IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 3 Ser. III (May – June 2019), PP 45-50
www.iosrjournals.org

Generalization of Enestrom Kakeya Theorem

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Abstract. *In this paper we will give generalizations of polynomials with complex coefficients when we have only real or imaginary parts of the coefficients.*
Keywords. *Enestrom-Kakeya Theorem, Maximum modulus principal, Schwarz's lemma.*

Date of Submission: 08-06-2019 Date of acceptance: 25-06-2019

I. Introduction

Suppose $F(t)=\sum_{v=0}^{n} b_v t^v$ is a polynomial of degree m whose coefficients satisfy $0 \le b_0 \le b_1 \le ... \le b_m$. Then $F(t)$ has all its zeros in the closed unit disk $|t| \le 1$

An equivalent but perhaps more useful statement of the above theorem due to in fact to Enestorm[3] is the following.

Theorem 1. Suppose $F(t)=\sum_{v=0}^{n} b_v t^v$, $n \ge 1$ be a polynomial of degree m with $b_v > 0 \;\; \forall \;\; 0 \le v \le n$. If

$$\beta=\beta[f]:=\min_{0\le v\le n}\left\{\frac{b_v}{b_v+1}\right\}, \gamma=\gamma[f]:=\max_{0\le v\le n}\left\{\frac{b_v}{b_v+1}\right\}$$

then all the zeros of $f(t)$ are contained in $\beta \le |t| \le \gamma$

Theorem 2. Let $F(t)=\sum_{v=0}^{n} b_v t^v, \mathrm{Re}\;\; b_{j^*}=\beta_{j^*} \;\; and \;\; \mathrm{Im}\;\; b_{j^*}=\gamma_{j^*}$ for $j^*=0,1,...,n, \;\; b_n \ne 0$ and for some k,

$$\beta_0 \le j^*\beta_1 \le j^{*2}\beta_2 \le \ldots \le j^{*k}\beta_k \ge j^{*k+1}\beta_{k+1} \ge j^{*k+2}\beta_{k+2} \ge \ldots \ge j^{*n}\beta_n$$

for some positive j^*.

Then $f(t)$ has all its zeros in $R_1 \le |t| \le R_2$ where

$$R_1=\frac{j^*|b_0|}{2j^{*k}\beta_k-\beta_0-j^{*n}\beta_n+j^{*n}|b_n|+|\gamma_0|+|\gamma_n|j^{*n}+2\sum_{i=1}^{n-1}|\beta_i|j^{*i}}$$

and

$$R_2=\max\frac{\begin{array}{c}|\beta_0|j^{*n+1}+(j^{*2}+1)j^{*n-k-1}\beta_k-j^{*n-1}\beta_0-j^*\beta_n+\\(j^*{}_2-1)\sum_{i=1}^{k-1}j^{*n-i-1}\beta_i+(1-j^{*2})\sum_{i=k+1}^{n-1}j^{n-i-1}\beta_i+\sum_{i=1}^{n}\left(|\gamma_i|+j^*|\gamma_i|\right)j^{*n-i}\end{array}}{|b_n|,\frac{1}{j^*}}$$

We do not know if the result is best possible, however if we take $k=n, \;\; j^*=1, \;\; \gamma_v=0 \;\; for \;\; 0 \le v \le n$

and $b_0 \ge 0,$ we get that all the zeros of the polynomial $f(t)$ lie in the annulus $\frac{b_0}{2b_n-b_0} \le |t| \le 1$ which is best possible in the sense that the inner and outer radii of the annulus here cannot be improved $\left(f(t)=t^m+t^{m-1}+...+t+1\right)$. If we take in the theorem 2 $k=n$ we get

Corollary1.1 Let $f(t)=\sum_{v=0}^{n} b_v t^v$, $\operatorname{Re}\ b_{j^*}=\beta_{j^*}$ *and* $\operatorname{Im} b_{j^*}=\gamma_{j^*}$ *for* $j^*=0,1,...,n, b_n\neq 0$ and

$$\beta_0 \le j^*\beta_1 \le j^{*2}\beta_2 \le \ldots \le j^{*n}\beta_n$$

for some positive j^*. Then $f(t)$ has all its zeros in $R_1 \le |t| \le R_2$ where

$$R_1=\frac{j^*|b_0|}{\left(j^{*n}\beta_n-\beta_0+j^{*n}|b_n|+|\gamma_0|+|\gamma_n|j^{*n}+2\sum_{i=1}^{n-1}|\gamma_i|j^{*i}\right)}$$

and $$R_2=\max\frac{\left[|b_0|j^{*n+1}+j^{*-1}\beta_n-j^{*n-1}\beta_0+(j^{*2}-1)\sum_{i=1}^{n} j^{*n-i-1}\beta_i+\sum_{i=1}^{n}\left(|\gamma_i-1|+j^*|\gamma_i|j^{*n-i}\right)\right]}{|\beta_n|,\frac{1}{j^*}}$$

