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Universal Portfolios Generated by the Extended f-Divergence

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Abstract: *The extended f-divergence between two functions of probability distributions is defined for a given convex function f and an increasing function g. A universal portfolio is generated from the zero gradient set of an objective function involving the estimated daily rate of wealth increase and the extended f-divergence. For specific convex functions f and increasing functions g the form of the universal portfolio is derived. There exists a convex function such that the Bregman universal portfolio generated by this convex function is similar to the universal portfolio generated by the extended f-divergence.*

Keywords-*universal portfolio, extended f-divergence,Bregman divergence*

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I. INTRODUCTION

Universal portfolios generated by different methods are of recent interest. One of early methods of generating a universal portfolio is that due to Cover and Ordentlich [1] using the moments of the Dirichlet distribution. Subsequently, Helmbold et al. [2] proposed a method of generating a time-cum-memory efficient method of generating a universal portfolio using an objective function containing the Kullback-Leibler divergence of two portfolio vectors. This method is extended by Tan and Kuang [4] to cover an objective function containing the f or Bregman divergence of two portfolio vectors.

A modification of the Cover-Ordentlich universal portfolio using only a finite number of recent price-relatives is time-cum-memory efficient [3]. Matrix-generated divergences (for example, the Mahalanobis squared-divergence) can also be applied to generate a universal portfolio [7]. Partially convex functions have been used to generate universal portfolios in [5]. The method of using inequality ratios to generate universal portfolios is discussed in [6]. In this paper, the Uchida and Shioya [8] extended f-divergence between two functions of probability distributions is proposed to generate a universal portfolio.

II. SOME PRELIMINARIES

The notation on portfolio vectors, price-relative vectors and wealth functions is similar to that in [4]. In particular, $\boldsymbol{b}_n$ is the investment portfolio on trading day n and $\boldsymbol{x}_n$ is the corresponding price-relative vector.

Let $f(t)$ be a convex function on $0 < t < \infty$ satisfying $f(1) = 0$ and is strictly convex at $t = 1$ (i.e. $f'(1) \neq 0$). Let $g(t)$ be a strictly increasing function of t for $0 \leq t < \infty$ and $\boldsymbol{p} = (p_i)$ and $\boldsymbol{q} = (q_i)$ be two probability functions. The f-divergence between $\boldsymbol{p}$ and $\boldsymbol{q}$ with respect to $g(t)$, denoted as $D_{f,g}(\boldsymbol{p}||\boldsymbol{q})$ is defined as:

$$D_{f,g}(\boldsymbol{p}||\boldsymbol{q}) = \sum_{j=1}^{m} g(q_j) f\left(\frac{g(p_j)}{g(q_j)}\right). \tag{1}$$

Let $f_\beta(t) = (1-\beta)\left[\frac{t-t^{1-\beta}}{\beta} - (t-1)\right]$ for $0 \leq \beta < 1$. Then

$$f_\beta'(t) = (1-\beta)\left[\frac{1-(1-\beta)t^{-\beta}}{\beta} - 1\right]$$

and $f_\beta''(t) = (1-\beta)^2 t^{-\beta-1} > 0$, for $t > 0, 0 \leq \beta < 1$. Therefore $f_\beta(t)$ is convex for $t \geq 0$. Let $g_\beta(t) = t^{\frac{1}{1-\beta}}$ and $g'(t) = \frac{1}{1-\beta} t^{\frac{\beta}{1-\beta}} > 0$ for $t > 0, 0 \leq \beta < 1$. Therefore $g_\beta(t)$ is increasing in t for $t > 0$.

