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Mathematical Model of Nutrition Transport in Cartilage

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Abstract: *Present herein are the analytical studies of nutrition transport in articular cartilage. These studies have enabled the researchers to analyzes the lubrication mechanism of joints, amount of nutrition being transported to the bone and thestructural behaviour of articular cartilage. The variation of concentration in the cartilage for various values of parameters . it has been observed that the concentration of synovial fluid increases when the gap increases between the bone surfaces.Again the concentration of articular cartilage decreases when the gap between the bone increases .*

Keywords: *Cartilage, Nutrition Transport, Synovial fluid , concentration .*

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I. Introduction

Scientists have been greatly attracted to the physical problems arising in mechanism of the body functioning and are trying to analyze analytically as well as experimentally, by applying classical Newtonian mechanism and various other problems arising in the body system. Body movement is totally dependent on the movability of the articulating joints because these are the functional connections between the different bones of Skelton.

Synovial fluid of joints normally functions as a biological lubricant providing low-friction and low-wear properties to articulating cartilage surface. These lubricant are secreted by chondrocytes in the articular cartilage and synoviocytes in synovium and concentrated in the synovial space by the semi-permeable synovial lining. Molecules postulated to play a key role, alone in lubrication are proteology. A variety of treatment has been developed to alter biologically the synovial joint environment in injury and arthritis.

The man component of synovial fluid is an ultrafiltrate of the blood plasma devoid of high molecular proteins, blood cells. While the most important component is actively added lubricant called hyaluronic acid.

Cartilage clearly performs a mechanical function. It provided a bearing surface with low friction and wear, it helps to distribute the loads between opposing bones in a synovial joint. If the cartilage were a stiff material like bone the contact stresses at a joint would be much higher. These mechanical functions alone would probably not be sufficient to justify an in-depth study of cartilage biomechanics. The apparent link between osteoarthrosis and mechanical factors in a joint adds a strong impetus for studying the mechanical behaviour of articular cartilage.

Articular cartilage is a living material composed of a relatively small number of cells known as chondrocytes surrounded by a multicomponent matrit. Mechanically articuler cartilage is a composite of materials with widely differing properties. Approximately 70% and 85% of the weight of the whole tissues is water. The remainder of the tissue is composed primarily of proteoglycan and collagen. Approximately 30% of the dry weight of articular cartilage is composed of proteoglycans. Proteoglycan concentration and water contant vary through the depth of the tissue.

Lai and Mow [1] studied similar problem considering the concentration of the synovial fluid and articular cartilage. They solved the concentration in the synovial fluid region and use the interphase boundary condition to evaluate the concentration in comparison to the fluid thickness in the cartilage region. They also considered the cartilageous thickness infinitely large in the fluid in cartilage region and simultaneously considered the cartilageous thickness large. Study of formation of H.A.P. gelon of the cartilage surface was developed by **Lai and Mow** [1]. **Mow et al and Mansur**[2] provided result for cartilage deformation produced by compressive stress. **Walker et al**[3] demonstrated the concentration of molecular weight constituent of synovial fluid increases due to filtration action of suspended medium . **Walker et al** [4] observed the similar result on frictional experiment. **Yadav A.K &Kumar S**.[5] studies the lubrication mechanism of knee joints.
In this research paper we have found the concentration equation of the fluid in cartilage region.

II. Formulation Of The Problem

The model being discussed in shown in figure . The cartilage surface is taken as the interphase and y co-ordinate is measured positively in the fluid region. Let $C_2(Y,t)$ be the concentration of solute of the synovial fluid at any time.

The governing equation in the cartilage is;

$$\frac{\partial C_1}{\partial t} - V\frac{\partial C_1}{\partial Y} = k\frac{\partial^2 C_1}{\partial Y^2} \quad (1)$$

and

$$\frac{\partial C_2}{\partial t} - V\frac{\partial C_2}{\partial Y} = \bar{k}\frac{\partial^2 C_2}{\partial Y^2} \quad (2)$$

Where v is downward squeezing speed and $k, \bar{k}$ be the diffusivities of the rotate in synovial fluid and cartilage respectively.

