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α-Convergence of Sequences in Fuzzy Normed Vector Spaces

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Abstract: *The notion of fuzzy normed vector space is an important notion in the fuzzy functional analysis with several properties and applications. The objective of this paper is to discuss some basic concept of fuzzy topological vector space together with α-convergence and α-completeness of sequences and also to establish some properties ofα-convergence in fuzzy normed vector space.*
***Key Word**: Fuzzy topology, fuzzy norm,α-norm, α-convergence, α-Cauchyness, α-completeness.*

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1. Introduction

Fuzzy set was introduced by L.A. Zadeh [8]. The notion of fuzzy topological vector spaces was given for the first time by A. K. Katsaras. Later, Katsaras [5]changed the definition of fuzzy topological space by assuming that fuzzy topology of such a space contains all constant fuzzy sets as Lowen [6] did in his definition of fuzzy topology. Felbin [3] introduced an idea of a fuzzy norm on a vector space by assigning a fuzzy real number to each element of the linear space so that the corresponding metric associated this fuzzy norm is of Kaleva type [4] fuzzy metric. Bag and Samanta [2] introduced the concepts norm in another way and also introduced α-norms and defined α-convergence, α-completeness and boundedness of sequences in fuzzy normed vector space. In this paper, we discuss some of propertiesα-convergence with respect to the induced α-norms in fuzzy normed vector space.

We start with the following basic definitions with examples [1, 2, 4, 5, 7].

1.1 Definition [7]:
Let X be universe of discourse. A fuzzy set $\widetilde{A}$ on X is defined as

$$\widetilde{A} = \left\{\left(x, \mu_{\widetilde{A}}(x)\right) : x \in X \, and \, \mu_{\widetilde{A}} : X \to I \right\}$$

where $I = [0,1]$ is closed unit interval. So, a fuzzy set on X is a set whose members are graded with grade valuesfrom 0 to 1. The set of all fuzzy sets on X is denoted by I^X.

1.1.1 Example:
Let X denote universe of all persons of Kathmandu and $x \in X$. Define $\mu_{\widetilde{A}} : X \to I$ by

$$\mu_{\widetilde{A}}(x) = \begin{cases} 1 & when \, x \le 20 \\ \dfrac{(35-x)}{15} & when \, 20 < x < 35 \\ 0 & when \, x \ge 35 \end{cases}$$

Then, $\widetilde{A}$ gives a fuzzy set of young persons of District- Kathmandu.

1.2 Definition [4]:
By a fuzzy topology on a set X we will mean a subset φ of I^X satisfying the following conditions:
(i) φ contains every constant fuzzy set in X;
(ii) if $\mu_{1,}\ \mu_2 \in \varphi$ then $\mu_1 \wedge \mu_2 \in \varphi$;
(iii) if $\mu_i \in \varphi$ for each $i \in A$, then $sup_{i \in A}\mu_i \in \varphi$.

1.3 Definition [5]:
A fuzzy linear topology on a vector space E over $\mathbb{K}$ is a fuzzy topology φ on E such that the two mappings:

$$+: E \times E \to E,\ (x, y) \mapsto x + y,$$
$$\cdot : \mathbb{K} \times E \to E,\ (t, x) \mapsto tx$$

are continuous when $\mathbb{K}$ has the usual fuzzy topology and $\mathbb{K} \times E$and $E \times E$ have the corresponding product fuzzy topologies. A vector space with a fuzzy linear topology is called a *fuzzy vector space* or *a fuzzy topological vector space*.

1.4 Definition [1]:
Let U be a vector space over a field F. A fuzzy subset N of $U \times \mathbb{R}$ is called a fuzzy norm on U iff $\forall\, x, u \in U$ and $c \in F$,

(1) $\forall\, t \in \mathbb{R}$ with $t \leq 0$, $N(x,t) = 0$;

(2) $(\forall\, t \in \mathbb{R}, t > 0, N(x,t) = 1)$ iff $x = \underline{0}$;

(3) $\forall\, t \in \mathbb{R}, t > 0, N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(4) $\forall\, s,t \in \mathbb{R}$ and $x,u \in U$; $N(x+u, s+t) \geq \min\{N(x,s), N(u,t)\}$;

(5) $N(x, .)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

The vector space U together with fuzzy norm $N$is denoted by ordered pair(U,N) and defined as a fuzzy normed vector space.