In particular, taking $j^*=1$ *and* $\gamma_v=0$ *for* $0\le v\le n$ in Corollary 1.1, if $f(t)=\sum_{v=0}^{n} b_v t^v$ is a polynomial with real coefficients satisfying $b_0 =\le b_1 \le ... \le b_n$ then $f(t)$ has all its zeros in

$$\frac{|b_0|}{b_n-b_0+|b_n|}\le |t| \le \frac{|b_0|+b_n-b_0}{|b_n|} \tag{1}$$

This result sharpen a result due to Joyal, Labelle and Rahman [1]. The Enestrom-Kakeya Theorem is implied by (1) when $b_0 \ge 0$

Corollary 1.2. Let $f(t)=\sum_{v=0}^{n} b_v t^v$, $\operatorname{Re}\ b_{j^*}=\beta_{j^*}$ *and* $\operatorname{Im}\ b_{j^*}=\gamma_{j^*}$ for $j^*=0,1,...,n,\ b_n\neq 0$ and

$\beta_0 \ge j^*\beta_1 \ge j^{*2}\beta_2 \ge \ldots \ge j^{*n}\beta_n$ for some positive j^*. Then $f(t)$ has all its zeros in $R_1 \le |t| \le R_2$ where

$$R_1=\frac{j^*|b_0|}{\left(\beta_0-j^{*n}\beta_n+j^{*n}|b_n|+|\gamma_0|+|\gamma_n|j^{*n}+2\sum_{i=1}^{n-1}|\gamma_i|j^{*i}\right)}$$

and

$$R_2=\max\frac{\left[|b_0|j^{*n+1}+j^{*n+1}\beta_0-j^*\beta_n+(1-j^{*2})\sum_{i=1}^{n} j^{*n-i-1}\beta_i+\sum_{i=1}^{n}\left(|\gamma_i-1|+j^*|\gamma_i|j^{*n-i}\right)\right]}{|\beta_n|,\frac{1}{j^*}}$$

In particular, if $f(t)=\sum_{v=0}^{n} b_v t^v$ is with real coefficients satisfying $b_0 \ge b_1 \ge ... \ge b_n$ then it has all its zeros in

$$\frac{|b_0|}{b_0-b_n+|b_n|}\le |t| \le \frac{|b_0|+b_0-b_0}{|b_n|} \tag{2}$$

Theorem 3 Let $f(t)=\sum_{v=0}^{n} b_v t^v$ $\operatorname{Re}\ b_{j^*}=\beta_{j^*}$ *and* $\operatorname{Im}\ b_{j^*_*}=\gamma_j^{\ *}$ for $j^*=0,1,...,n\ \ b_n\neq 0$ and for some k, $j^{*n}\beta_0 \le j^{*n-1}\beta_1 \le j^{*n-2}\beta_2 \le ... \le j^{*k}\beta_{n-k} \ge j^{*k-1}\beta_{n-k+1} \ge ... \ge j^*\beta_{n-1} \ge \beta_n$ for some positive j^*.

Then $f(t)$ has all its zeros in $R_1 \le |t| \le R_2$ where

$$R_1 \ \min = \left(\frac{|b_0|}{\left(|b_n| j^{*n+1} + (j^{*2}+1)\right) j^{*n-k-1}\beta_{n-k} - j^{*n-1}\beta_n - j^{*}\beta_0 + (j^{*2}-1)\sum_{i=1}^{k-1} j^{*n-j-1}\beta_{n-j^*}\beta_{n-j^*}} \right.$$
$$\left. + (1-j^{*2})\sum_{i=k+1}^{n-1} j^{*n-i-1}\beta_{n-j} + \sum_{i=1}^{n}\left(|\gamma_{n-j^*+1}| + j^*|\gamma n-i| j^{*n-j}\right), j^* \right)$$

and

$$R_2 = \left(\frac{2j^{*k}\beta_{n-k} - \beta_n - j^{*n}\beta_0 + j^{*n}|\beta_0| + |\gamma_0| j^{*n} + |\gamma_n| + 2\sum_{i=1}^{n-1}|\gamma_{n-i}| j^{*i}}{\left(j^*|\beta_n|\right)} \right)$$