The boundary conditions of the fluid in the cartilageous regions are given by;

$$\frac{\partial C_1}{\partial Y} = 0 \text{ at } Y = H \text{ when } H = H_0 - vt \quad (3)$$

$$\left.\begin{aligned} & C_1(0,t) = \lambda C_2(0,t) \\ & k\frac{\partial C_1}{\partial Y} = k\frac{\partial C_2}{\partial Y} - vNC_1 \qquad at\ Y = 0 \end{aligned}\right] \quad (4)$$

$$C_1(Y,0) = C_0 \quad \text{and} \quad C_2(Y,0) = \frac{C_0}{\lambda} \quad (5)$$

λ is partition constant and $N = 1 - \frac{1}{\lambda}$

We introduce the following non-dimensional quantities;

$$\left.\begin{aligned} & C_1 = C_0 C_1^*, && C_2 = C_0 C_2^*, \\ & Y = Y^0 H_0, && t = \frac{H_0^2 t}{k}, \\ & P_c = \frac{UH_0}{k}, && \frac{\bar{k}}{k} = \varepsilon \end{aligned}\right] \quad (6)$$

The equation (1) and (2) reduce as

$$\frac{\partial C_1}{\partial t^*} - P_c\frac{\partial C_1^*}{\partial Y^*} = \frac{\partial^2 C_1^*}{\partial Y^{*2}} \quad (7)$$

$$\frac{\partial C_2}{\partial t^*} - P_c\frac{\partial C_2^*}{\partial Y^*} = \frac{\partial^2 C_2^*}{\partial Y^{*2}} \quad (8)$$

The trans formed boundary conditions are;

$$\frac{\partial C_1^*}{\partial Y^*} = 0 \text{ at } Y^* - H^* = 1 - P_c t^* \quad (9)$$

$$\left.\begin{aligned} & C_1^*(0,t^*) = \lambda C_2^*(0,t*) \\ & \frac{\partial C_1}{\partial Y^*} = \in \frac{\partial C_2}{\partial Y^*} - NP_c C_1 \end{aligned}\right] \ at\ Y^* = 0 \quad (10)$$

Where

$$N = 1 - \frac{1}{\lambda}$$

$$C_1^*\left(Y^0,0\right)=1, C_2^*\left(Y,^*0\right)=\frac{1}{\lambda} \quad (11)$$

SOLUTION OF THE PROBLEM

C_1^* To solve the above equation we first decouple C_1 and C_2 we solve C_1^* in equation (7) independently with the following conditions

$$\frac{\partial C_1}{\partial Y^*}=0 \qquad at\ Y=H^* \qquad (12)$$

$$\frac{\partial C_1}{\partial Y^*}=-NP_cC_1 \qquad at\ Y=0 \quad (13)$$

and

$$C_1^*\left(Y,0\right)=C_0 \qquad And\ C_1^*=1,\ at\ t^*=0 \quad (14)$$

Using Laplace transform method and dropping *, we have

$$\frac{d^2\bar{C}_1}{dY^2}=\in\frac{d\bar{C}_2}{\partial Y}-sC_1=1 \qquad (15)$$

The transformed boundary conditions are:

$$\frac{d\bar{C}_1}{dY}=0 \qquad at\ \ Y=H^* \qquad (16)$$

$$\frac{d\bar{C}_1\left(0,s\right)}{dY}=-NP_c\bar{C}_1\left(0,s\right) \qquad at\ \ Y=0 \quad (17)$$

Solution of equation (15) is

$$\bar{C}_1\left(Y,s\right)=e^{-P_cY}\left[A\cosh\frac{\sqrt{P_c^2+us}}{2}Y+B\sinh\sqrt{P_C^2+us}\,Y\right]+\frac{1}{s} \qquad (18)$$

Using the boundary condition (16) and (17), we get

$$A=\frac{NP_c\left[P_c\tanh\frac{\sqrt{P_C^2+us}}{2}H-\sqrt{P_C^2+us}\,Y\right]}{s\left[-\left(NP_c^2+2s\right)\tanh\frac{\sqrt{P_C^2+us}}{2}H+2NP_c\sqrt{P_C^2+us}\right]} \qquad (19)$$

and

$$B=\frac{NP_c\left[\sqrt{\left(P_c^2+us\right)}\tanh\frac{\sqrt{P_C^2+us}}{2}H-P_C\right]}{s\left[-\left(NP_c^2+2s\right)\tanh\frac{\sqrt{P_C^2+us}}{2}H+2NP_c\sqrt{P_C^2+us}\right]} \qquad (20)$$

