Riemannian Curvature Tensor on Trans-Sasakian Manifold

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Abstract:

Background: Oubina, J.A.[1] defined and initiated the study of Trans-Sasakian manifolds. Blair [2], Prasad and Ojha [3], Hasan Shahid [4] and some other authors have studied different properties of C-R-Sub – manifolds of Trans-Sasakian manifolds. Golab, S. [5] studied the properties of semi-symmetric and Quarter symmetric connections in Riemannian manifold. Yano, K.[6] has defined contact conformal connection and studied some of its properties in a sasakian manifold. Mishra and Pandey [7] have studied the properties in Quarter symmetric metric F-connections in an almost Grayan manifold.

Result: In this paper we have studied Riemannian curvature tensor on Trans-Sasakian manifold. Following the patterns of Yano [6], we have proved that a Trans-Sasakian manifold admitting a killing structure vector is an (α, 0) type Trans-Sasakian manifold. Further we have proved that a Trans-Sasakian manifold with structure 1-form A is closed, becomes (β, 0) type Trans-Sasakian manifold.

Conclusion: Trans-Sasakian manifold admitting a killing structure vector is an (α, 0) type Trans-Sasakian manifold. And a Trans-Sasakian manifold with structure 1-form A is closed, becomes (β, 0) type Trans-Sasakian manifold.

Key words: Riemannian curvature tensor, Trans-Sasakian manifold, C-R-Sub –manifolds of Trans-Sasakian manifolds, semi-symmetric and Quarter symmetric connections in Riemannian manifold, almost Grayan manifold.

I. Introduction

Let $M_n$ ($n = 2m + 1$) be an almost contact metric manifold endowed with a (1,1)-type structure tensor $F$, a contravariant vector field $T$, a 1-form $A$ associated with $T$ and a metric tensor $g$ satisfying:

\[ F^2X = -X + A(X)T \]
\[ FT = 0 \]
\[ A(FX) = 0 \]
\[ A(T) = 1 \]

and

\[ g(X, Y) = g(Y, X) - A(X)A(Y) \]

Where

\[ X \equiv FX \]

And

\[ A(T, X) \equiv A(X) \]

For all $\infty$-vector fields $X,Y$ in $M_n$ also, a fundamental 2-form $\varphi$ in $M_n$ is defined as

\[ \varphi(X, Y) = g(X, Y) = \varphi(Y, X) \]

Then, we call the structure bundle $\{F,T,A,g\}$ an almost contact-metric structure [1].

An almost contact metric structure is called normal [1], if

\[ (dA)(X,Y)T + N(X,Y) = 0 \]

Where

\[ N(X,Y) = (D_X - F)(Y) - (D_Y - F)(X) - (D_X F)(Y) + (D_Y F)(X) \]

Is Nijenhuis tensor in $M_n$.

An almost contact metric manifold $M_n$ with structure bundle $\{F,T,A,g\}$ is called a Trans-Sasakian manifold [3],[1], if

\[ (DF)(Y) = \alpha g(Y, T) - A(Y)X + \beta \varphi(Y, X) \]

Where $\alpha$, $\beta$ are non-zero constants.

It can be easily seen that a Trans-Sasakian manifold is normal. In view of (1.6) one can easily obtain in $M_n$, the relations
(1.7) \( N(X, Y) = 2\alpha F(X, Y) T \)

(1.8) \((dA)(X,Y) = -2\alpha' F(X,Y)\)

(1.9) \((D_\alpha A)(Y) + (D_Y A)(X) = 2\beta \{g(Y, X) - A(Y)A(X)\} \)

(1.10) \((D_X F(Y, Z)) + (D_Y F(Z, X)) + (D_Z F(X, Y)) = 2\beta \{A(Z)F(Y, X) + A(X)F(Y, Z) + A(Y)F(Z, X)\} \)

(1.11)(a) \((D_X A)(Y) = -\alpha F(X, Y) + \beta \{g(X, Y) - A(X)A(Y)\} \)

(1.11)(b) \((D_X T) = -\alpha X + \beta \{X - A(X)T\} \)

REMARK (1.1): In the above and in what follows, the letters \(X, Y, Z \ldots \) etc. are \(C^\infty\) vector fields in \(M_n\).