Given $f_\beta(t) = (1-\beta)\left[\frac{t-t^{1-\beta}}{\beta} - (t-1)\right]$ and $g_\beta(t) = t^{\frac{1}{1-\beta}}$ for $t > 0, 0 \leq \beta < 1$, then changing the parameter β to α where $\alpha = \frac{\beta}{1-\beta}$, results in $f_\alpha(t) = \frac{1}{\alpha}\left[t - t^{\frac{1}{1+\alpha}}\right] - \frac{1}{1+\alpha}[t-1]$ and $g_\alpha(t) = t^{1+\alpha}$, where $0 \leq \alpha < \infty$. For the given $f(.)$ and $g(.)$, from (1),

$$D_{f,g}(\boldsymbol{p}||\boldsymbol{q}) = \sum_{j=1}^{m} g(q_j) f\left(\frac{g(p_j)}{g(q_j)}\right)$$

$$= \sum_{j=1}^{m} q_j^{1+\alpha} \left\{\frac{1}{\alpha}\left[\frac{g(p_j)}{g(q_j)} - \left(\frac{g(p_j)}{g(q_j)}\right)^{\frac{1}{1+\alpha}}\right] - \frac{1}{1+\alpha}\left[\frac{g(p_j)}{g(q_j)} - 1\right]\right\}$$

$$= \sum_{j=1}^{m} q_j^{1+\alpha} \left\{\frac{1}{\alpha}\left[\frac{p_j^{1+\alpha}}{q_j^{1+\alpha}} - \left(\frac{p_j}{q_j}\right)\right] - \frac{1}{1+\alpha}\left[\left(\frac{p_j}{q_j}\right)^{1+\alpha} - 1\right]\right\}$$

$$= \sum_{j=1}^{m} \left\{\frac{1}{\alpha}\left[p_j^{1+\alpha} - p_j q_j^{\alpha}\right] - \frac{1}{1+\alpha}\left[p_j^{1+\alpha} - q_j^{1+\alpha}\right]\right\}$$

$$= \sum_{j=1}^{m} \left\{\frac{1}{\alpha} p_j \left[p_j^{\alpha} - q_j^{\alpha}\right] - \frac{1}{1+\alpha}\left[p_j^{1+\alpha} - q_j^{1+\alpha}\right]\right\}, \qquad (2)$$

where $0 \leq \alpha < \infty$.
For $\boldsymbol{p} = \boldsymbol{b}_{n+1}$ and $\boldsymbol{q} = \boldsymbol{b}_n$,

$$D_{f,g}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n) = \sum_{j=1}^{m} \left\{\frac{1}{\alpha} b_{n+1,j}\left[b_{n+1,j}^{\alpha} - b_{nj}^{\alpha}\right] - \frac{1}{1+\alpha}\left[b_{n+1,j}^{1+\alpha} - b_{n,j}^{1+\alpha}\right]\right\} \qquad (3)$$

is known as the discrete density power divergence.

III. MAIN RESULTS

With reference to the extended f-divergence (3), the following result is obtained.

Proposition 3.1: Let the convex function $f_\alpha(t) = \frac{1}{\alpha}\left[t - t^{\frac{1}{1+\alpha}}\right] - \frac{1}{1+\alpha}[t-1]$ and the increasing function $g(t) = t^{1+\alpha}$ be given for $\alpha \geq 0$. For the objective function

$$\hat{F}(\boldsymbol{b}_{n+1};\lambda) = \xi\left[\log(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1\right] - D_{f,g}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n) + \lambda\left[\sum_{j=1}^{m} b_{n+1,j} - 1\right] \qquad (4)$$

where $D_{f,g}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n)$ is given by (3), $\xi > 0$ and λ is the Lagrange multiplier, the universal portfolio generated is given by

$$b_{n+1,i} = \left\{\alpha\eta + b_{ni}^{\alpha} + \alpha\xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1\right]\right\}^{\frac{1}{\alpha}}$$

for $i = 1,2,\cdots,m$ where η is a real parameter.

Proof: The objective function (4) can be written as:

$$\hat{F}(\boldsymbol{b}_{n+1};\lambda) = \xi\left[\log(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1\right] - \sum_{j=1}^{m} \frac{1}{\alpha}\left[b_{n+1,j}^{1+\alpha} - b_{n+1,j} b_{nj}^{\alpha}\right]$$

$$+ \frac{1}{1+\alpha}\sum_{j=1}^{m}\left[b_{n+1,j}^{1+\alpha} - b_{n,j}^{1+\alpha}\right] + \lambda\left[\sum_{j=1}^{m} b_{n+1,j} - 1\right]$$

$$\frac{\partial \hat{F}}{\partial b_{n+1,i}} = \xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] - \frac{1+\alpha}{\alpha} b_{n+1,i}^{\alpha} + \frac{1}{\alpha} b_{ni}^{\alpha} + b_{n+1,i}^{\alpha} + \lambda$$

