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Conformallyberwald Finsler Space Withspecial (α, β)- Metric

Gayathri. K[1]

1.Assistant Professor, REVA University , Kattigenahalli, Bengaluru-560064

Abstract: *In this paper, we find the necessary and sufficient conditions for a Finsler space with the metric $L = \alpha - \frac{\beta^2}{\alpha}$ to be a Berwald space and also to be a Berwald space, where α is a Riemannian metric and β is a differential one-form. Further, we study the conformal change of Berwald space with the above mentioned special $(\alpha, \beta) - metric$.*

Keywords: *FinslerSpace, Berwald Space, Conformal change.*
AMS Subject Classification (2010): *53A20, 53B40, 53C60, 58J60.*

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I. Introduction

The conformal theory of Finsler spaces has been introduced by M. S. Knebelman in 1929 and later on M. Hashiguchi developed such theory which was based on Matsumoto's approach to Finsler geometry. The conformal theory of two-dimensional Finsler space has been studied by M. Matsumoto and based on the above work, B. N. Prasad and D. K. Diwedi discussed the theory of conformal change of three-dimensional Finsler space.

Nabil L. Youssef, S. H. Abed and A. Soleiman investigated intrinsically conformal changes in Finsler geometry. Also they studied conformal change of Barthel connection and its curvature tensor, the conformal changes of Cartan and Berwald connections as well as their curvature tensors. Shun-ichi Hojo, M. Matsumoto and K. Okubo discussed the theory of conformally Berwald Finsler spaces and its applications to $(\alpha, \beta) -$ metrics.

The notion of Douglas space has been introduced by M. Matsumoto and S. Bacso as a generalization of Berwald space from the view point of geodesic equations. It is remarkable that a Finsler space is a Douglas space or is of Douglas type if and only if the Douglas tensor vanishes identically.

M. Matsumoto studied on Finsler spaces with $(\alpha, \beta) -$ metric of Douglas type. Hong-Suh Park and Eun-Seo Choi explained Finsler spaces with an approximate Matsumoto metric of Douglas type. The authors S. Bacso and I. papp studied on a generalized Douglas space. Also the team of authors Benling Li, Yibing Shen and Zhongmin Shen studied on a class of Douglas metrics.

The first part of the present paper is devoted to study the condition for the Finsler space with special $(\alpha, \beta) -$ metric to be a Berwald type (Theorem 3.1). The second part is devoted to find the condition for the Finsler space with special $(\alpha, \beta) -$ metric to be Berwald space (Theorem 4.1) and finally, we apply the conformal change of Finsler space with the special $(\alpha, \beta) -$ metric of Douglas type (Theorem 5.1). We study conformal change of Berwald space with special $(\alpha, \beta) -$ metric $L = \alpha - \frac{\beta^2}{\alpha}$ and also we obtain the conditions for Finsler space with an $(\alpha, \beta) -$ metric $L^2 = 2\alpha\beta$ to be conformally Berwald.

II. Preliminaries

To find the condition for the Finsler space with special $(\alpha, \beta) -$ metric to be Berwald space (Theorem 4.1) and finally, we apply the conformal change of Finsler space with the special $(\alpha, \beta) -$ metric of Douglas type (Theorem 5.1). We study conformal change of Berwald space with special $(\alpha, \beta) -$ metric $L = \alpha - \frac{\beta^2}{\alpha}$ and also we obtain the conditions for Finsler space with an $(\alpha, \beta) -$ metric $L^2 = 2\alpha\beta$ to be conformally Berwald.
The geodesics of a Finsler space $F^n = (M^n, L)$ are given by the differential equations:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

Where $2G^i(x, y) = \gamma_j{}^i{}_k(x, y)y^j y^k$ *and* $\gamma_j{}^i{}_k(x, y)$ are Christoffel symbols constructed from $g_{ij}(x, y)$ with respect to x^i.
A Finsler space F^n is said to be of Douglas type if

$$D^{ij} \equiv G^i(x, y)y^j - G^j(x, y)y^i \tag{2.1}$$

are homogeneous polynomials in (y^i) of degree three.