In particular, if we take $k = 0$ and $\gamma_v = 0$ for $0 \le v \le n$, we get that if $f(t) = \sum_{v=0}^{n} b_v t^v$ is a polynomial of degree m with real coefficients satisfying $j^{*n}b_0 \le j^{*n-1}b_1 \le ... \le j^* b_{n-1} \le b_n$ for some positive j^*, then all the zeros of $f(t)$ lie in

$$\min\left[\frac{|b_0|}{|b_n| j^{*n+1} + j^{*n+1}b_n - j^* b_o + (1-j^{*2})\sum_{i=1}^{n} j^{*n-i-1} b_n - i}, j^* \right] \le |t| \le \frac{b_n - j^{*n}b_0 + |b_0| j^{*n}}{j^*|b_n|}$$

This result hold good due to Kovacevic and Milovanovic [6] for $j^* = 1$, this further reduces to (1) when $b_0 \ge 0$, reduces to the Enestrom-Kakeya Theorem.

If we have information only about the imaginary parts of the coefficients we have the following theorem which is of interest and follows by applying theoem1 to $-if(t)$.

Theorem 4 Let $f(t) = \sum_{v=0}^{n} b_v t^v$, $\text{Re} \ \ b_{j^*} = \beta_{j^*} \ \ and \ \ b_{j^*} = \gamma_{j^*}$ for $j^* = 0,1,..., n,\ \ b_n \ne 0$ and for some k,

$\gamma_0 \le j^*\gamma_1 \le j^{*2}\gamma_2 \le ... \le j^{*k}\gamma_k \ge j^{*k+1}\gamma_{k+1} \ge j^{*k+2}\gamma_{k+2} \ \ge ... \ge j^{*n}\gamma_n$ for some positive j^*. Then $f(t)$ has all its zeros in $R_1 \le |t| \le R_2$ where

$$R_1 = \frac{j^*|b_0|}{\left(2j^*\gamma_k - \gamma_0 - j^{*n}\gamma_n + j^{*n}|b_n| + |\beta_0| + |\beta_n| j^{*n} + 2\sum_{i=1}^{n-1}|\beta_i| j^{*i}\right)}$$

and

$$R_2 = \max \frac{\left[\begin{array}{l}\left(|b_0| j^{*n+1} + (j^{*2}+1)\right) j^{*n-k-1}\gamma_k - j^{*n-1}\gamma_0 - j^{*}\gamma_n + \\ (j^{*2}-1)\sum_{i=1}^{k-1} j^{*n-i-1}\gamma_j + (1-j^{*2})\sum_{i=k+1}^{n-1} j^{*n-i-1}\gamma_i + \sum_{i=1}^{n}\left(|\beta_i - 1| + j^{*}|\beta_i| j^{*n-i}\right)\end{array}\right]}{|b_n|, \ 1/j^{*}}$$

By making suitable choice of j^* and k in the above theorems, one can also obtain the following corollaries which appear to be interesting and useful. In each of these

$f(t) = \sum_{v=0}^{n} b_v t^v$, Re $\quad b_j^{\ *} = \gamma_{j^*}$ *and* Im $\quad b_{j*} = \gamma_{j*}$ for $j^* = 0,1,\ldots, n$ and $b_n \neq 0$.

Corollary 1.3 Let $\beta_0 \le \beta_1 \le \ldots \le \beta_n$ then all the zeros of $f(t)$ lie in $R_1 \le |t| \le R_2$ where

$$R_1 = \frac{|b_0|}{\left\{\beta_n - \beta_0 + |\beta_n| + |\gamma_0| + \gamma_n + 2\sum_{i=1}^{n-1}|\gamma_i|\right\}}$$

and
$$R_2 = \frac{\left[|b_0| - \beta_0 + \beta_n + |\gamma_0| + |\gamma_n| + 2\sum_{i=1}^{n-1}|\gamma_i|\right]}{|b_n|}$$

Corollary 1.4 Let $\beta_0 \ge \beta_1 \ge \ldots \ge \beta_n$ then all the zeros of $f(t)$ lie in $R_1 \le |t| \le R_2$ where

$$R_1 = \frac{|b_0|}{\left[\beta_0 - \beta_n + |b_n| + |\gamma_0| + |\gamma_n| + 2\sum_{i=1}^{n-1}|\gamma_i|\right]}$$

and
$$R_2 = \frac{\left(|b_0| + \beta_0 - \beta_n + |\gamma_n| + +2\sum_{i=1}^{n-1}|\gamma_i|\right)}{|b_n|}$$