$$= \xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] - \frac{1}{\alpha} b_{n+1,i}^{\alpha} + \frac{1}{\alpha} b_{ni}^{\alpha} + \lambda = 0 \qquad (5)$$

for $i = 1,2,\cdots,m$.
Multiplying (5) by b_{ni} and sum over i to get

$$\xi + \frac{1}{\alpha}\sum_{j=1}^{m} b_{nj}\left(b_{nj}^{\alpha} - b_{n+1,j}^{\alpha}\right) + \lambda = 0 \tag{6}$$

Substracting (6) from (5),

$$\xi\left[1 - \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] + \frac{1}{\alpha}\sum_{j=1}^{m} b_{nj}\left(b_{nj}^{\alpha} - b_{n+1,j}^{\alpha}\right) + \frac{1}{\alpha} b_{n+1,i}^{\alpha} - \frac{1}{\alpha} b_{ni}^{\alpha} = 0 \tag{7}$$

for $i = 1,2,\cdots,m$.

Let $y_i = \frac{1}{\alpha} b_{n+1,i}^{\alpha}$. Then from (7)

$$y_i + \frac{1}{\alpha}\sum_{j=1}^{m} b_{nj}^{1+\alpha} - \sum_{j=1}^{m} b_{nj}\, y_j = \frac{1}{\alpha} b_{ni}^{\alpha} + \xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1\right] \tag{8}$$

for $i = 1,2,\cdots,m$.

Rearranging (8),

$$y_i - \frac{1}{\alpha} b_{ni}^{\alpha} + \xi\left[1 - \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] = \sum_{j=1}^{m} b_{nj}\, y_j - \frac{1}{\alpha}\sum_{j=1}^{m} b_{nj}^{1+\alpha}$$

$$= \text{constant, say } \eta \text{ not depending on } i$$

The solution to (8) is of the form

$$y_i = \eta + \frac{1}{\alpha} b_{ni}^{\alpha} + \xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1\right], \quad i = 1,2,\cdots,m. \tag{9}$$

Any y_i of the form (9) satisfies Eqn. (8). Multiply (9) by b_{ni} and sum over i to get

$$\sum_{j=1}^{m} b_{nj}\, y_j = \eta + \frac{1}{\alpha}\sum_{j=1}^{m} b_{nj}^{1+\alpha}$$

$$\eta = \sum_{j=1}^{m} b_{nj}\, y_j - \frac{1}{\alpha}\sum_{j=1}^{m} b_{nj}^{1+\alpha} \tag{10}$$

Replacing η in (9) by η in (10),

$$y_i - \eta = y_i + \frac{1}{\alpha}\sum_{j=1}^{m} b_{nj}^{1+\alpha} - \sum_{j=1}^{m} b_{nj}\, y_j$$

$$= \frac{1}{\alpha} b_{ni}^{\alpha} + \xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1\right]$$

which is Eqn (8).

The general solution to (8) is $y_i = \frac{1}{\alpha} b_{n+1,i}^{\alpha} = \eta + \frac{1}{\alpha} b_{ni}^{\alpha} + \xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1\right]$ for $i = 1,2,\cdots,m$ and any real η.

$$b_{n+1,i} = \left\{\alpha\eta + b_{ni}^{\alpha} + \alpha\xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1\right]\right\}^{\frac{1}{\alpha}}, \quad for\ i = 1,2,\cdots,m. \tag{11}$$

The universal portfolio generated by the extended f-divergence of two probability distribution where $f_\alpha(t) = \frac{1}{\alpha}\left[t - t^{\frac{1}{1+\alpha}}\right] - \frac{1}{1+\alpha}[t-1]$ and $g_\alpha(t) = t^{1+\alpha}$ for $0 \le \alpha$ is given (11). The extended f-divergence $D_{f,g}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n)$ is given (3).