Now using the interphase condition

$$C_1\left(0,s\right)=\lambda C_2\left(0,s\right) \quad (21)$$

We have

$$C_1\left(0,s\right)=\frac{1}{\lambda}\frac{NP_c\left[P_c\tanh\frac{\sqrt{P_C^2+us}}{2}H-\sqrt{P_C^2+us}\right]}{s\left[-\left(NP_c^2+2s\right)\tanh\frac{\sqrt{P_C^2+us}}{2}H+2NP_c\sqrt{P_C^2+us}\right]} \qquad (22)$$

Solve equation (19), we get;

$$C_2(Y,s) = Ce^{\left(\frac{-P_c+\sqrt{\left(P_C^2+us\right)}}{2}\right)}Y + De^{\left(\frac{-P_c+\sqrt{\left(P_C^2+us\right)}}{2}\right)}Y - \frac{1}{\lambda s} \quad (23)$$

Now as Y→ ∞, the second term tend to zero hence vanish D, the solution is

$$\bar{C}_2(Y,s) = Ce^{\left(\frac{-P_c+\sqrt{\left(P_C^2+us\right)}}{2}\right)} - \frac{1}{\lambda s} \quad (24)$$

We first obtain the approximate value of $C_1(0,s)$ in equation (21) we get;

$$\bar{C}_1(0,s) = \frac{1}{\lambda s}\frac{\left[NP_c\left\{P_c\tanh\frac{\sqrt{\left(P_C^2+4s\right)}}{2}H\right\} - \sqrt{\left(P_C^2+4s\right)}\right]}{s\left[-\left(NP_c^2+2s\right)\tanh\frac{\sqrt{P_C^2+4s}}{2}H + 2NP_c\sqrt{P_C^2+4s}\right]} \quad (25)$$

Hence the solution $\bar{C}_1(Y,s)$ in (18) and $\bar{C}_2(Y,s)$ in (19) reduces to are given by;

$$\bar{C}_1(Y,s) = e^{-\frac{P_CY}{2}}\left[\frac{NP_C\left(-P_C+\frac{2}{H}\right)}{s\left\{NP_C\left(-P_C+\frac{2}{H}\right)+2s\right\}}\right]\cosh\frac{\sqrt{P_C^2+4s}}{2}Y +$$

$$e^{-\frac{P_CY}{2}}\left[\frac{NP_C^2\left(-P_C+\frac{2}{H}\right)+4s}{s\left\{NP_C\left(-P_C+\frac{2}{H}\right)+2s\right\}\sqrt{\left(P_C^2+4s\right)}}\right]\sinh\frac{\sqrt{P_C^2+4s}}{2}Y + \frac{1}{s} \quad (26)$$

and

$$\bar{C}_2(Y,s)e^{-\frac{P_CY}{2}}\frac{1}{\lambda}\left[\frac{NP_C\left(-P_C+\frac{2}{H}\right)}{s\left\{NP_C\left(-P_C+\frac{2}{H}\right)+2s\right\}}\right]e^{\left(\frac{\sqrt{\left(P_C^2+4s\right)}}{2}\right)} + \frac{1}{\lambda s} \quad (27)$$

Solution are valid for first expansion. We write the solution of above equation (26) and (27) as;

$$\bar{C}_1(Y,s) = e^{-\frac{P_CY}{2}}\left[\frac{A_{11}}{s\left\{A_{11}+2s\right\}}\right]\cosh\frac{\sqrt{P_C^2+4s}}{2}Y + e^{-\frac{P_CY}{2}}\left[\frac{P_CA_{11}+4s}{s\left\{A_{11}+2s\right\}}\right]\sinh\frac{\sqrt{P_C^2+4s}}{2}Y + \frac{1}{s} \quad (28)$$

$$\bar{C}_2(Y,s) = e^{-\frac{P_CY}{2}}\left[\frac{A_{11}}{s\left\{A_{11}+2s\right\}}\right]e^{\left(\frac{\sqrt{\left(P_C^2+4s\right)}}{2}\right)Y} + \frac{1}{\lambda s} \quad (29)$$

