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Cut Points and Punctured Points in Topological Manifold M

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Abstract: *The object in the paper is to introduce the concept of cut point and punctured space in the Topological spaces, like Topological Manifold M of dimension n , Tangent bundle of dimension – 2n , fiber bundle . Let* ∪ *be subset of M. We defined cut point* $p \in M$ *of order K , if* $f: \cup \rightarrow R^n$ *is a differential map of U into* R^n *and f is represented by n-co-ordinates functions* $f^1 ,\ldots\ldots\ldots, f^n : \cup \rightarrow R$ *(* $f(p)=f^1(p)\ldots\ldots f^n(p)$ *for each* $p \in \cup$*) all of which are differentials . The main object is to define these sequence of function* $\{f_n\}$ *are differentiable, also inverses. These inverse functions image, forms a open sets* $\cup_i \in M$ *forms a cover for M , which remove the cut point on M i.e. M becomes connected Topological Manifold.*
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I. Introduction:

Topological Manifold M of dimension n is c∞ Manifold. The sequences $\{f_n\}$ of functions are differentiable on each point P∈M.

An n-dimensional topological Manifold is a Topological space which is path connected (connected) if it is not the disjoint union of non empty open subsets. A subset U of a Topological space (X,T) is connected if every pair of points of subset U is path connected.
If space is not path connected across all subset U_i of (X,T) then it can be decomposed into components/cells.

Definition 1.1 [6]

Let x be a Topological space and x ∈ X. The component containing x, denoted C(x) is defined to be the union of all connected subsets of X that contain x. A subset of Topological space X is a (connected) component if it is C(x) for same x ∈ X.

If U is some open subset of a smooth Manifold M with atlas A_M then U is it self a differentiable Manifold with an atlas of charts being given by all the restrictions ($U_P \cap U$, $X_p / U_p \cap U$) where (U_p, X_p) ∈ A_M . we shall refer to such an open subset ∪ ⊂ M with this differentiable structure as an open sub manifold of M. Open structure as an open subset of R^n might seem to be very uninteresting Manifold but in fact they can be quite complex.

A function f:M→ R is C^k differentiable at P∈M if and only if it is continuous and $f \circ x^{-1} : x(U) \rightarrow R$ is C^K differentiable for some admissible chart (U, x) with P ∈ U, and f is of class C^k if it is of class C^k at every P. The set of all C^k maps M→N is denoted by C^k (M, N). The addition, scaling and multiplication are defined point-wise so that $(f + g)p=f_{(p)} +g_{(p)}$.

If f is a C^k function then $\partial f/\partial x^i$ is clearly C^{k-1}. Notice also that f recally only needs to be defined in a neighborhood of P and differentiable at P for the expression $\partial f/\partial x$ (p) to make sense.

E Kreyszig (Erwin)[1] introduce the convergence of a sequence limit as :-

Definition 1.2

A sequence $[x_n]$ in a metric space X =(x, d) is said to be converge or to be convergent if there is an x ∈ X such that

$$\lim_{n\rightarrow\infty} d(x_n, x)=0, \quad \text{x is called the limit of } \{x_n\}.$$

i.e. $\lim(n\rightarrow\infty) x_n \rightarrow x$

Definition 1.3, Cauchy sequence completeness:

A sequence $\{x_n\}$ in a metric space X=(x,d) is said to be Cauchy(or fundamental) if for every ϵ>0 there is an N=N(ϵ) such that $d(x_m,x_n)< \epsilon$ ∀m,n>N.
The space X is said to be complete if every Cauchy's sequence in X is converges.

Theorem 1.5 [1]
Every convergent sequence in a metric space is a Cauchy sequence.

We introduce the concept of cut points and punctured point [2][3][4][5] on Topological Manifold . In this paper we explained the main concept of cut points and punctured point with the help of Jacobian matrix .The sequences of functions $\{f_n\}$ converges to f which gives the connectedness property.

In section 2 we recalled some basic concept like definition, theorem, properties of metric space and Cauchy's Convergent Conditions, also we redefine the some required definition with as your requirement for research wok.