1.4.1 Example:
Let $(U, \|.\|)$ be a normed vector space. For $x \in U$ and for $t \in \mathbb{R}$, define

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|} & for\ t > 0 \\ 0 & for\ t \leq 0 \end{cases}$$

Then, N defines a fuzzy norm on U and hence $(U, \|.\|)$ is a fuzzy normed vector space.

1.5 Definition [2]:
Let (U,N) be a fuzzy normed linear space. Let$\{x_n\}$ be a sequence in U. Then $\{x_n\}$ is said to be convergent if $\exists\, x \in U$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1\ \ \forall\, t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} x_n$.

1.6 Definition [2]:
A sequence $\{x_n\}$ in U is said to be a Cauchy-sequence if

$$\lim_{n\to\infty} N\big(x_{n+p} - x_n,\ t\big) = 1\ \ \forall\, t > 0 \text{ and p = 1, 2, 3, … .}$$

1.7 Definition [1]:
Let (U,N) be a fuzzy normed linear space. For each $\alpha \in (0,1)$, define$\|x\|_\alpha = \wedge\{t > 0 : N(x,t) \geq \alpha\}$, provided $\forall\, t > 0, N(x,t) > 0$ implies $x = \underline{0}$.Then, $\{\|\ \|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on U and for each fixed $\alpha \in (0,1)$, $\|.\|_\alpha$is called α-norm on U corresponding to the fuzzy norm N on U.

2. α-convergence of Sequences:

2.1 Definition [2]:
Let (U,N) be a fuzzy normed linear space and$\alpha \in (0,1)$. A sequence $\{x_n\}$ in U is said to be α-convergent in U if $\exists\, x \in U$ such that $\lim_{n\to\infty} N(x_n - x,\ t) > \alpha\ \ \forall\, t > 0$ and x is called the limit of $\{x_n\}$.

2.2 Definition [2]:
Let (U,N) be a fuzzy normed linear space and $\alpha \in (0,1)$. A sequence $\{x_n\}$ is said to be α-Cauchy if $\lim_{n\to\infty} N\big(x_{n+p} - x_n,\ t\big) \geq \alpha\ \ \forall\, t > 0$ and p = 1, 2, 3, …

2.3 Definition [2]:
Let (U,N) be a fuzzy normed vector space and $\alpha \in (0,1)$. It is said to be α-complete if any α-Cauchy sequence in U, α-converges to a point in U.

2.3.1 Example:
Let us consider the linear space U whose elements are sequences of complex numbers of the form $(\xi_1, \xi_2, \xi_3, \dots)$ where, $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$.

We define $\|x\|_0 = \{\sum_{i=1}^{\infty} |\xi_i|^2\}^{1/2}$ and $\|x\| = sup\{|\xi_i|\}$ where $x = (\xi_1, \xi_2, \xi_3, \dots) \in U$.

Then it is known that $\|.\|_0$ and $\|.\|$ are norms on U and $(U, \|.\|_0)$, $(U, \|.\|)$ are Banach spaces.
Now, we define a function $N: U \times \mathbb{R} \to [0, 1]$ by

$$N(x,t) = \begin{cases} 1 & if\ t > \left\{\sum_{i=1}^{\infty} |\xi_i|^2\right\}^{1/2} \\ 0.5 & if\ sup\{|\xi_i|\} < t \leq \left\{\sum_{i=1}^{\infty} |\xi_i|^2\right\}^{1/2} \\ 0 & if\ t \leq sup\{|\xi_i|\} \end{cases}$$

We can easily verify that N is a fuzzy norm on U.

Now α-norms of N are given by $\|x\|_\alpha = \begin{cases} \|x\| & \text{when } 0 < \alpha \leq 0.5 \\ \|x\|_0 & \text{when } 0.5 < \alpha < 1 \end{cases}$

Then, every α-Cauchy sequence in U, α-converges in U. Hence, U is complete w.r.t. $\|.\|_{0.5}$.

