Weak separation axioms in terms of R-I-open sets

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Abstract: Throughout this paper, we study some weak separation axioms in ideal topological spaces using R-I-open sets and certain properties of the same.

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I. Introduction

The concept of ideal in topological space was first introduced by Kuratowski and Vaidyanathswamy [8]. They have also defined local function in ideal topological space. Further Hamlett and Jankovic in [2] studied the properties of ideal topological spaces. The notion of R₀ topological spaces was introduced by Shanin [7] in 1943. In 1961, Davis [1] studied some properties of the same and also introduced the notion of R₁ topological space. Further investigations of the properties of R₀ topological spaces were carried out by many topologists as in [3, 5, 9]. In this paper we define weak separation axioms using the notion of R-I-open sets and certain of its properties.

II. Preliminaries

By a space (X, τ), we mean a topological space with a topology τ defined on X on which no separation axioms are assumed unless otherwise explicitly stated. For a given point x in a space (X, τ), the system of open neighborhoods of x is denoted by N(x) = {U ∈ τ : x ∈ U}. For a given subset A of a space (X, τ), Cl(A) and Int(A) are used to denote the closure of A and interior of A, respectively, with respect to the topology.

A nonempty collection of subsets of a set X is said to be an ideal I on X, if it satisfies the following two conditions: (i) If A ∈ I and B ⊂ A, then B ∈ I; (ii) If A ∈ I and B ∈ I, then A ∪ B ∈ I. An ideal topological space (or ideal space) (X, τ, I) means a topological space (X, τ) with an ideal I defined on X. Let (X, τ) be a topological space with an ideal I defined on X. Then for any subset A of X, A∗(I, τ) = {x ∈ X/A ∩ U = ∅ for every U ∈ N(x)} is called the local function of A with respect to I and τ. If there is no ambiguity, we will write A∗(I) or simply A∗ for A∗(I, τ). Also, Cl∗(A) = A ∪ A*. It defines a Kuratowski closure operator for the topology τ ∗(I) (or simply τ ∗) which is finer than τ. [2, 4, 6]

A subset A of an ideal topological space (X, τ, I) is said to be R-I-open (resp. regular open) if Int(Cl∗(A)) = A (resp. Int(Cl(A)) = A). We call a subset A of (X, τ, I) is R-I-closed if its complement is R-I-open. The intersection of all R-I-closed sets containing A is called the R-I-closure of A and is denoted by R − I − Cl(A). The R-I-interior of A is defined by the union of all R-I-open sets contained in A and is denoted by R − I − Int(A). The family of all R-I-open (resp. R-I-closed) sets of (X, τ, I) containing a point x ∈ X is denoted by RIO(X, x) (resp.RIC(X, x)). A subset N of X is called an R-I-nbd of a point x ∈ X if there exists an R-I-open set U of (X, τ, I) such that x ∈ U ⊂ N.

Definition 2.1. [1] A topological space (X, τ, I) is said to be:
(i) R₀ if every open set contains the closure of each of its singletons.
(ii) R₁ if for x, y ∈ X with Cl((x)) = Cl((y)) there exists disjoint open sets U, V such that Cl((x)) ⊂ U and Cl((y)) ⊂ V.

III. R-I-R₀ spaces

Definition 3.1. A topological space (X, τ, I) is said to be:
(i) R-I-T₀ if for each pair of distinct points x and y in X, there exists R-I-open set containing x but not y.
(ii) R-I-T₁ if for each pair of distinct points x and y in X, there exist R-I-open sets U and V of X, such that x ∈ U, y ∉ U and y ∈ V, x ∉ V.
(iii) R-I-T₂ if for each pair of distinct points x and y in X, there exist disjoint R-I-open sets U and V in X such that x ∈ U and y ∈ V.
Definition 3.2. Let $(X, τ, I)$ be an ideal topological space and $A ⊂ X$. The R-I-kernel of $A$ is denoted by $IgKer(A)$ and is defined to be the set $IgKer(A) = \∩ \{ G ∈ RIO(X) : A ⊂ G \}$.

Lemma 3.3. For subsets $A, B$ of an ideal topological space $(X, τ, I)$, the following properties hold:
1. $A ∈ IgKer(A)$.
2. If $A ⊂ B$, then $IgKer(A) ⊂ IgKer(B)$.
3. If $A$ is R-I-open, then $IgKer(A) = A$.
4. $x ∈ IgKer(A)$ if and only if $A ∩ D ≠ φ$ for any R-I-closed set $D$ of $X$ such that $x ∈ D$.