Where

$$A_{11} = NP_C\left(-P_C+\frac{2}{H}\right) \quad (30)$$

Taking Laplace inverse, we get;

$$C_1\left(Y,t\right)=1-\frac{\phi\left(0\right)}{\Psi_1'\left(0\right)}+\sum_{1}^{\infty}\frac{\phi\left(\mu_n\right)e^{-\frac{\left(\mu_n^2 t+P_C Y\right)}{2}}}{\Psi_1'\left(\mu_n\right)} \quad (31)$$

$$C_2\left(Y,t\right)=1-\frac{\phi\left(0\right)}{\Psi_1'\left(0\right)}+\sum_{1}^{\infty}\frac{\phi_a\left(\mu_n\right)e^{-\frac{\left(\mu_n^2 t+P_C Y\right)}{2}}}{\Psi_1'\left(\mu_n\right)} \quad (32)$$

Where

$$\phi\left(0\right)=\phi_1\left(0\right)\cosh\frac{P_C Y}{2}+\phi_2\left(0\right)\sinh\frac{P_C Y}{2} \quad (33)$$

$$\Psi_1\left(0\right)=2P_C\left(\lambda-1\right)\left\{P_C\sinh\frac{P_C h}{2}-P_C\cosh\frac{P_C h}{2}\right\}\left[P_C\cosh\frac{P_C H}{2\varepsilon}-P_C\sinh\frac{P_C H}{2\varepsilon}\right] (34)$$

$$\phi_a\left(0\right)=-2NP_C^3\left(\sinh\frac{P_C h}{2}-\cosh\frac{P_C h}{2}\right)\left(\cosh\frac{P_C H}{2}-\sinh\frac{P_C H}{2}\right)\left[\cosh\frac{P_C Y}{2\varepsilon}+\sinh\frac{P_C Y}{2\varepsilon}\right] \quad (35)$$

$$\phi\left(\mu_n\right)=2P_c\left(\lambda-1\right)\left\{P_C\sinh\frac{\mu_n h}{2}-\mu_n\cosh\frac{\mu_n h}{2}\right\}\left[\left(\mu_n\cosh\frac{\mu_n H}{2\varepsilon}+P_C\sinh\frac{\mu_n H}{2\varepsilon}\right)\cos\frac{\mu_n Y}{2\varepsilon}-\left(\mu_n\sin\frac{\mu_n h}{2}+P_C\cos\frac{\mu_n h}{2\varepsilon}\right)\left(\mu_n\cosh\frac{\mu_n H}{2\varepsilon}+P_C\cos\frac{\mu_n H}{2\varepsilon}\right)\sin\frac{\mu_n Y}{2}\right] \quad (36)$$

$$\phi_a\left(\mu_n\right)=2NP_c\left[\left\{P_C\sinh\frac{\mu_n h}{2}-\mu_n\cosh\frac{\mu_n h}{2}\right\}\left(\mu_n\cosh\frac{\mu_n H}{2\varepsilon}+P_C\sinh\frac{\mu_n H}{2\varepsilon}\right)\cos\frac{\mu_n Y}{2\varepsilon}-\left(P_C\sin\frac{\mu_n h}{2}-\mu_n\cos\frac{\mu_n h}{2}\right)\left(\mu_n\sinh\frac{\mu_n H}{2\varepsilon}+P_C\cos\frac{\mu_n H}{2\varepsilon}\right)\sin\frac{\mu_n Y}{2\varepsilon}\right] (37)$$

III. Results And Discussions

Previous section outlines the silent properties of the solution. Some numerical results are presented in the following table for complete discussion. The table describes the variation of concentration in the cartilage for various values of parameters. The values of different parameters are chosen so as to lie within the range of human joints. It is clear that the concentration of synovial fluid increases when the gap increases between the bone surfaces as the viscosity increases.

Further the value of concentration in articular cartilage increases when the gap between the bone surface increases as well as velocity increases.