Note:

$$f_\alpha(t) = -\frac{1}{\alpha} t^{\frac{1}{1+\alpha}} + \frac{1}{\alpha(1+\alpha)} t + \frac{1}{1+\alpha}$$

$$= \frac{1}{\alpha}\left[-t^r + rt + \frac{\alpha}{1+\alpha}\right] \text{ where } r = \frac{1}{1+\alpha}$$

Remark. The convex function $f_\alpha(t)$ in Proposition 3.1 can be written as:

$$f_\alpha(t) = -\frac{1}{\alpha} t^{\frac{1}{1+\alpha}} + \frac{1}{\alpha(1+\alpha)} t + \frac{1}{1+\alpha}$$

$$= \frac{1}{\alpha}\left[-t^r + rt + \frac{\alpha}{1+\alpha}\right] \text{ where } r = \frac{1}{1+\alpha} \tag{12}$$

Proposition 3.2. The universal portfolio generated by $D_{f_{\alpha-1},g_{\alpha-1}}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n)$ with respect to the convex function $f_{\alpha-1}(t) = \frac{1}{\alpha-1}\left[-t^{\frac{1}{\alpha}} + \frac{1}{\alpha}t + \frac{\alpha-1}{\alpha}\right]$ and $g(t) = t^\alpha$ for $\alpha \geq 1$ is similar to the universal portfolio generated by the Bregman divergence $B^{f_\alpha^*}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n)$ with respect to the convex function $f_\alpha^*(t) = t^\alpha - \alpha t$ for $\alpha > 1$.
*Proof.*The Bregman divergence with respect to the convex function f is given by

$$B^f(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n) = \sum_{j=1}^{m}\left[f(b_{n+1,j}) - f(b_{nj}) - f'(b_{nj})(b_{n+1,j} - b_{nj})\right]$$

and

$$B^{f^*}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n) = \sum_{j=1}^{m}\left[b_{n+1,j}^\alpha + (\alpha-1)b_{nj}^\alpha - \alpha b_{nj}^{\alpha-1}b_{n+1,j}\right] \quad (13)$$

corresponding to $f^* = t^\alpha - \alpha t, \alpha > 1$. On the other hand,

$$D_{f_\alpha,g_\alpha}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n) = \sum_{j=1}^{m}\left[\left(\frac{1}{\alpha} - \frac{1}{1+\alpha}\right)b_{n+1,j}^{\alpha+1} + \frac{1}{1+\alpha}b_{nj}^{\alpha+1} - \frac{1}{\alpha}b_{nj}^\alpha\, b_{n+1,j}\right]$$
$$= \sum_{j=1}^{m}\left[\frac{1}{\alpha(\alpha+1)}b_{n+1,j}^{\alpha+1} + \frac{1}{\alpha+1}b_{nj}^{\alpha+1} - \frac{1}{\alpha}b_{nj}^\alpha\, b_{n+1,j}\right]$$

and hence,

$D_{f_{\alpha-1},g_{\alpha-1}}(\boldsymbol{b}_{n+1}\|\|\boldsymbol{b}_n) = \sum_{j=1}^{m}\left[\frac{1}{(\alpha-1)\alpha}b_{n+1,j}^\alpha + \frac{1}{\alpha}b_{nj}^\alpha - \frac{1}{(\alpha-1)}b_{nj}^{\alpha-1}b_{n+1,j}\right]$	(14)

Differentiating (13) and (14),

$\frac{\partial}{\partial b_{n+1,i}}B^{f^*}(\boldsymbol{b}_{n+1}\|\|\boldsymbol{b}_n) = \alpha b_{n+1,i}^{\alpha-1} - \alpha b_{ni}^{\alpha-1} = \alpha\left(b_{n+1,i}^{\alpha-1} - b_{ni}^{\alpha-1}\right)$	(15)
$\frac{\partial}{\partial b_{n+1,i}}D_{f_{\alpha-1},g_{\alpha-1}}(\boldsymbol{b}_{n+1}\|\|\boldsymbol{b}_n) = \frac{1}{\alpha-1}b_{n+1,i}^{\alpha-1} - \frac{1}{\alpha-1}b_{ni}^{\alpha-1} = \frac{1}{\alpha-1}\left[b_{n+1,i}^{\alpha-1} - b_{ni}^{\alpha-1}\right]$	(16)

The derivative (15) and (16) are the same, except for the coefficients. Hence the universal portfolios generated by (15) and (16) are similar.

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