II. Riemannian Curvature Tensor On Trans-Sasakian Manifold:

From (1.11)(b) given by

\[(D_X T) = -\alpha X + \beta \{X - A(X)T\} \]

we obtain, in view of (1.6)

\[ (2.1) K(X,Y,T) = D_X D_Y T - D_Y D_X T - D[X,Y]T \]

Where \(K(X, Y, Z)\) is the Riemannian curvature tensor with respect to the Riemannian connection \(D\). From (2.1), we have the following relations

(2.2)(a) \(K(X,T,T) = -\alpha (\alpha^2 - \beta^2)X + 2\alpha\beta \{A(Y)X - A(X)Y\} \)

(2.2)(b) \(K(T,T,T) = 0 \)

(2.2)(c) \('K(X,Y,T) = g(K(X,Y,T), T) = 0 \)

Also by contracting (2.1) with respect to \(X\), we get

(2.3)(a) \(\text{Ric}(Y,T) = (n-1)(\alpha^2 - \beta^2)A(Y) \)

Further, putting \(T\) for \(Y\) in (2.3)(a), we get

(2.3)(b) \(\text{Ric}(T,T) = (n-1)(\alpha^2 - \beta^2) \)

Again, barring \(Y\) in (2.3)(a), we can get

(2.3)(c) \(\text{Ric}(Y,T) = 0 \)

Also (2.3)(a) gives

(2.3)(d) \(\text{R}(T) = (n-1)(\alpha^2 - \beta^2)T \)

Thus, we have

THEOREM (2.1): In a Trans-Sasakian manifold \(M_n\), the equation (2.1), (2.2) and (2.3) hold good.

Now, differentiating covariantly the equation (2.1) with respect to a vector field \(Z\), we obtain, in view of the equation (1.6), (1.11)(b)

\[(2.4) (D_Z K)(X,Y,T) - \alpha K(X,Y,Z) + \beta K(X,Y,T) = - \alpha (\alpha^2 - \beta^2)Y + 2\alpha\beta \{A(Y)X - A(X)Y\} \]

Now, putting \(T\) for \(Z\) in (2.4), we get

(2.5) \( (D_T K)(X,Y,T) = 0 \)

Also, contracting (2.4) with respect to \(Z\), we obtain

(2.6) \((D_T K)(X,Y,T) = -2\alpha (\alpha^2 - \beta^2)Y F(X,Y) \)

Thus, we have

THEOREM (2.2): In a Trans-Sasakian manifold \(M_n\), we have

(2.7) \((D_T K)(X,Y,T) = 0 \)

Now, suppose \(T\) is a Killing vector, i.e.

(2.7) \((D_T A)(Y) + (D_Y A)(X) = 0 \)

Then, in view of (1.8) and (2.7), we easily get

(2.8)(a) \((D_T A)(Y) = -\alpha F(X,Y) \)

(2.8)(b) \((D_T T) = -\alpha X \)

from which, we have

COROLLARY (2.1): A Trans-Sasakian manifold \(M_n\) admitting a Killing structure vector \(T\) is of \((\alpha, 0)\) type Trans-Sasakian manifold.

COROLLARY (2.2): In a \((\alpha, 0)\) type Trans-Sasakian manifold, we have

(2.9)(a) \((D_T K)(X,Y,T) = -\alpha (\alpha^2 - \beta^2)Y F(X,Y) \)

(2.9)(b) \((D_T K)(X,Y,T) = -\alpha^2 F(X,Y) \)

PROOF: Putting \(\beta = 0\) in (2.4) and (2.6), we immediately obtain the above result in (2.9).
COROLLARY (2.3): A Trans-Sasakian manifold $M_\alpha$ with structure 1-form $A$ is closed, becomes $(\alpha,0)$ type Trans-Sasakian manifold.

**PROOF:** The 1-form $A$ is closed, i.e.

\[
(2.10) \ (\alpha A)(X,Y) = (D_X A)(Y) - (D_Y A)(X) = 0
\]

Using this in (1.8), we easily get $\alpha = 0$, so that $M_\alpha$ becomes $(0, \beta)$ type Trans-Sasakian manifold.