In section 3 contains the main research work. In this section we prove the connectivity's of Topology Manifold using the sequence of functions we prove that how to remove cut points and punctured points in a space M?. The Regular function and Jacobian matrix Relation, The rank of Jacobin matrix we defined in different ways.

Lastly, contains conclusion of this paper. i. e. Removing cut points and puncture point to make Topological Manifold stronger and stronger.

II. Some Basic Concepts (Definitions)

In this section We recalled some concepts on M ,

Definitions : 2.1

Let f(x) be continuous function of x at a is said to be differential coefficient of f(x) write x if

$\lim_{x \to a} \left(\frac{f(x+a)-f(x)}{x-a} \right)$ exists.

Let M be Topological Manifold of dimension n . Define a sequence {fn} of functions .which are continuously differentiable functions on U∈M.

Definition 2.2

Let the sequences {fn} of differential functions of order c^k fn : U →R^n , denote the set T - the set of differential functions then set T is a family if fn are converges to f for f ϵ T

i.e. $\lim_{n\to\infty} f_n = f$ Therefore, f_n, f ∈ T .

Definition 2.3

If family of differential functions of the order K then sequence $\{f_n\}$ is convergent sequence converges to f ϵ T .

Definition 2.4

If fj ϵ T then we call fj : U → R^n is regular if

(i) Each function fj is 1-1 (one,-one)

(ii) Rank of Jacobian matrix is 'n'

Definition :- 2.5

A point P ϵ M (P ϵ U ⊆ M) is called cut point if T is a set of functions fi : U →R^n

1. f_i are not differential in neighborhood of P ϵ U ⊂ M
2. There does not exists functions f_i on U, i < n ,
   If i=1, the cut point is one,
   i=2 there are two cut points,
   i<n there are i cut points in U.
3. The Rank of Jacobi matrix of Remaining function is n- i

OR we call a point P ϵ U ⊂ M is a cut point than all functions f_i are divergent

$\lim_{i\to\infty} f_i \neq f$.

Definition :- 2.6

The functions which are defined on cut points are called cut functions

Let P ϵ U ⊂ M be a cut point of M

The function f_i are defined on cut point of M than f_i are called cut functions.

Definition_: 2.7 weakly connected.

A Topological Manifold M is said to be weakly connected if there exists only one point P ϵ U.
Such that all paths joining between any two point x and y ϵ M , otherwise strongly connected .

III. Theorems on M

Let P ϵ U ⊂ M be a cut point M, then U_A and U_B are two components of M by the cut point P
As M (without cut point) is connected Topological Manifold of dimension n.4 Then function f:U →R^n is C^K differentiable similarly The family of open subset U_i ⊂ M of M , the function fi are defined .
Consider a subset V of M which is open of P ϵ U ⊂ M . As V is subset of M which is open neighborhood of P ϵ V ⊂ U ⊂ M , that V is path connected .
Let any two point x and y ϵ V ⊂ M
(P-h =x, P + h= y) there exists a path γ : [0,1] → V such that γ (0) =x , γ(1) =y

Where the function f_i are not differentiable.

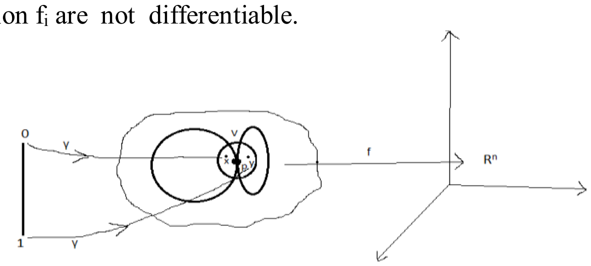

fig.1

Definition:- 3.1

A Topological space U ⊆ M is locally connected if every P ϵ U has a connected neighborhood. A space U is a locally path connected if every P ϵ U has a path connected neighborhood.