Theorem 3.4. Let $(X, τ, I)$ be an ideal topological space and $x, y ∈ X$. Then $y ∈ IgKer(\{x\})$ if and only if $x ∈ R - I - Cl(\{y\})$.

Proof. Let $x, y ∈ X$. Suppose $y ∈ IgKer(\{x\})$. Then there exists $U ∈ RIO(X, x)$ such that $y ∉ U$. So $X - U$ is a R-I-open set containing $y$ but not $x$. Therefore $x ∉ R - I - Cl(\{y\})$. Conversely suppose $x ∉ R - I - Cl(\{y\})$. Then there exists $V ∈ RIC(X, y)$ such that $x ∉ V$. So $X - V$ is a R-I-open set containing $x$ but not $y$. Hence $y ∉ IgKer(\{x\})$.

Theorem 3.5. Let $(X, τ, I)$ be an ideal topological space and $S$ a subset of $X$. Then $IgKer(S) = \{ x ∈ X / R - I - Cl(\{x\}) ∩ S ≠ φ \}$.

Proof. Let $S ⊂ X$ and let $x ∈ IgKer(S)$. Suppose $S ∩ R - I - Cl(\{x\}) = φ$. Hence $X - (R - I - Cl(\{x\}))$ is an R-I-open set not containing $x$. But $S ⊂ X - (R - I - Cl(\{x\}))$. This implies $x ∉ IgKer(S)$, which is a contradiction. Hence $S ∩ R - I - Cl(\{x\}) ≠ φ$. Now suppose $x ∈ X$ and $S ∩ R - I - Cl(\{x\}) ≠ φ$ and suppose that $x ∉ IgKer(S)$. Then there exists an R-I-open set $S$ such that $S ⊂ U$ and $x ∉ U$. Let $y ∈ S ∩ R - I - Cl(\{x\})$. Thus $y ∈ S ∪ y ∈ U$ and so $y ∈ R-I-ndb of x and x ∉ U$. But this will make a contradiction that $y ∈ R - I - Cl(\{x\}) ⊂ X - U$. Hence the proof.

Definition 3.6. An ideal topological space $(X, τ, I)$ is called an R-I-Ro space if every R-I-open set contains the R-I-closure of each of its singletons.

Theorem 3.7. Let $(X, τ, I)$ be an ideal topological space. Then $X$ is R-I-$T_1$ if and only if it is R-I-$T_0$ and R-I-$R_0$.

Proof. Let $X$ be a R-I-$T_1$ space. Then clearly $X$ is a R-I-$T_0$ space and also $X$ is a R-I-$R_0$ space. Conversely, let $X$ be both R-I-$T_0$ and R-I-$R_0$. Let $x, y$ be any two distinct points of $X$. Since $X$ is R-I-$T_0$, there exists a R-I-open set $U$ such that $x ∈ U$ and $y ∉ U$ or there exists a R-I-open set $V$ such that $y ∈ V$ and $x ∉ V$. Since $X$ is R-I-$R_0$, then $R - I - Cl(\{x\}) ⊂ U$ for $x ∈ U$. Since $y ∉ U$, $y ∉ R - I - Cl(\{x\})$. So $y ∈ X - (R - I - Cl(\{x\})) = W$, say. Thus $U$ and $W$ are R-I-open sets containing $x$ and $y$ respectively. Hence $x ∉ U$ and $y ∉ U$. Hence $X$ is R-I-$T_1$.

Remark 3.8. Every R-I-$T_1$ space is R-I-$R_0$ space, since in R-I-$T_1$ space every singleton R-I-closed. The converse is not true in general.

Example 3.9. Let $X = \{a, b, c\}$, $τ = \{φ, X, \{a, b\}, \{c\}\}$, $I = \{φ, \{a\}, \{b\}, \{c\}\}$. $(X, τ, I)$ is R-I-$R_0$ but not R-I-$T_1$.

Example 3.10. Let $X = \{a, b, c\}$, $τ = \{φ, X, \{a, b\}\}$, $I = \{φ, \{a\}, \{b\}\}$. $(X, τ, I)$ is not R-I-$T_0$ and not R-I-$R_0$.

Remark 3.11. R-I-$T_0$ and R-I-$R_0$ are independent, which is clear from the above two examples.

