de donner une simple démonstration et d’en déduire une preuve de la conjecture de Beal.

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I. Introduction

Set out by Pierre de Fermat [2], it was not until more than three centuries ago that Fermat's great theorem was published, validated and established by the British mathematician Andrew Wiles [4] in 1995. In mathematics, and more precisely in number theory, the last theorem of Fermat [2], or Fermat's great theorem, or since his Fermat-Wiles theorem demonstration [4], is as follows: There are no non-zero integers a, b, and c such that:

\[ a^n + b^n = c^n \]

as soon as n is an integer strictly greater than 2.

The Beal conjecture is the following conjecture in number theory: If

\[ a^x + b^y = c^z \]

where a, b, c, x, y and z are positive integers with x, y, z > 2, then a, b, and c have a common prime factor. Equivalently, There are no solutions to the above equation in positive integers a, b, c, x, y, z with a, b and c being pairwise coprime and all of x, y, z being greater than 2.

If the famous Fermat-Wiles theorem has been demonstrated in 110 pages by A. Wiles [4], the purpose of this article is to give a simple proof and deduce a proof of the Beal conjecture.

II. The proof of Fermat 's last theorem

Theorem: There are no non-zero integers a, b, and c such that:

\[ a^n + b^n = c^n \]

with n an integer strictly greater than 2.

Lemma 1: If n, a, b and c are a non-zero integers with and \( a^n + b^n = c^n \) then:

\[ \int_0^b x^{n-1} - \left( \frac{a}{b} x + a \right)^{n-1} \frac{c-a}{b} \, dx = 0 \]

Proof:

\[ a^n + b^n = c^n \iff \int_0^a n x^{n-1} \, dx + \int_a^b n x^{n-1} \, dx = \int_b^c n x^{n-1} \, dx \]

But as:

\[ \int_a^b n x^{n-1} \, dx = \int_0^a n x^{n-1} \, dx + \int_a^c n x^{n-1} \, dx \]

So:

\[ \int_0^b n x^{n-1} \, dx = \int_a^c n x^{n-1} \, dx \]

And as by changing variables we have:

\[ \int_a^c n x^{n-1} \, dx = \int_0^b \left( \frac{c-a}{b} x + a \right)^{n-1} \frac{c-a}{b} \, dy \]
Then :

$$\int_0^b x^{n-1} dx = \int_0^b \left( \frac{c-a}{b} y + a \right)^{n-1} \frac{c-a}{b} dy$$

It results:

$$\int_0^b x^{n-1} - \left( \frac{c-a}{b} x + a \right)^{n-1} \frac{c-a}{b} dx = 0$$

**Corollary 1 :** If N, n, a, b and c are a non-zero integers with and $a^n + b^n = c^n$ then :

$$\int_0^N x^{n-1} - \left( \frac{c-a}{b} x + a \right)^{n-1} \frac{c-a}{b} dx = 0$$

**Proof :** It results from the proof of lemma 1 by replacing a, b and c respectively by $\frac{a}{N}$, $\frac{b}{N}$ and $\frac{c}{N}$.

**Lemma 2 :**

If $a^n + b^n = c^n$, where n, a, b and c are a non-zero integers with $n > 2$ and $a \leq b \leq c$. Then for an integer N big enough we have : $x^{n-1} - \left( \frac{c-a}{b} x + a \right)^{n-1} \frac{c-a}{b} \leq 0 \forall x \in \left[0, \frac{b}{N}\right]$.

**Proof :**

Let $f(x,a,b,c,y) = x^{n-1} - \left( \frac{c-a}{b} x + y \right)^{n-1} \frac{c-a}{b}$, with $x, y \in \mathbb{R}^+$.

We have:\n
$$\frac{\partial f}{\partial x} = (n-1)x^{n-2} - (n-1)\left( \frac{c-a}{b} x + y \right)^{n-2} \left( \frac{c-a}{b} \right)^2, f(0,a,b,c,y) < 0$$ and $\frac{\partial f}{\partial x} \big|_{x=0} < 0$.

So, by continuity, $\exists \epsilon > 0$ such that $\forall u \in [0, \epsilon]$ we have $\frac{\partial f}{\partial x} \big|_{x=u} < 0$. So the function f is decreasing in $[0, \epsilon]$ and $\exists \epsilon' > 0, \epsilon \geq \epsilon' > 0$ such that we have : $f(x,a,b,c,y) \leq 0 \forall x \in [0, \epsilon'] \forall y \in [0, \epsilon']$.

As $\frac{b}{N} \in [0, \epsilon]$ for an integer N big enough, It follows that $\forall x \in \left[0, \frac{b}{N}\right]$ we have :

$f(x,a,b,c,\frac{a}{N}) \leq 0 \forall x \in \left[0, \frac{b}{N}\right]$.  