Definition:-3.2

A Topological space M is said to be path connected space if
(i) M contains cut point P
(ii) Sequence $\{f_i\}$ sequence of functions defined in neighborhood of cut point P ϵ U ⊂ M
(iii) The family (set) of neighborhood V_i of cut point are path connected.
(iv) The set $U_i \subseteq$ M are locally path connected.
(v) Each path from U_i to U_j is passing through cut point P ϵ U

<u>Examples</u> 3.3

Any open subset $U \subseteq R^n$ is locally path connected for each P ϵ U, there is an open ball Br(p) ⊆ U and this ball is a path connected neighborhood of P .

The space {U} ∪{1/n ϵ R/n ϵ N } is not locally connected or locally path connected.

<u>Theorem</u> 3.4 [7]

Every Manifold is locally path connected.

<u>Theorem</u> 3.5 [8]

Let U be a open subset of R^n. if U is connected then U is path connected.

<u>Theorem</u> 3.6

Let M be Topological Manifold of dimension n. if M contains at least one cut point ct(p) ≥ 1 then M is weakly connected .

<u>Proof</u> :

Let M be Topological Manifold of dimension n.

Let A point P ϵ U ⊂ M is cut point of M

The ∃ is a set of cut point of M and some function f are not differential in neighborhood of P ϵ U ⊂ M and there does not exists function f_j on U, j<I

if j=1 is the cut point of M

Therefore say 'P'

Therefore As P is cut point, M-p becomes disconnect

Therefore M-P has two components M_1 and M_2 satisfies the condition that

$M=M_1 \cup M_2 \cup P$

$M_1 \cap M_2 = P$ or $M_1 \cap M_2 \cap \{P\} = \{P\}$.

Therefore we have M is connected Topological Manifold

Therefore there exists a path between any two points in M if p is only point connects the components of M ($M_1$and $M_2 \subseteq M$)

Therefore As M_1 ,$M_2 \subseteq M$ and M is connected, there exists a path $\gamma:[0,1]\to V$, where V is nbd of point P ϵ ∪ ⊂ M, defined as $\gamma(0)=x$ $\gamma(1)=y$ for all x and y ϵ V ⊂ ∪ ⊂ M, which passing throughout the Point P.

There does not exists a path in M other than γ which joining point and x ϵ M, and y_2 ϵ M_2.

Therefore V is connected neighborhood in M and M_1 M_2 are connected containing some point of V.

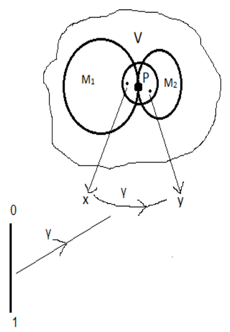

fig.2

Therefore, V is locally connected by path γ.
Therefore path γ is bridge between M_1 and M_2 passing through P □ ∪ ⊂ M.
Therefore these two sub manifold M_1 and M_2 are connected by bridge path, which is unique.
Therefore M is called weakly connected Manifold. As we have each function f_j: ∪ → R^n are differentiable on ∪ with the Jacobian matrix is defined as.
$\partial(f_i.....f_i)$ / $\partial(x_i.....x_i)$ whose determinant is zero
Therefore, Rank is n-1
In a Jacobian matrix
Any function f_i is not defined on any cut points p_j then the Jocobian row j contains all entry is 1
The Jacobian Determinan value is zero
Jacobian matrix for function f_i with respect to parameter x_i is

$$J_{ct}M=\begin{bmatrix} \frac{\partial f1}{\partial x1} & & \frac{\partial f2}{\partial x1} & \cdots & \frac{\partial fi}{\partial x1} \\ \frac{\partial f1}{\partial x2} & \vdots & \frac{\partial f2}{\partial x2} & \cdots & \frac{\partial fi}{\partial x2} \\ \frac{\partial f1}{\partial xi} & & \frac{\partial f2}{\partial xi} & \cdots & \frac{\partial fi}{\partial xi} \end{bmatrix}$$

Each entry $\partial f_i / \partial x_j = \delta_{ij}$, When δ_{ij} = { 0 if each f_i f^n is defined at x_j, $x_i \neq x_j$
{1, if each f_i f is defined at x_i

Suppose x_i is cut point M.
The corresponding function f_i is defined on M at x_i $\partial f_i/\partial x_i = 0$

$$J_{ctp}M=\begin{bmatrix} 1 & 0\cdots & 0 \\ \vdots 0 & 1 \ddots & 0 \vdots \\ 0 & 0\cdots & 0 \end{bmatrix}$$

Therefore, i^{th} row of matrix, all entries are 0.
The rows contains at least one entry is unit.
Therefore, rank of $J_{ct}M$ is i-1 for i =1,2,...,n
Therefore, rank of $J_{ct}M$ is n-1 .

