A Multi – Timing Perturbation Analysis of the Deformation and Dynamic Buckling Of A Viscously Damped Toroidal Shell Segment Stressed By a Step Load

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Abstract:
This paper is concerned with analytical determination of (a) the normal displacement and Airy stress function of an imperfect viscously damped toroidal shell segment subjected to a step load and (b) the dynamic buckling load of the structure using perturbation technique in asymptotic procedures. The continuously differentiable imperfection is assumed in the form of a two – term Fourier series expansion while homogeneous initial and boundary conditions are assumed. The formulation contains a small parameter depicting the amplitude of imperfection and upon which a multi – timing regular perturbation procedure is initiated, while the light viscous damping is of some order of imperfection. Simply – supported boundary conditions are assumed and in the final analysis, a uniformly valid asymptotic formula for determining the dynamic buckling load is determined nontrivially. The dynamic buckling load is related to the corresponding static buckling load and the relationship is independent of imperfection parameter. Besides, the dynamic load is found to depend, among other things, on the Fourier coefficients while the formula for determining the dynamic buckling load is implicit in the load parameter.

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I. Introduction
As in the case of elastic cylindrical shell segments, elastic toroidal segments are imperfection – sensitive structural materials that don’t seem to attract as much attention as other imperfection – sensitive materials such as columns, plates and even spherical shells. To our knowledge, strictly analytical investigations into the dynamic behavior of toroidal shell segments appear rather scanty and far between. Perhaps, one of the earliest investigations into toroidal shell segments was that by Stein and Mc Elman [1] whereby the buckling of toroidal shells (static buckling) was discussed. Later, Hutchinson [2] studied the initial postbuckling behavior of toroidal shell segment. In yet another investigation, Oyesanya [3] investigated the asymptotic analysis, a uniformly valid asymptotic formula for determining the dynamic buckling load is determined nontrivially. The dynamic buckling load is related to the corresponding static buckling load and the relationship is independent of imperfection parameter. In a similar investigation, Ette et al. [5] analyzed the normal response and buckling of a toroidal shell segment pressurized by a static load, while Ette et al. [6], in the like manner, studied the deformation and static buckling of a toroidal shell segment using a two – term Fourier series imperfection.

II. Dynamic buckling Criteria
Globally, the Budiansky / Roth criterion [8] and the Budiansky / Hutchinson criterion [9] appear to be popular and of wider applicability. In the first case (ieBudiansky / Roth criterion), the response (displacement) is plotted against the applied load and the particular load that initiates a sudden jump in the displacement is regarded as the dynamic buckling load. This criterion easily lends itself to easy application of numerical techniques such as Finite element method and to easy computer application. However, in the Budiansky / Hutchinson criterion, the dynamic buckling load is defined as the largest load parameter for the solution of the problem to remain bounded and the condition [9] for dynamic buckling is

$$\frac{d\lambda}{dU_a} = 0$$

where \(\lambda\) is the load parameter and \(U_a\) is the maximum displacement. This second criterion easily lends itself to easy application, through phase plane portriate, of the dynamic buckling analyses of some simple elastic

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one-dimensional materials under step load. It also lends itself to dynamic buckling investigations of some much more complicated multi-dimensional elastic structures such as cylindrical, spherical and even toroidal shells, hence its preference in this investigation.

In order to utilize equation (1), our initial aim will be to apply a two-timing regular perturbation technique and obtain a uniformly valid asymptotic expansion of the maximum displacement \( U_x \) subsequent upon which the invocation of (1) is initiated to obtain the dynamic buckling load. Similar perturbation techniques were initiated by Kervokian [10], Kuzmak [11], Luke [12], Li and Kervokian [13], Danielson [14] and Lockhart and Amazigo [15]. Mention must also be made of investigations by Reda and Forbes [16], Priyardarsini et al. [17], Mc Shane et al. [18], Kubiak [19] and Kolakowski and Mania [20].

III. Formulation of the Problem
By adjusting the associated Karman – Donnell equilibrium and compatibility equations in [3, 4], to the dynamic setting, the equations satisfied by the normal displacement \( W(X, Y, T) \) and Airy stress function \( F(X, Y, T) \) of the undamped toroidal shell segments are respectively

\[
\rho W_{TT} + D \nabla^4 W + \frac{1}{a} F_{,XX} + \frac{1}{b} F_{,YY} + P \left[ \frac{1}{2} (W + \overline{W})_{,XX} + \left( 1 - \frac{1}{2} \frac{a}{b} \right) (W + \overline{W})_{,YY} \right] = \hat{S}(W + \overline{W}, F) \tag{2}
\]

and

\[
\frac{1}{Eh} \nabla^4 F - \frac{1}{a} W_{,XX} - \frac{1}{b} F_{,YY} = -\frac{1}{2} \hat{S}(W + \overline{W}, W) \tag{3}
\]

\[
W = W_{,XX} = F = F_{,XX} = 0 \quad \text{at} \quad X = 0, L
\]

\[
W(X, Y, 0) = W_T(X, Y, 0) = 0
\]

\[
0 < X < L, \quad 0 < Y < a
\]