Table (1)

VARIATION OF CONCENTRATION C_1 FOR DIFFERENENT VALUE OF

λ=.5 h=.4, H=.8, ε=.25, t=.2 P=1 ε=.25 N=1

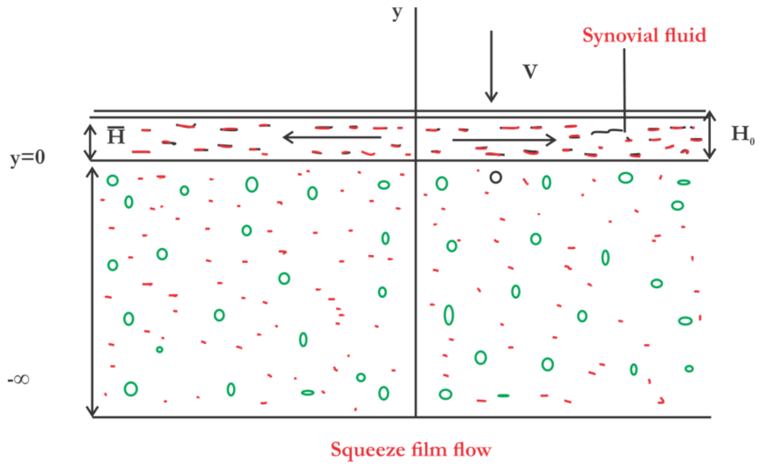
Y ↓	C_1				
	μ→.2	.4	.6	.8	1.0
1.1	.722754313	.763497975	.773750319	.838124280	.907672478
1.2	.789007150	.859163396	.869404595	.930708002	1.014788302
1.3	.887372210	.955391086	.964837715	1.022792273	1.086256678
1.4	.986512770	1.051196549	1.060261943	1.115443721	1.192035968
1.5	1.131644806	1.147211625	1.155915718	1.208543737	1.265975818
1.6	1.230645606	1.243689508	1.252036814	1.300052034	1.356822623

Table (2)

VARIATION OF CONCENTRATION C2 FOR DIFFERENENT VALUE OF

l=.5 h=.4, H=.8, e=.25, t=.2 P=1 e=.25 N=1

Y ↓	C_2				
	μ→.2	.4	.6	.8	1.0
1.1	14.43139549	14.43462069	14.43767813	14.44055371	14.44326978
1.2	17.00390071	17.00677205	17.00951870	17.01211162	17.01456891
1.3	20.17916386	20.18173111	20.18417817	20.18649511	20.18869644
1.4	24.08401919	24.08628659	24.08845203	24.09050720	24.09246393
1.5	28.87517798	28.87716962	28.87907468	28.88287437	28.88280632
1.6	32.74452346	32.74637724	32.74804448	32.75125491	32.75193892