COROLLARY (2.4): In a $(0, \beta)$ Trans-Sasakian manifold, we have

\[
(2.11)(a) \ K(X,Y,Z) = \beta \{A(Y)X - A(X)Y\}
\]

\[
(2.11)(b) \ (D_X K)(Y,Z,T) + \beta K(X,Y,Z) = \alpha \{A(Z)X - A(X)Z\}
\]

\[
= \beta \{g(Z,Y)X - g(Z,X)Y - A(Z)A(Y,X) + A(Z)A(Y)X\}
\]

\[
(2.11)(c) \ (D_X K)(Y,T) = 0
\]

**PROOF:** The above results are also immediate consequence of (2.1), (2.4) and (2.6) for $\alpha = 0$. Now, we have

\[
(2.12) \ K(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z
\]

\[
= D_X \{(D_Y F)(Z)\} + D_Y \{(D_X F)(Z)\} - (D_{[X,Y]} F)(Z) - D_{[X,Y]} Z
\]

\[
= D_X \{(D_Y F)(Z)\} + D_Y \{(D_X F)(Z)\} + D_{[X,Y]} F(Z) - D_{[X,Y]} Z
\]

\[
- [D_{[X,Y]} F](Z) - D_{[X,Y]} Z]
\]

Using (1.6) in the above equation, we get

\[
K(X,Y,Z) = D_X \{\alpha \{g(Y,Z)X - g(Z,Y)X\} + \beta \{F(Y,Z)T - A(Z)Y\} + \alpha \{g(X,D_Z)Y - A(D_Z)X\}
\]

\[
+ \beta \{F(X,D_Z)T - A(D_Z)X\} - D_D \{\alpha \{g(X,Z)T - A(Z)X\} + \beta \{F(X,Z)T - A(Z)X\}\}
\]

\[
- \alpha \{g(Y,D_Z)T - A(D_Z)Y\} - \beta \{F(Y,D_Z)T - A(D_Z)Y\} + K(X,Y,Z) - \alpha \{g(X,Y,Z)T - A(D_Z)Y\}
\]

\[
\text{Again using (1.6), (1.11)(b) in this result, we obtain}
\]

\[
K(X,Y,Z) = K(X,Y,Z) - (\alpha^2 - \beta^2) \{g(Y,Z)X - g(Z,Y)X\} + 2\alpha \beta \{g(Y,Z)X - g(Z,Y)X\}
\]

\[
+ \alpha \{F(X,Z)Y - F(Y,Z)X\} - \beta \{A(Y)F(X,Z)T - A(X)F(Y,Z)T\} - 2\alpha \beta \{F(X,Z)Y - F(Y,Z)X\}
\]

From which, we easily obtain

\[
K(X,Y,Z,U) = K(X,Y,Z,U) + K(X,Y,Z,U)
\]

\[
= \{\alpha^2 - \beta^2\} \{g(Y,Z)F(X,U) - g(Z,Y)F(Y,U)\} + 2\alpha \beta \{g(Y,Z)g(X,U) - g(Z,Y)g(X,U)\}
\]

\[
+ \alpha \{F(X,Z)g(Y,U) - F(Y,Z)g(X,U)\} - \beta \{A(Y)A(U)F(X,Z) - A(X)A(U)F(X,Z)\}
\]

\[
+ \alpha \beta \{F(X,Z)F(Y,U) - F(Y,Z)F(X,U)\}
\]

Putting $U$ for $V$ in the above and then barring $X$ and $Y$, we easily get

\[
(2.14)(a) \ K(X,Y,Z,T) = 2\alpha \beta \{A(X)g(Y,Z) - A(Y)g(X,Z)\} - (\alpha^2 - \beta^2) \{A(X)F(Y,Z) - A(Y)F(X,Z)\}
\]

And

\[
(2.14)(b) \ K(X,Y,Z,Z,T) = 0
\]

Thus, we have

**THEOREM (2.3):** In a Trans-Sasakian manifold $M_\alpha$, we have

\[
K(X,Y,Z,T) = 2\alpha \beta \{A(X)g(Y,Z) - A(Y)g(X,Z)\} - (\alpha^2 - \beta^2) \{A(X)F(Y,Z) - A(Y)F(X,Z)\}
\]

And

\[
K(X,Y,Z,Z,T) = 0
\]

III. Conclusion

Trans-Sasakian manifold admitting a killing structure vector is an $(\alpha, 0)$ type Trans-Sasakian manifold. And a Trans-Sasakian manifold with structure 1-form $A$ is closed, becomes $(\beta, 0)$ type Trans-Sasakian manifold.

**References**