Theorem 3.12. Let $(X, τ, I)$ be an ideal topological space. Then for $x, y$ in $X$, $IgKer(\{x\}) ≠ IgKer(\{y\})$ if $⇒ R - I - Cl(\{x\}) ≠ R - I - Cl(\{y\})$.

Proof. Let $IgKer(\{x\}) ≠ IgKer(\{y\})$. Then there exists $z ∈ X$ such that $z ∈ IgKer(\{x\})$ and $z ∉ IgKer(\{y\})$. Also, by theorem 3.4, $y ∉ R - I - Cl(\{z\})$ and $x ∉ R - I - Cl(\{z\})$. So $R - I - Cl(\{x\}) ∩ R - I - Cl(\{y\})$ and so $y ∉ R - I - Cl(\{x\})$ and $x ∉ R - I - Cl(\{y\})$. Hence $R - I - Cl(\{x\}) ≠ R - I - Cl(\{y\})$. Now, let $R - I - Cl(\{x\}) ≠ R - I - Cl(\{y\})$. Then there exists $z ∈ X$ such that $z ∈ R - I - Cl(\{x\})$ and $z ∉ R - I - Cl(\{y\})$. This implies there exists a R-I-open set containing $z$ and $x$ but not $y$. So $y ∉ IgKer(\{x\})$. Hence $IgKer(\{x\}) ≠ IgKer(\{y\})$.

Theorem 3.13. Let $(X, τ, I)$ be an ideal topological space. Then the following are equivalent.

(i) $(X, τ, I)$ is a R-I-Ro space.
(ii) For any $P ∈ RIC(X)$, $x ∉ P$ implies $P ⊂ U$ and $x ∉ U$ for some $U ∈ RIO(X)$.
(iii) For any $P ∈ RIC(X)$, $x ∉ P$ implies $P ∩ R - I - Cl(\{x\}) = φ$.
(iv) For any distinct points $x$ and $y$ of $X$, $R - I - Cl(\{x\}) = R - I - Cl(\{y\})$ or $R - I - Cl(\{x\}) ∩ R - I - Cl(\{y\}) = φ$.

Proof. (i) ⇒ (ii) Let $P$ be a R-I-closed set of $X$ and $x ∉ P$. Since $X$ is R-I-Ro, $R - I - Cl(\{x\}) ⊂ X - P$. Denote $U = X - (R - I - Cl(\{x\}))$. Then $U$ is a R-I-open set and $P ⊂ U$ and $x ∉ U$.

(ii) ⇒ (iii) Let $P ∈ RIC(X)$ and $x ∈ P$. Then there exists $U ∈ RIO(X)$ such that $P ⊂ U$ and $x ∉ U$. Since $U ∈ RIO(X)$, $U ∩ R - I - Cl(\{x\}) = φ$ and so $P ∩ R - I - Cl(\{x\}) = φ$.

(iii) ⇒ (iv)
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Suppose $R - I - Cl((x)) \neq R - I - Cl((y))$ for $x \neq y \in X$. Then there exists $z \in X$ with $z \in R - I - Cl((x))$ and $z \notin R - I - Cl((y))$. So there exists a $R$-I-open set $V$ in $X$ such that $y \notin V$ and $z \in V$ and so $x \in V$. Also, we get, $x \notin R - I - Cl((y))$. Hence $R - I - Cl((x)) \cap R - I - Cl((y)) = \emptyset$. The other statement can be proved in a similar way.

(iv) $\Rightarrow$ (i)

Let $V$ be a $R$-I-open set in $X$. For each $y \notin V, x \neq y$ and $x \notin R - I - Cl((y))$. So, by (iv), $R - I - Cl((x)) \cap R - I - Cl((y)) = \emptyset$ for each $y \in V$. Hence $(R - I - Cl((x))) \cap (U_{y\in Y} R - I - Cl((y))) = \emptyset$. Since $V$ is $R$-I-open and $y \in X - V, R - I - Cl((y)) \subset X - V$ and so $X - V = y \in X - V R - I - Cl((y))$. Therefore $(X - V) \cap (R - I - Cl((x))) = \emptyset$ or $R - I - Cl((x)) \subset V$. Hence $(x, y, I)$ is a $R$-I-R0 space.

**Theorem 3.14.** An ideal topological space $(x, y, I)$ is $R$-I-R0 if and only if for any two points $x, y \in X$, $I_{Ker}(x) \neq I_{Ker}(y)$ implies $I_{Ker}(x) \cap I_{Ker}(y) = \emptyset$.