Let $hp(r)$ denote the homogeneous polynomial in y^i of degree r.
We use the following definition in future.
Definition 2.1: *The Finsler space F^n is of Douglas type if and only if the Douglas tensor*

$$D_{i\ jk}^{h} = C_{i\ jk}^{h} - \frac{1}{n+1}(G_{ijk}\, y^h + G_{ij}\,\delta_k^h + G_{jk}\,\delta_i^h + G_{ki}\,\delta_j^h)$$

Vanishes identically, where $G_{i\ jk}^{h} = \dot{\partial}_k G_{i\ j}^{h}$ is $hv-curvature$*tensor of the Berwald connection.*

The covariant differentiation with respect to the Levi-Cavita connection $\{_j{}^i{}_k\}(x)$ of R^n is denoted by (|).
We use the symbols as follows:

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i})$$
$$s_j = b_r s^{rj}, \qquad s_j^i = a^{ir} s_{rj}$$

The functions $G^i(x,y)$ of F^n with $(\alpha,\beta)-$ metric are written in the form

$$2G^i = \left\{{}_0{}^i{}_0\right\} + 2B^i,$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha} s_0^i + C^*\left[\frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i\right)\right], \tag{2.2}$$

Where $L_\alpha = \partial L/\partial\alpha, L_\beta = \partial L/\partial\beta, L_{\alpha\alpha} = \partial L/\partial\alpha\partial\alpha,$ the subscript 0 means contraction by y^i and we put $C^* = \frac{\alpha\beta\,(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})},$
Where $\gamma^2 = b^2\alpha^2 - \beta^2, b^i = a^{ij}\, b_j$ *and* $b^2 = a^{ij}\, b_i b_j.$
Since $\left\{{}_0{}^i{}_0\right\}(x)$ are $hp(2), F^n$ with $(\alpha,\beta)-$ metric is a Douglas space if and only if $B^{ij} \equiv B^i y^j - B^j y^i$ are $hp(3)$. From (1.2.1) and (1.2.2), we have

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^i y^j - b^j y^i) \tag{2.3}$$

We use the following lemma.
Lemma 2.1*If $\alpha^2 \equiv 0(mod\beta)$, that is, $a_{ij}(x)y^i y^j$ contains $b_i y^i$ as a factor, then the dimension n is equal to 2 and b^2 vanishes. In this case, we have 1-form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

III. Berwald Space With Special Metric $L = \alpha - \frac{\beta^2}{\alpha}$

In this section, we find the condition for a Finsler space F^n with the metric $L = \alpha - \frac{\beta^2}{\alpha}$ to be a Berwald space.
For the special $(\alpha,\beta)-$ metric $L = \alpha - \frac{\beta^2}{\alpha}$, the above equation is written as

$$(\alpha^2+\beta^2)B_{jki}\, y^j y^k + \alpha^2(\alpha - 2\beta)(B_{jki}\, b^k - b_{j|i})y^j = 0 \tag{3.1}$$

Where $B_{jki} = a_{kr} B_{j\ i}^{r}$.

We suppose that F^n is a Berwald space. Then $B_{j\ k}^{i}$ and $b_{i|j}$ are functions of position alone. Therefore, we separate (3.1) as the rational and irrational terms in y^i as:

$$(\alpha^2+\beta^2)B_{jki}\, y^j y^k - 2\alpha^2\beta(B_{jki}\, b^k - b_{j|i})y^j + \alpha\{\alpha^2(B_{jki}\, b^k - b_{j|i})y^j\} = 0,$$

Which yields two equations

$$(\alpha^2+\beta^2)B_{jki}\, y^j y^k - 2\alpha^2\beta(B_{jki}\, b^k - b_{j|i})y^j = 0, \tag{3.2}$$
$$\alpha^2(B_{jki}\, b^k - b_{j|i})y^j = 0. \tag{3.3}$$

Substituting (3.3) in (3.2), we have

$$B_{jki}\, y^j y^k = 0$$

and hence $B_{jki} + B_{kji} = 0$. Since B_{jki} is symmetric in (j,i), we get $B_{jki} = 0$ and from (3.3), we have $b_{j|i} = 0$.

Conversely, if $b_{j|i} = 0$, then $B_{jki} = 0$ are uniquely determined from (3.1).