Corollary 1.5 Let $\gamma_0 \le \gamma_1 \le \ldots \le \gamma_n$ then all its zeros of $f(t)$ lie in $R_1 \le |t| \le R_2$ where

$$R_1 = \frac{|b_0|}{\left[\gamma_n - \gamma_0 + |b_n| + |\beta_0| + |\beta_n| + 2\sum_{i=1}^{n-1}|\beta_i|\right]}$$

And
$$R_2 = \frac{\left[\gamma_n - \gamma_0 + |b_0| + |\beta_0| + |\beta_n| + 2\sum_{i=1}^{n-1}|\beta_i|\right]}{|b_n|}$$

Corollary 1.6 Let $\gamma_0 \ge \gamma_1 \ge \ldots \ge \gamma_n$ then all the zeros of $f(t)$ lie in $R_1 \le |t| \le R_2$

$$R_1 = \frac{|b_0|}{\left[\gamma_0 - \gamma_n + |b_n| + |\beta_0| + |\beta_n| + 2\sum_{i=1}^{n-1}\beta_i\right]}$$

and
$$R_2 = \frac{\left[\gamma_0 - \gamma_n + |b_0| + |\beta_0| + |\beta_n| + 2\sum_{i=1}^{n-1}\beta_i\right]}{|b_n|}$$

Proof of Theorem 2

Let the polynomial $F(t)=(j^{*}-t)\,f(t)=j^{*}b_{0}+\sum_{i=1}^{n}(j^{*}b_{i}-b_{i}-1)t^{j^{*}}-b_{n}j^{*n+1}=-b_{n}j^{*n+1}+G_{2}^{*}(t)$

We first note that

$$\left|b_{i-1}-j^{*}b_{i}\right|=\left|b_{i-1}-j^{*}\beta_{i}+i(\gamma_{i-1}-j^{*}\gamma_{i}\right| \qquad (3)$$

Then

$$\left|t^{n}G_{2}^{*}\left(\frac{1}{t}\right)\right|=\left|j^{*}b_{0}t^{n}+\sum_{i=1}^{n}\left(j^{*}b_{i}-b_{i-1}\right)t^{n-j^{*}}\right| \text{ and on } |t|=j^{*} \text{ by (3)}$$

$$\left|t^{n}G_{2}^{*}\left(\frac{1}{t}\right)\right|\leq\left|j^{*}b_{0}\right|j^{*n}+\sum_{i=1}^{n}\left|j^{*}a_{i}-a_{i-1}\right|j^{*n-i}$$

$$\leq\left|b_{0}\right|j^{*n+1}+\sum_{j=1}^{n}\left|j^{*}\beta_{j}-\beta_{j-1}\right|j^{*n-j}+\sum_{j=1}^{n}\left(\left|\gamma_{j-1}\right|+j^{*}\left|\gamma_{j}\right|\right)j^{*n-j}$$

$$=\left|b_{0}\right|j^{*n+1}+\sum_{i=1}^{k}\left(j^{*}\beta_{i}-\beta_{i-1}\right)j^{*n-i}+\sum_{i=k+1}^{n}\left(\beta_{i-1}-j^{*}\beta_{i}\right)j^{*n-i}+\sum_{i=1}^{n}\left(\left|\gamma_{i-1}\right|+j^{*}\left|\gamma_{i}\right|\right)j^{*n-i}$$

$$=\left|b_{0}\right|j^{*n+1}+(j^{*2}+1)\,j^{*n-k-1}\beta_{k}-j^{*n-1}\beta_{0}-j^{*}\beta_{n}+(j^{*2}-1)\sum_{i=1}^{k-1}j^{*n-i-1}\beta_{i}$$

$$+(1-j^{*2})\sum_{i=k+1}^{n-1}j^{*n-i-1}\beta_{i}+\sum_{i=1}^{n}\left(\left|\beta_{i-1}\right|+j^{*}\left|\gamma_{i}\right|\right)j^{*n-i}$$

Hence, by the Maximum Modulus Principal

$$\left|t^{n}G_{2}^{*}\left(\frac{1}{t}\right)\right|\leq M_{2}\quad for\ \ |t|\leq i$$