**Proof.** Let $(x, y, I)$ be $R$-I-R0. Then by theorem 3.13, the statement holds. Conversely let $U$ be a $R$-I-open set of $X$ containing $x$. We claim $R - I - Cl((x)) \subset U$. For that let $y \in X - U$. So, $x \notin R - I - Cl((y))$. This implies $R - I - Cl((x)) \neq R - I - Cl((y))$. By assumption, $R - I - Cl((x)) \cap R - I - Cl((y)) = \emptyset$. Thus $y \notin R - I - Cl((x))$ and hence the claim.

**Theorem 3.15.** An ideal topological space $(x, y, I)$ is $R$-I-R0 if and only if for any two points $x, y \in X$, $I_{Ker}(x) \neq I_{Ker}(y)$ implies $(I_{Ker}(x) \cap I_{Ker}(y))$. Let $x \in R - I - Cl((y))$. Assume the contrary and suppose $x \notin I_{Ker}(x) \cap I_{Ker}(y)$. Since $z \in I_{Ker}(x), x \in R - I - Cl((z))$. Also, $x \in R - I - Cl((x))$. Then by theorem 3.13(iv), $R - I - Cl((x)) \neq R - I - Cl((z))$. In a similar way, we get $R - I - Cl((x)) = R - I - Cl((z))$. From this contradiction, we have $I_{Ker}(x) \cap I_{Ker}(y) = \emptyset$. Now assume the converse. By theorem 3.12, if $R - I - Cl((x)) \neq R - I - Cl((y))$, then $I_{Ker}(x) \cap I_{Ker}(y)$. So by assumption, $I_{Ker}(x) \cap I_{Ker}(y) = \emptyset$. Now let $z \in R - I - Cl((x))$ and this implies $x \notin I_{Ker}(z)$. Therefore $I_{Ker}(x) \cap I_{Ker}(y) = \emptyset$. Then by hypothesis, $I_{Ker}(x) \cap I_{Ker}(z)$. So $z \in R - I - Cl((x)) \cap R - I - Cl((y))$ will imply that $I_{Ker}(x) = I_{Ker}(z) = I_{Ker}(y)$. From this contradiction, we get $R - I - Cl((x)) \cap R - I - Cl((y)) = \emptyset$. Thus, by theorem 3.14, $(x, y, I)$ is a $R$-I-R0 space.

**Theorem 3.16.** Let $(x, y, I)$ be a $R$-I-R0 space. Then $x \in R - I - Cl((y))$ if and only if $y \in R - I - Cl((x))$ for any $x, y \in X$. The converse is also true.

**Proof.** Let $(x, y, I)$ be $R$-I-R0. Let $x \in R - I - Cl((y))$ and let $U$ be a $R$-I-open set of $X$ containing $y$. Then $R - I - Cl((y)) \subset U$. So, by hypothesis, $x \in R - I - Cl((y))$ implies $x \in U$. That means, every $R$-I-open set containing $y$ contains $x$. Hence $y \in R - I - Cl((x))$.

Assume the converse. Let $U$ be a $R$-I-open set in $X$ containing $x$. If $y \notin U$, then $x \notin R - I - Cl((y))$ and hence $y \notin R - I - Cl((x))$. This means $R - I - Cl((x)) \subset U$. Then $(X, y, I)$ is $R$-I-R0.

**Theorem 3.17.** Let $(x, y, I)$ be an ideal topological space. Then the following are equivalent:

(i) $(x, y, I)$ is a $R$-I-R0 space.

(ii) For any $\varphi \neq P \in X$ and $U \in RIO(X)$ with $P \cap U \neq \varphi$, there exists $V \in RIC(X)$ such that $P \cap V \neq \varphi$ and $V \subset U$.

(iii) For any $U \in RIO(X), U = \cup \{ V \in RIC(X) : V \subset U \}$.

(iv) For any $V \in RIC(X), V = \cap \{ U \in RIO(X) : V \subset U \}$.

(v) For any $x \in X, R - I - Cl((x)) \subset Ker((x))$.

**Proof.** (i) $\Rightarrow$ (ii)

Let $\varphi \neq P \in X$ and $U$ be an $R$-I-open set with $P \cap U \neq \varphi$. Let $x \in P \cap U$. Since $x \in U$ and $X$ is $R$-I-R0, $R - I - Cl((x)) \subset U$. Let $V = R - I - Cl((x))$. Then $V \in RIC(X)$ and $V \subset U$ and $P \cap V \neq \varphi$.