Remark

1) If i = 1,2,3,…,n. and the Manifold M contains some m cut points (m < n) then the rank of Jacobian matrix in n-m.
2) If x_i = 1,2,3,…,m. be cut points of M then Jacobian matrix contain m-rows whose entries are 0.

Theorem 3.7

If p ∈ U ⊆ M is a cut point and function f_i are invertible functions which are continuous, also f_p cut function then p is removable cut point if and only if

1) $|f_i|$ is bounded.
2) U is compact subset of M.
3) If U ⊆ M is connected subset of M.

Proof

Let M be a Topological Manifold of dimension. If any p ∈ U ⊆ M is a cut point and functions f_i are invertible and continuous also the function f_p at p if cut functions then p is removable cut point.
This theorem is proved by both ways.
Let assume that
P is removable cut point.
To prove that $|f_i|$ are bounded.

We have a set in a metric space is bounded if it has finite diameter equivalently a set is bounded if it is contained in some open ball of finite radius. A function taking values in a metric space is bounded if it's image is a bounded set.

Let U be subset of M which is bounded in M. Consider the V_i be subset of U ⊆ M and the function $\{f_i\}$ are function defined on each subset V_i of ∪ respectively.

Let $P_1, P_2, \ldots, P_i$ be points in U i.e. $p_1, \in U_1$, $p_2 \in U_2 \ldots p_i \in U_i$
The function f_i defined U_i as
$f_1: U_1 \rightarrow R^n$, $f_i(U_i)=n_i$
$f_2:U_2 \rightarrow R^n$ i=1,2,3,…,i.
$f_i=U_i \rightarrow R^n$
Here each U_i's are bounded set.
U= $\bigcup_i^n U_i$ for each i
As each sequence of function f_i converges to f,

$$\lim_{i\to\infty} f_i = f$$

Therefore, each convergent sequence is bounded in M.
Therefore $|f_i| < \in$
→$|f_i|$ are bounded functions
→Each image of $f_i(U_i) \epsilon R^n$ is bounded.
Now, To prove U is a compact subset of M.
As U ⊆ M is subset of M.
To prove that U is compact space .
As f_n are bounded functions and each f_n is invertible function there exists $f_n(U_n)= \eta_n$ such that $f_n^{-1}(\eta_n)=U_n$ for finite n.
Therefore, each image $f_1^{-1}(\eta_1)=U_1$,
$f_2^{-1}(\eta_2)=U_2, \ldots,$
$f_n^{-1}(\eta_n)=U_n$
U=$\bigcup_{i=1}^n U_i \subseteq$ M, also all U_i are satisfies finite intersection property (f.i.p) in
$\bigcap_{i=1}^n U_i \neq \emptyset$
→ Shows that, each open cover U_i for the cover for U
Therefore every open cover has a finite sub cover which shows that U is compact space.
This proves the 2) condition U is compact subset of M.
Lastly,

we will prove third condition, suppose U⊆M be compact subset of M then to prove that U is connected subset of M.
This is proved in proof of (2) condition.
As U ⊆ M be compact subset of M, also $\{f_n\}$ are bijective functions.
The inverse image are $f_n^{-1}(\eta_n)$ are connected, they forms a cover for U as

$$U=f_1^{-1}(\eta_1) \cup f_2^{-1}(\eta_2) \cup, \ldots, \cup f_n^{-1}(\eta_i)$$

U implies, compact and connected which cover by U_i's
But U is subset of M.
This prove that U is connected subset of M.