**Proof of Theorem :**

If $a^n + b^n = c^n$, where n, a, b and c are a non-zero integers with $n > 2$ and $a \leq b \leq c$. Then for an integer N big enough, it results from the lemma 2 that we have :

$$f(x,a,b,c,\frac{a}{N}) = x^{n-1} - \left( \frac{c-a}{b} x + a \right)^{n-1} \frac{c-a}{b} \leq 0 \forall x \in \left[0, \frac{b}{N}\right]$$

And by using the corollary 1, we have $\int_0^b x^{n-1} - \left( \frac{c-a}{b} x + a \right)^{n-1} \frac{c-a}{b} dx = 0$.

So : $x^{n-1} - \left( \frac{c-a}{b} x + a \right)^{n-1} \frac{c-a}{b} = 0 \forall x \in \left[0, \frac{b}{N}\right]$.

And therefore $\frac{c-a}{b} = 1$ because $f(x,a,b,c,\frac{a}{N})$ is a null polynomial as it have more than n zeros. So $c = a+b$ and $a^n + b^n \neq c^n$ which is absurde.

**Corollary :** [Beal conjecture]

If $a^x + b^y = c^z$ where a, b, c, x, y and z are positive integers with $x, y, z > 2$, then a, b, and c have at least a common prime factor.

III. The proof of Beal conjecture:

If $a^x + b^y = c^z$ where a, b, c, x, y and z are positive integers with $x, y, z > 2$, then a, b, and c have at least a common prime factor.
Equivalently, there are no solutions to the above equation in positive integers a, b, c, x, y, z with a, b and c being pairwise coprime and all of x, y, z being greater than 2.

**Proof:**

Let \( a^x + b^y = c^z \).

If a, b and c are not pairwise coprime, then by posing \( a = ka' \), \( b = kb' \), and \( c = kc' \).

Let \( a' = u^{yz} \), \( b' = v^{yz} \), \( c' = w^{yz} \) and \( k = u^x \), \( k = v^y \), \( k = w^z \).

As \( a^x + b^y = c^z \), we deduce that \( (uu')^{yz} + (vv')^{yz} = (ww')^{yz} \).

So: 

\[ k^z u^{xyz} + k^y v^{xyz} = k^z w^{xyz} \]

This equation does not look like the one studied in the first theorem. But if a, b and c are pairwise coprime, we have \( k = 1 \) and \( u = v = w = 1 \) and we will have to solve the equation: \( u^{xyz} + v^{xyz} = w^{xyz} \).

The equation \( u^{xyz} + v^{xyz} = w^{xyz} \) have a solution if at least one of the equations:

\( (u^{xyz})^z + (v^{xyz})^z = (w^{xyz})^z \), \( (u^{xyz})^z + (v^{xyz})^z = (w^{xyz})^z \), \( (u^{xyz})^y + (v^{xyz})^y = (w^{xyz})^y \), have a solution.

So by the proof given in the proof of the first Theorem we must have: \( x \leq 2 \) or \( y \leq 2 \), or \( z \leq 2 \).

We therefore conclude that if \( a^x + b^y = c^z \) where a, b, c, x, y, z are positive integers with \( x, y, z > 2 \), then a, b, and c have a common prime factor.

**IV. Important notes:**

1- If a, b, and c are not pairwise coprime, someone, by applying the proof given in the corollary like this:

\[ a = u^x \], \( b = v^y \), \( c = w^z \] will have \( u^{xyz} + v^{xyz} = w^{xyz} \), and could say that all the x, y, and z are always smaller than 2. What is false: \( 7^3 + 7^4 = 14^3 \).

The reason is simple: it is the common factor k which could increase the power, for example if \( k = c^{z+1} \) in the proof, then \( c^z = (kc)z = c^{k(z+1)} \). You can take the example: \( 2^2 + 2^2 = 2^{2+1} \) where \( k = 2^{z+1} \).

2- These techniques do not say that the equation \( a^x + b^y = c^z \) where \( a, b, c \in [0, +\infty[ \), has no solution since in the proof the equation \( x^2 + y^2 = z^2 \) can have a sloution. We take \( a = X^2 \), \( b = Y^2 \) and \( C = Z^2 \).

3- In [3] I proved the abc conjecture which implies only that the equation \( a^x + b^y = c^z \) has only a finite number of solution with a, b, c, x, y, z a positive integers, a, b and c being pairwise coprime and all of x, y, z being greater than 2.

**V. Conclusion:**

The techniques used in this article have allowed to prove both the Fermat’ last theorem and the Beal’ conjecture and have shown that the Beal conjecture is only a corollary of the Fermat’ last theorem.

**Bibliography:**