(ii) $\Rightarrow$ (iii)

Let $U \in RIO(X)$. Then clearly, $U \cup \{ V \in RIC(X) : V \subset U \} \subset U$. Now let $x \in U$. Then there exists $V \in RIC(X)$ such that $x \in V \subset U$. Thus, $x \in V \cup \{ V \in RIC(X) : V \subset U \}$. Hence $U = \cup \{ V \in RIC(X) : V \subset U \}$.

(iii) $\Rightarrow$ (iv)

Let $V \in RIC(X)$. Consider all $R$-I-open sets containing $V$. Then $\cap \{ U \in RIO(X) : V \subset U \} \subset V$. Now let $x \in V$. Then $x \in U$ for all $U \in RIO(X)$ with $V \subset U$. So $x \in \cap \{ U \in RIO(X) : V \subset U \}$. Thus $V \cap \{ U \in RIO(X) : V \subset U \}$.

(iv) $\Rightarrow$ (v)

Let $x \in X$ and let $y \in Ker((x))$. Then there exists $G \in RIO(X)$ such that $x \in G$ and $y \notin G$. So $R - I - Cl((y)) \cap G = \varphi$. Then by (iv), $(\cap \{ U \in RIO(X) : V \subset U \}) \cap G = \varphi$. So there exists
an R-I-open set $U$ such that $x \notin U$ and $R - I - Cl((y)) \subset U$. Hence $R - I - Cl((x)) \cap U = \emptyset$ and $y \notin R - I - Cl((x))$. Thus $R - I - Cl((x)) \subset I_R Ker((x))$.

(v) $\Rightarrow$ (i) Let $U$ be an R-I-open set in $X$ and $x \in U$. Let $y \in I_R Ker((x))$. Then $x \in R - I - Cl((y))$. Also $y \in U$. Then $I_R Ker((x)) \subset U$. Thus $x \in R - I - Cl((x)) \subset I_R Ker((x)) \subset U$. Hence $X$ is an R-I-$R_0$ space.

**Theorem 3.18.** Let $(X, \tau, I)$ be an ideal topological space. Then the following are equivalent:

(i) $(X, \tau, I)$ is an R-I-$R_0$ space.

(ii) If $V$ is a R-I-closed subset of $X$, then $V = I_R Ker(V)$.

(iii) If $V$ is a R-I-closed subset of $X$ and $x \in V$, then $I_R Ker((x)) \subset V$.

(iv) If $x \in X$ such that $I_R Ker((x)) \subset R - I - Cl((x))$.

**Proof.** (i) $\Rightarrow$ (ii) Let $V$ be a R-I-closed subset of $X$ and let $x \in X - V$. Since $X$ is a R-I-$R_0$ space and $X - V \in RIO(X, x)$, $R - I - Cl((x)) \subset X - V$. By theorem 3.5, $I_R Ker(V) \subset X - (R - I - Cl((x)))$. Also $x \notin I_R Ker(V)$. Thus $I_R Ker(V) = V$.

(ii) $\Rightarrow$ (iii) Since $U \subset V$ implies $I_R Ker(U) \subset I_R Ker(V)$, it follows that $I_R Ker((x)) \subset I_R Ker(V)$ for $x \in V$. Therefore $I_R Ker((x)) \subset V$ from (ii).

(iii) $\Rightarrow$ (iv) Let $x \in X$ and clearly $x \in R - I - Cl((x))$. From (iii) $I_R Ker((x)) \subset R - I - Cl((x))$.

(iv) $\Rightarrow$ (i) Let $x \in R - I - Cl((y))$. Then by theorem 3.4, $y \in I_R Ker((x))$. Thus we get $y \in I_R Ker((x)) \subset R - I - Cl((y))$. Hence $x \in R - I - Cl((y))$ implies $y \in R - I - Cl((x))$. Clearly the reverse implication holds. Thus, by theorem 3.16, $X$ is a R-I-$R_0$ space.

**Corollary 3.19.** Let $(X, \tau, I)$ be an ideal topological space. If $(X, \tau, I)$ is R-I-$R_0$, then $I_R Ker((x)) = R - I - Cl((x))$ for all $x \in X$. The converse is also true.