3. Main Results:

3.1 Theorem:
Let (U, N) be a fuzzy normed linear space satisfying

(i) $\forall\, t > 0, N(x,t) > 0$ implies $x = \underline{0}$ and,

(ii) for $x \neq \underline{0}$, $N(x,.)$ is a continuous function of $\mathbb{R}$ and strictly increasing on the subset

$\{t : 0 < N(x,t) < 1\}$ of $\mathbb{R}$.

Then, for any increasing (or decreasing) sequence $\{\alpha_n\}$ in (0,1), $\alpha_n \to \alpha \in (0,1)$ implies $\|x\|_{\alpha_n} \to \|x\|_\alpha$ $\forall\, x \in U$.

Proof.
For $x = \underline{0}$, it is clear that $\alpha_n \to \alpha \Rightarrow \|x\|_{\alpha_n} \to \|x\|_\alpha$.
Also, for $x \neq \underline{0}$, $\alpha \in (0,1)$ and $t' > 0$ we have, $\|x\|_\alpha = t'$ iff $N(x, t') = \alpha$.
Let $\{\alpha_n\}$ be an increasing sequence in (0,1) such that $\alpha_n \to \alpha \in (0,1)$.
Let $\|x\|_{\alpha_n} = t_n$ and $\|x\|_\alpha = t$. Then, $N(x, t_n) = \alpha_n$ and $N(x,t) = \alpha$ …(1)
Since $\{\|\ \|_\alpha : \alpha \in (0,1)\}$ is an increasing family of norms $\{t_n\}$ is an increasing sequence of real numbers and it is bounded above by t (since $\|x\|_{\alpha_n} \leq \|x\|_\alpha \forall\, n$) so, $\{t_n\}$ is convergent.
Thus, $\lim_{n\to\infty} N(x, t_n) = \lim_{n\to\infty} \alpha_n \Rightarrow N(x, \lim_{n\to\infty} t_n) = \alpha$ …(2)
From (1) and (2) we have, $N(x, \lim_{n\to\infty} t_n) = N(x,t)$.

$$\Rightarrow \lim_{n\to\infty} t_n = t \quad \text{by (ii).}$$

Therefore, $\lim_{n\to\infty} \|x\|_{\alpha_n} \to \|x\|_\alpha$.
Similarly, if $\{\alpha_n\}$ is a decreasing sequence in (0,1) and $\alpha_n \to \alpha \in (0,1)$ then it can be shown that $\lim_{n\to\infty} \|x\|_{\alpha_n} \to \|x\|_\alpha$ $\forall\, x \in U$.

3.2 Theorem:
If (U, N) is a fuzzy normed vector space such that $\forall\, t > 0, N(x,t) > 0 \Rightarrow x = \underline{0}$ then, the limit of an α-convergent sequence $\{x_n\}$ is unique in U.

Proof.
Let $\{x_n\}$ be an α-convergent and suppose it converges to x and y in U.
Then, $\lim_{n\to\infty} N(x_n - x,\ t) > \alpha\ \ \forall\, t > 0$ and $\lim_{n\to\infty} N(x_n - y,\ t) > \alpha\ \ \forall\, t > 0$.

Now, $N(x-y, t) = N\left(x - x_n + x_n - y,\ \frac{t}{2} + \frac{t}{2}\right) \forall\, n$

$\geq\ min\left\{N\left(x - x_n,\ \frac{t}{2}\right), N\left(x_n - y,\ \frac{t}{2}\right)\right\} \forall\, n$

$\Rightarrow\ N(x-y,t) \geq\ min\left\{\lim_{n\to\infty} N\left(x - x_n,\ \frac{t}{2}\right), \lim_{n\to\infty} N\left(x_n - y,\ \frac{t}{2}\right)\right\}$

$\Rightarrow\ N(x-y,t) > \alpha > 0\ \forall\, t > 0$
$\Rightarrow\ x-y=\underline{0} \quad \Rightarrow\ x=y.$
Hence, sequence $\{x_n\}$, α-converges to unique limit in U.