Hence, we conclude the following

Theorem 3.1: *A Finsler space with a special $(\alpha,\beta)-metric$ $L = \alpha - \frac{\beta^2}{\alpha}$ is Berwald if and only if* $b_{j|i} = 0$.

IV. Conformally Berwald Finsler Space

In this section, we find the conditions for Finsler space with an $(\alpha,\beta)-$ metric $L^2 = c\alpha\beta$ to be conformally Berwald.
Definition 4.1: *A Finsler space $F^n = (M^n, L)$ is called conformally Berwald if there exists a conformal change $L \to \bar{L} = e^{\sigma(x)}L$ such that the changed space $\bar{F}^n = (M^n, \bar{L})$is a Berwald space, that is, $G_{j\ k}^{i}$are functions of position alone.*

V. Conclusion

M. Matsumoto and S. Bacso introduced the notion of Douglas space as a generalization of Berwald space from the view point of geodesic equations. Douglas metrics can be viewed as generalized Berwald metrics. The study on Douglas metrics will enhance our understanding on the geometrical meaning of non-Riemannian quantities. In this paper, we found the conditions for a Finsler space with special $(\alpha, \beta)-$ metric $L = \alpha - \frac{\beta^2}{\alpha}$ to be a Douglas space and also to be a Berwald space. Further, we found the conditions for a conformally transformed Douglas space with the above mentioned special $(\alpha, \beta)-$metric to be a Douglas space.

References

[1]. Benling Li, Yibing Shen and Zhongmin Shen, On a class of Douglas metrics, Comm. Korean Math. Soc., 14(3)(1999), 535-544. B.N Prasad. B.N Gupta and D.D. Singh, Conformal transformation in Finsler spaces with $(\alpha, \beta) -$ metric, Indian J. Pure and Appl. Math., 18(4)(1961), 290-301.

[2]. Hong-Suh Park and Eun-Seo Choi, Finsler spaces with an approximate Matsumoto metric of Douglas type, Comm. Korean Math. Soc., 14(3)(1999), 535-544.

[3]. M. Hashiguchi, S. Hojo and Matsumoto, On Landsberg spaces of dimension two with $(\alpha, \beta) -$ metric,Tensor, N.S., 57(1996), 145-153.

[4]. M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, Kaiseisha press, Otsu, Saikawa, Japan, (1986).

[5]. M. Matsumoto, The Berwald connection of a Finsler space with an $(\alpha, \beta) -$ metric, Tensor, N.S., 50(1991),18-21.

[6]. M. Matsumoto, On Finsler spaces with $(\alpha, \beta) -$ metric of Douglas type, Tensor, N.S. 60(1998).

[7]. Pradeep Kumar, S.K. Narasimhamurthy, H.G. Nagaraja, S.T. Aveesh, On a special hypersurface of a Finsler space with $(\alpha, \beta) -$ metric, Tbilisi Mathematical Journal, 2(2009), 51-60.

[8]. P.L. Antonelli, R.S. Ingarden and M. Matsumoto, The theory of sprays and Finsler spaces with applications in Physics and Biology, Kluwer academic publishers, Netherlands (1993).

[9]. S. Bacso, X. Cheng and Z. Shen, Curvature properties of $(\alpha, \beta) -$ metrics, Advanced studies in Pure Mathematics, Math. Soc. Of Japan, 48(2007), 73-110.

[10]. S. Bacso and M. Matsumoto, On Finsler spaces of Douglas type. A generalization of the notion of Berawld spae, Publ. Math. Debrecen, 51(1997), 385-406.

[11]. S. Bacso and I. Papp, On Landsberg spaces of dimension two with $(\alpha, \beta) -$ metric, Periodica Mathematica Hungarica, 48(1-2)(2004), 181-184.

[12]. S.K. Narasimhamurthy and D.M. Vasantha, Projective change between two Finsler spaces with $(\alpha, \beta) -$ metric, Kyunpook Mathematical Journal.

[13]. S.K. Narasimhamurthy and G.N. Latha Kumari, On a hypersurface of a special Finsler space with a metric $L = \alpha + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2} + \beta$, ADJM, 9(1)(2010), 36-44.

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