**Proof.** Suppose $(X, \tau, I)$ is a R-I-$R_0$ space. By theorem 3.17 and theorem 3.18 the statement is obvious. The converse is trivial by theorem 3.18.

IV. R-I-$R_1$ spaces

**Definition 4.1.** An ideal topological space $(X, \tau, I)$ is called an R-I-$R_1$ space if for $x, y \in X$ with $R - I - Cl((x)) \neq R - I - Cl((y))$ there exist disjoint R-I-open sets $U, V$ such that $R - I - Cl((x)) \subset U$ and $R - I - Cl((y)) \subset V$.

**Theorem 4.2.** Every R-I-$R_1$ space is R-I-$R_0$.

**Proof.** Let $(X, \tau, I)$ be a R-I-$R_1$ space and $x, y \in X$. Let $U$ be a R-I-open set containing $x$ but not $y$. So, $x \in R - I - Cl((y))$. Then we have $R - I - Cl((x)) \neq R - I - Cl((y))$. By hypothesis, then there exists R-I-open set $V$ such that $R - I - Cl((y)) \subset V$. Therefore $x \notin V$ and this implies $y \notin R - I - Cl((x))$. Thus $R - I - Cl((x)) \subset U$. Hence $(X, \tau, I)$ is R-I-$R_0$.

**Remark 4.3.** The converse of the above theorem is not true in general.

**Example 4.4.** Let $X = \{a, b, c\}, \tau = \{\varphi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$, $I = \{\varphi, \{a\}\}$. The R-I-open sets are $\{a\}, \{b, c\}, X$. Then $(X, \tau, I)$ is R-I-$R_0$ but not R-I-$R_1$.

**Remark 4.5.** R$_0$ implies R-I-$R_1$ but the converse is not true.

**Example 4.6.** Consider the same example written above (Example 4.4). $(X, \tau, I)$ is R-I-$R_0$ but not R$_0$.

**Remark 4.7.** R$_1$ implies R-I-$R_1$ but the converse is not true.

**Example 4.8.** Let $X = \{a, b, c\}, \tau = \{\varphi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$, $I = \{\varphi, \{a\}\}$. The R-I-open sets are $\{a\}, \{b, c\}, X$. Then $(X, \tau, I)$ is R-I-$R_1$ but not R$_1$.

**Theorem 4.9.** An ideal topological space $(X, \tau, I)$ is R-I-$R_1$ if and only if for any two points $x, y \in X$, $I_R Ker((x)) \neq I_R Ker((y))$ implies there exists disjoint R-I-open sets $U, V$ such that $R - I - Cl((x)) \subset U$ and $R - I - Cl((y)) \subset V$.

**Proof.** By theorem 3.12, theorem directly follows.

**Theorem 4.10.** Let $(X, \tau, I)$ be a R-I-$R_1$ space. Then for $x, y \in X$ with $R - I - Cl((x)) \neq R - I - Cl((y))$, there exists R-I-closed sets $K_1$ and $K_2$ such that $x \in K_1$, $y \in K_2$, $y \notin K_1$, $x \notin K_2$ and $K_1 \cup K_2 = X$.

**Proof.** Let $(X, \tau, I)$ be R-I-$R_1$. Suppose $x, y \in X$ with $R - I - Cl((x)) \neq R - I - Cl((y))$. Then there exist disjoint R-I-open sets $U, V$ such that $R - I - Cl((x)) \subset U$ and $R - I - Cl((y)) \subset V$. Let $K_1 = X - V$ and $K_2 = X - U$. Then $K_1$ and $K_2$ are R-I-closed sets such that $x \in K_1$, $y \in K_2$, $y \notin K_1$, $x \notin K_2$ and $K_1 \cup K_2 = X$.

Assume the converse. To show $X$ is R-I-$R_1$, we first prove $X$ is R-I-$R_0$. For that suppose $U$ be an R-I-open set containing $x$ and suppose $R - I - Cl((x))$ is not a subset of $U$. So, $R - I - Cl((x)) \cap U^c \neq \emptyset$. Let

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Assume the converse. If $\varphi \notin U$, then $\varphi \in V_1 = \{ \varphi \}$. For each $x, y \in X$ either (a) or (b) holds. (a) if $U$ is R-I-open, then $x \in U$ if and only if $\varphi \notin U$. (b) there exist disjoint R-I-open sets $U$ and $V$ such that $x \in U$ and $\varphi \notin V$. Thus $x \in U$ if and only if $y \in V$. Then there exists an element $\varphi \notin U$ and $\varphi \notin V$. Let $K_1$ and $K_2$ be R-I-closed sets such that $x \in K_1$, $y \in K_2$, and $\varphi \notin K_1 \cup K_2$.