y
Synovial fluid
V
$\overline{H}$
H_0
y=0
-∞
Squeeze film flow

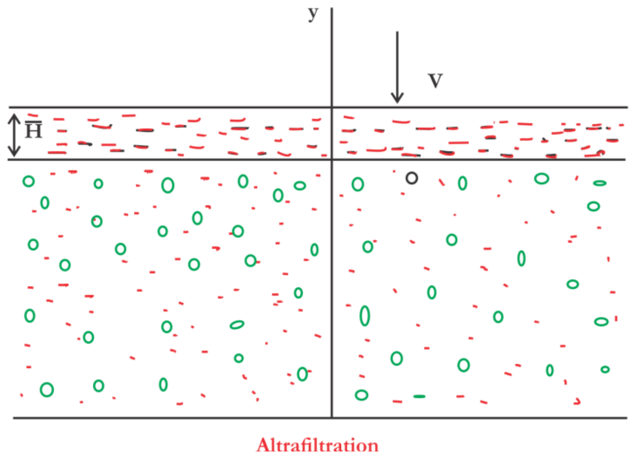

y
V
$\overline{H}$
Altrafiltration

FIG.-1

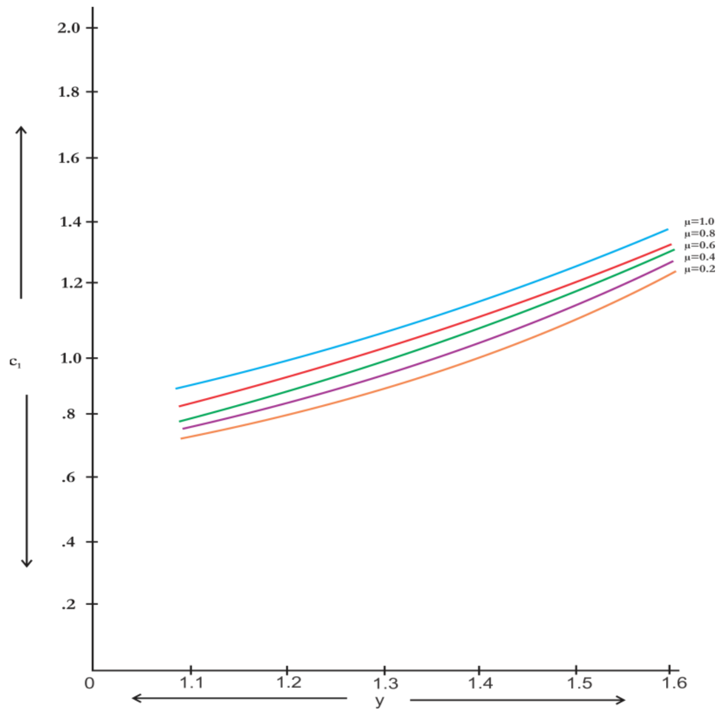

2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
c_1
μ=1.0
μ=0.8
μ=0.6
μ=0.4
μ=0.2
0
1.1
1.2
1.3
1.4
1.5
1.6
y

FIG.-2

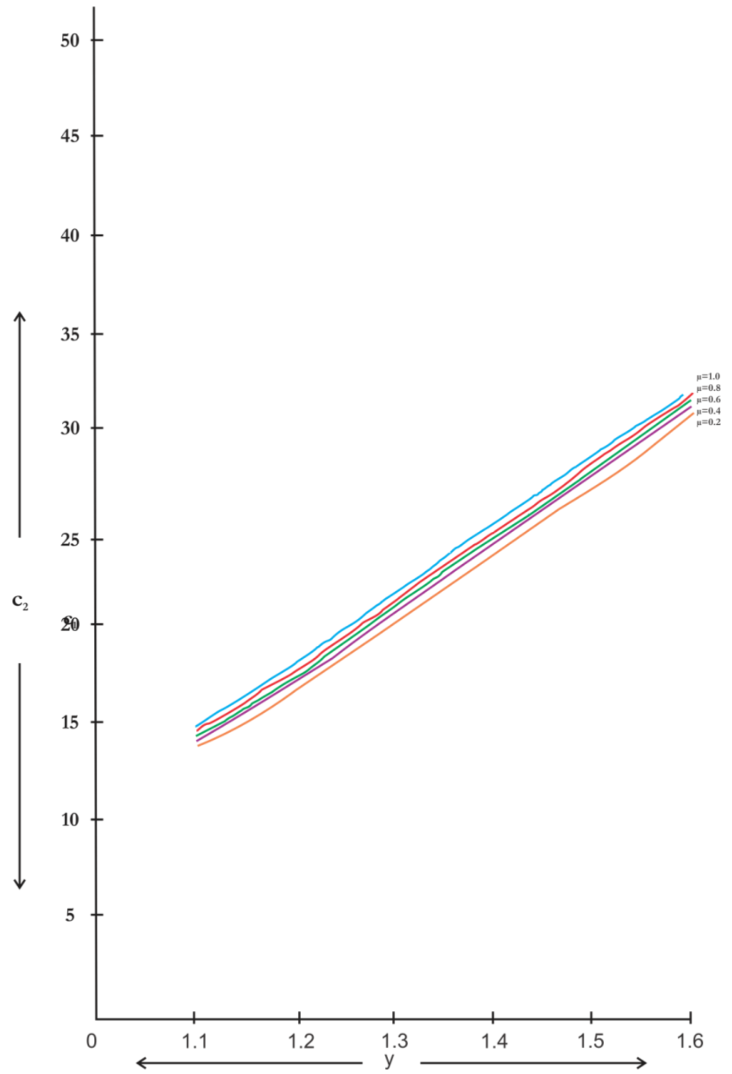


FIG.-3: Variatin of concentration of articular cartilage for different value of y and μ

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