3.3 Theorem:
Let (U,N) be a fuzzy normed linear space such that $\forall\, t>0, N(x,t)>0 \Rightarrow x=\underline{0}$. If $\{x_n\}$ be an α-convergent sequence in (U,N). Then, $\|x_n-x\|_\alpha \to 0$ as n→∞ but converse is not necessarily true.

Proof.
Let $\{x_n\}$ be an α-convergent sequence in (U,N) and suppose, it converges to x. Then,
$\lim_{n\to\infty} N(x_n-x,t) > \alpha\ \ \forall\, t>0$
$\Rightarrow\ \forall\, t>0, \exists\, n_0(t) :\ N(x_n-x,t) > \alpha\ \ \forall\, n \ge n_0(t)$
$\Rightarrow\ \forall\, t>0, \exists\, n_0(t) :\ \|x_n-x\|_\alpha \le t\ \ \forall\, n \ge n_0(t)$
Since $t>0$ is arbitrary, $\|x_n-x\|_\alpha \to 0$ asn $\to \infty$.
The converse is not necessarily true. For this, we consider the following example:
Let$U=l^\infty$ be the sequence space. We define$\|x\| = sup\{\,|x_n|\,\}$ and $\|x\|_0 = sup\left\{\left|\frac{x_n}{n}\right|\right\}$
where $x=(x_1,x_2,x_3,\dots,x_n,\dots)$. Then, $\|.\|$ and $\|.\|_0$ are norms on U.
We now define a functionN : U $\times \mathbb{R} \to [0,1]$ by

$$N(x,t)=\begin{cases} 1 & if\ t > sup\{\,|x_n|\,\} \\ 0.5 & if\ sup\left\{\left|\frac{x_n}{n}\right|\right\} < t \le sup\{\,|x_n|\} \\ 0 & if\ t \le sup\left\{\left|\frac{x_n}{n}\right|\right\} \end{cases}$$

Then clearly, N is a fuzzy norm on U.
The α-norms of N are given by $\|x\|_\alpha = \begin{cases} \|x\|_0 & when\ 0<\alpha\le 0.5 \\ \|x\| & when\ 0.5<\alpha<1 \end{cases}$
Now, consider the sequence $\{e_n\}$ in U where

$$e_1=(1,0,0,\dots\dots\dots),\ e_2=(0,1,0,\dots\dots\dots),\ \dots\ \dots\ \dots,\ e_n=(\underbrace{0,0,0,\dots,1}_{n\ terms},\dots\dots).$$

Then,$\{e_n\}$ converges to the point $x=(0,0,0,\dots\dots\dots)$with respect to $\|x\|_{0.5}$.
For $\|e_n-x\|_{0.5} = \|e_n\|_{0.5} = sup\left\{\frac{0}{1},\frac{0}{2},\frac{0}{3},\dots\dots,\frac{0}{n-1},\frac{1}{n},\frac{0}{n+1},\dots\dots\right\} = \frac{1}{n}$
Therefore, $\lim_{n\to\infty}\|e_n-x\|_{0.5}=0$. Taking $t=\frac{1}{2}$, we have
$sup\left\{\frac{0}{1},\frac{0}{2},\frac{0}{3},\dots\dots,\frac{0}{n-1},\frac{1}{n},\frac{0}{n+1},\dots\dots\right\} = 1 < sup\{0,0,0,\dots\dots,0,1,0,\dots\dots\}$ if $n\ge 3$.
So, $\lim_{n\to\infty} N(e_n-x,\frac{1}{2}) = \lim_{n\to\infty} N\left(e_n,\frac{1}{2}\right) = 0.5 \ngtr 0.5.$
Thus, $\{e_n\}$is not 0.5-convergent although $\{e_n\}$ converges to x with respect to $\|x\|_{0.5}$.

4. Conclusion:

In this paper, we have shown that for $\alpha \in (0,1)$, the α-convergencein fuzzy normed vector spaces of a sequence provides the truth level α to what a sequence is convergent. The 0-convergence actually tells the sequence is not convergent and 1-convergence tells it is definitely convergent. Thus, α-convergence gives a generalization of convergence. Now, the question arises that whether it is possible define α-metric with help of α-norms defined by Bag and Samanta [2].

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