Which implies

$$\left|G_{2}^{*}\left(\frac{1}{t}\right)\right|\leq M_{2}|t|^{n}\quad for\ \ |t|\geq\frac{1}{j^{*}}$$

This follows

$$\left|F(t)\right|=\left|-b_{n}t^{n+1}+G_{2}^{*}(t)\right|$$

$$\geq\left|b_{n}\right|\left|t\right|^{n+1}-M_{2}|t|^{n}=|t|^{n}\left(\left|b_{n}\right|t\right|-M_{2}\left|\right)\qquad for\ \ |t|\geq\frac{1}{j^{*}}\ .$$

So if $|t|>\max\left[\frac{M_{2}}{|b_{n}|},\frac{1}{j^{*}}\right]=R_{2},\ \ then\ \ F(t)\neq 0$ and in turn $f(t)\neq 0$, thus establishing the outer radii for the theorem.

For the inner bound, Let us consider

$$F(t)=(j^{*}-t)\,f(t)=j^{*}b_{0}+\sum_{i=1}^{n}(j^{*}b_{i}-b_{i-1})t^{i^{*}}-b_{n}t^{n+1}=j^{*}b_{0}+G_{1}(t)$$

Then for $|t|=j^{*}$ by (2)

$$\left|G_{1}(t)\right|\leq\sum_{i=1}^{n}\left|b_{i-1}-j^{*}b_{i}\right|j^{*i}+\left|b_{n}\right|j^{*n+1}$$

$$=\sum_{i=1}^{n}\left|\beta_{i-1}-j^{*}\beta_{i}\right|j^{*i}+\sum_{i=1}^{n}\left(\left|\gamma_{i-1}\right|+j^{*}\left|\gamma_{i}\right|\right)j^{*i}+\left|b_{n}\right|j^{*n+1}$$

$= -j^*\beta_0 + 2j^{*k+1}\beta_k - j^{*n+1}\beta_n + |b_n| j^{*n+1} + |\gamma_0| j^* + |\gamma_n| j^{*n+1} + 2\sum_{i=-1}^{n-1} |\gamma_i| j^{*i+1} = M_1$ Applying Schwartz's Leema [2] to $G_1(t)$, we get

$$G_1(t) \le \frac{M_1(t)}{j^*} \quad for \quad |t| \le j^*$$

So

$$|F(t)| = \left| j^* b_0 + G_1(t) \right| \ge j^* |b_0| - |G_1(t)| \ge j^* |b_0| - \frac{M_1(t)}{j^*}$$

$\frac{j^*|b_0|}{M_1} \le j^*$. So if $|t| < \frac{j^{*2}|b_0|}{M_1} = R_1$ then $F(t) \ne 0$ and in turn $f(t) \ne 0$.

References

[1]. A.Joyal,G.Labelle, and Q.I.Rahman, On the location of zeros of polynomials, Canadian Math. Bull.,10(1967),55-63.
[2]. E.Titchmarsh, The Theory of Functions, Oxford Univeristy Pree (Oxford, 1932).
[3]. G.Enestrom, Harledning af en allamn formel for antalet pensionarer…, Ofv. Af. Kungl. Vetenskaps-Akademiens Forhandlingen, No.6 (Stockholm,1893)
[4]. K.K.Dewan and N.K Govil, On the Enestorm-kakeya Theorem, J.Approx. Theory,42 (1984),239-244.
[5]. M.Marden, Gepmetry of polynomials, Amer. Math. Soc. Math. Surveya 3 (1996).
[6]. M.Kovacevic and I.Milovanovic, On a generalization of the Enestrom-kakeya Theorem, Pure Math. And Applic.,Ser.A,3(1992),43-47.
[7]. N.Anderson,E.B.saff and R.S varga, An extension of the Enestrom-kakeya theorem and its sharpnsess, SIAM J.Math. Anal.,12(1981),10-22
[8]. N.Anderson, E.B.Saff,and R.S varga, On the Enestrom-kakeya theorem and its sharpness, Linerar Algebra and its Application,2891979),5-16.
[9]. R.Gardner and N.K.Govil, On the Enestrom-Kakeya theorem, J.Approx. Theory, 42(1984),239-244.
[10]. S.kakeya, On the limits of the roots of an algebraic equation with positive coefficients, Tohoku Math.J., 2(1912-13),140-142.

Jahangeer Habibullah Ganai. " Generalization of Enestrom Kakeya Theorem." IOSR Journal of Mathematics (IOSR-JM) 15.3 (2019): 45-50.