\textbf{Theorem 4.11.} The following statements are equivalent:

(i) $(X, \tau, I)$ is a R-I-R$_1$ space.

(ii) For each $x, y \in X$ either (a) or (b) holds. (a) if $U$ is R-I-open, then $x \in U$ if and only if $\varphi \notin U$. (b) there exist disjoint R-I-open sets $U$ and $V$ such that $x \in U$ and $\varphi \notin V$.

(iii) If $x, y \in X$ with $R - I - Cl((x)) \neq R - I - Cl((y))$, there exists R-I-closed sets $K_1$ and $K_2$ such that $x \in K_1$, $y \in K_2$, $\varphi \notin K_1 \cup K_2$.

\textbf{Proof.} (i) $\Rightarrow$ (ii)

Let $x, y \in X$. Case 1: $R - I - Cl((x)) = R - I - Cl((y))$. Let $U$ be an R-I-open set. Then $x \in U$ implies $y \in R - I - Cl((x)) \subset U$ and $\varphi \notin U$ implies $x \in R - I - Cl((y)) \subset U$. Thus $x \in U$ if and only if $y \in V$. Case 2: $R - I - Cl((x)) \neq R - I - Cl((y))$. Then there exist disjoint R-I-open sets $U$ and $V$ such that $x \in R - I - Cl((x)) \subset U$ and $y \in R - I - Cl((y)) \subset V$.

(ii) $\Rightarrow$ (iii)

Let $x, y \in X$ such that $R - I - Cl((x)) \neq R - I - Cl((y))$. Then either $x \notin R - I - Cl((y))$ or $y \notin R - I - Cl((x))$. Suppose $x \notin R - I - Cl((y))$. Then there exists a R-I-open set $S$ such that $x \in S$ and $\varphi \notin S$. So, by (ii) there exists disjoint R-I-open sets $U$ and $V$ such that $x \in U$ and $\varphi \notin V$. Let $K_1 = V^c$ and $K_2 = U^c$. Then $K_1$ and $K_2$ are R-I-closed sets such that $x \in K_1$, $y \in K_2$, and $\varphi \notin K_1 \cup K_2$.

(iii) $\Rightarrow$ (i)

This is the statement of theorem 4.10.

\textbf{Theorem 4.12.} An ideal topological space is R-I-R$_1$ if and only if $x \in X - R - I - Cl((x))$ implies that $x$ and $y$ have disjoint R-I-open neighbourhoods.

\textbf{Proof.} Let $(X, \tau, I)$ be R-I-R$_1$ and let $x \in X - R - I - Cl((x))$. Then $R - I - Cl((x)) \neq R - I - Cl((y))$. Then $x$ and $y$ have disjoint R-I-open neighbourhoods.

Assume the converse. First, we prove that $(X, \tau, I)$ is R-I-R$_0$. Let $U$ be a R-I-open set containing $x$. Suppose $\varphi \notin U$. Then $R - I - Cl((y)) \subset U = \varphi$. Also, $x \notin R - I - Cl((y))$. Then there exists disjoint R-I-open sets $V_1$ and $V_2$ such that $x \in V_1$ and $y \in V_2$. Then $R - I - Cl((x)) \subset R - I - Cl((y)) \subset R - I - Cl((x)) \subset V_1 \cap V_2 \neq \varphi$. Thus $y \notin R - I - Cl((x))$. Hence $R - I - Cl((x)) \subset V_1 \subset U$ and $(X, \tau, I)$ is R-I-R$_0$. Now suppose $R - I - Cl((x)) \neq R - I - Cl((y))$. Then there exists an element $w \in R - I - Cl((x))$ and $w \notin R - I - Cl((y))$. By assumption there exists disjoint R-I-open sets $W_1$ and $W_2$ such that $w \in W_1$ and $y \notin W_2$. Since $w \notin R - I - Cl((x))$, $w \notin W_1$. Since $(X, \tau, I)$ is R-I-R$_0$, $R - I - Cl((x)) \subset W_1$ and $R - I - Cl((y)) \subset W_2$. Thus $(X, \tau, I)$ is R-I-R$_1$.

\textbf{References}