Conversely:

Assume that, $|f_i|$ is bounded function and U ⊆M be connected subset of M which is compact.
Therefore to prove that, p ∈ U ⊆ M is removable cut point.
The sequence of function $\{f_i\}$ defined on $U_i \subseteq U$, $f_i(U_i)= \eta_i$,there exist a inverse functions $f_n^{-1}(\eta_i) =U_i \subseteq U$.
U_i are subset of U also U=$\bigcup_{i=1}^n U_i$ form a cover to M and each sequence f_n^{-1} is converges to f^{-1},
$\lim_{i\to\infty} f_i^{-1}=f^{-1}$
Implies $f_i^{-1} \rightarrow f^{-1}$
Therefore,$\bigcup_{i=1}^n U_i$ is converse U ⊆ M implies p is removable cut point.

Theorem 3.8

If p ∈ U ⊆ M is removable cut point then f_I are convertible to regular function at p if and only if

I. $\lim_{i\to\infty} f_i^{-1}=f$ or $|f_i-f| < \in$ where $|f_i| < M$, U ⊆ M

II. If U compact closed subset of M.

Proof

Let p ∈ U ⊆ M be a cut point of M if p is removable cut point then f_i are convertible to regular function.
As p is removable cut point.
By theorem 3.7 there exists invertible function $f_1^{-1}, \ldots, f_i^{-1}$ on subset of R^n.
Which are defined and continuous image in U , U ⊆M which forms a open cover for U.
i.e. $U=\cup_{i=1}^{n} f_i^{-1}(\eta_i)$ ∀ η ∈ R^n , i for finite i n > N.
Therefore, each point of p cover by U.
Therefore, the sequence of function defined on U which are continuous and also one – one.
Therefore, sequence of function f_i ∈ T
f_i:U →R^n is defined and regular at p.
That proves that $\lim_{i\to\infty} f_i=f$ or $|f_i-f| < \in$.
Therefore A p ∈ U ⊆ M be removable cut point U is closed compact subset of M.
Each function defined on U is bounded, (see theorem 3.7)
Therefore, $f_i(U_i) \subseteq R^n$ which are open subset, as U_i are open, But U is closed compact subset of M.
Therefore, $f_i(U) \subseteq R^n$ is closed bounded in R^n
Therefore, Image of U is bounded under $\{f_i\}$
⇨ f_i bounded and converges to f
I f_i - f I $< \epsilon$

⇨ $f_i \to f$
⇨ $\lim_{i\to\infty} f_i = f$
(ii) To prove that, U is compact closed subset of M
Therefore, By definition of compactness
A space is compact if every open has a finite sub cover.
This is proved in theorems 3.7.

<u>Conversely</u>

Assume that the sequence $\{f_i\}$ of function are bounded and converges to f on a compact closed subset of M .
T.P.T P ϵ U subset of M is removable cut point .
The f_i are convertible to regular function at P.
The $\{f_i\}$ are bounded function in R^n defined on bounded set U subset M is closed .
By closed map theorem, there exists a inverse function f_i whose images are closed and bounded functions.
but U is compact subset of M which has finite sub cover $\bigcup_{i=1}^{n} U_i$
Therefore $U=\bigcup_{i=1}^{n} U_i$ which remove the cut points P ϵ U subset M .
The function $\{f_i\}$ are bounded and defined each upon subset U_i of U which are 1-1 whose Jacobi matrix Rank is equal to n .

<u>Proposition</u> 3.9

The phenomena of converting cut function into regular function is called transmisition phase.
If P ϵ U subset of M is removable cut point than ∃ (is set of function $\{f_i\}$) is a transmisition phase on M , M is called ***Organic Manifold.***

IV. Conclusion:

This paper contains sequence of functions which removes cut points and punctured points in M. This article gives the compactness property of a Topological Manifold. Sequence of functions defined on neighborhood of cut points and punctured points converges to cover the whole space to make space bounded and compact. This concepts we are applying on neurology science also this concepts also use in police department for lock down and social distances to prevent corona virus in world details in the next session

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