Here, equations (2) and (3) are the equations of motion and compatibility respectively, \( X, Y \) and \( T \) are the axial, circumferential and time variables respectively while \( \rho \) is the mass per unit area. Similarly, \( E \) and \( h \) are the Young’s modulus and thickness respectively, \( a \) and \( b \) are the two radii of the toroidal shell. \( P(T) \) is the step load, while \( D = \frac{Eh^3}{12(1-\sigma^2)} \) is the bending stiffness where \( \sigma \) is the Poisson’s ratio. \( \nabla^4 \) is the two-dimensional biharmonic operator defined as \( \nabla^4 \equiv \left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right)^2 \), while \( \hat{S} \) is a symmetric bilinear differential functional defined as

\[
\hat{S}(P, Q) = P_{XX} Q_{YY} + P_{YY} Q_{XX} - 2P_{XY} Q_{XY}.
\]

In addition, \( \overline{W} \) is a time-independent stress-free continuously differentiable normal displacement acting as the imperfection and a subscript following a comma denotes partial differentiation.

IV. Nondimensionalization of relevant equations
As in [3, 4] and [15], we now let

\[
P(t) = \begin{cases} \bar{P}, & T > 0, \\ 0, & T < 0, \end{cases} \quad x = \frac{\pi X}{L}, \quad y = \frac{2\pi Y}{a}, \quad \varepsilon \overline{W} = \frac{W}{h},
\]

\[
w = \frac{W}{h}, \quad \lambda g(t) = \frac{L^2 a \bar{P}}{\pi^2 D}, \quad g(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0 \end{cases}, \quad A = \frac{L^2 \sqrt{12(1-\sigma^2)}}{\pi \varepsilon ah},
\]

\[
H = \frac{h}{a}, \quad \xi = \frac{L^2}{(\pi a)^2}, \quad K(\xi) = -\left( \frac{A}{1+\xi} \right)^2, \quad t = \frac{T \pi^2 D / h}{L^2}, \quad 0 < \varepsilon \ll 1, \quad 0 < \xi \ll 1.
\]

Here, we have neglected the axial and circumferential inertia and have similarly neglected the boundary-layer effect by assuming that the pre-buckling deflection is constant so that

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The classical buckling load approximations, while the parameter \( \alpha = 1 \) if pressure contributes to axial stress through and plates otherwise \( \alpha = 0 \) if pressure acts laterally. By introducing the light viscous damping term, namely

\[
\begin{align*}
F &= \frac{1}{2} \bar{p} a \left( X^2 + \frac{1}{2} a Y^2 \right) + \frac{E h^2 L^2}{\pi^2 a (1 + \xi)^2} \bar{f} \\
W &= \frac{\bar{p} a^2 (1 - \alpha v)}{E h} + h w
\end{align*}
\]

(8)

(9)

The first terms in (8) and (9) are the pre–buckling approximations. On substituting (17a) into (15) and simplifying, yields

\[
\begin{align*}
w_{tt} + 2 \epsilon^2 w_t + \nabla^4 w - K(\xi) \left( f_{xx} + \xi f_{yy} \right) + \lambda \left[ \frac{\alpha}{2} (w + \epsilon \bar{w})_{xx} + \xi \left( 1 - \frac{\alpha}{2} \right) (w + \epsilon \bar{w})_{yy} \right] &= -K(\xi) H S (f, w + \epsilon \bar{w}), \\
t > 0
\end{align*}
\]

(10)

and

\[
\begin{align*}
\nabla^4 \tilde{f} - (1 + \xi)^2 (w_{xx} + \xi w_{yy}) &= - \frac{1}{2} H (1 + \xi)^2 S (w + \epsilon \bar{w}, w) \\
w &= w_{xx} = \tilde{f} = \tilde{f}_{xx} = 0 \text{ at } x = 0, \pi, \quad 0 < x < \pi, \quad 0 < y < 2\pi, \quad t > 0, \quad r = \frac{a}{b}
\end{align*}
\]

(11)

(12)

\[
w(x, y, 0) = w_t(x, y, 0) = 0
\]

(13)

Here,

\[
S(p, q) = p_{xx} q_{yy} + p_{yy} q_{xx} - 2 p_{xy} q_{xy} \text{ and } \nabla^4 = \left( \frac{\partial^2}{\partial x^2} + \xi \frac{\partial^2}{\partial y^2} \right)^2
\]

V. Classical Theory

The necessary equations in this case are obtained from (9) and (10) by neglecting all the time related terms and all forms of nonlinearity as well as imperfection. The relevant equations are

\[
\begin{align*}
\nabla^4 w - K(\xi) \left( f_{xx} + \xi f_{yy} \right) + \lambda \left[ \frac{\alpha}{2} w_{xx} + \xi \left( 1 - \frac{\alpha}{2} \right) w_{yy} \right] &= 0 \\
\nabla^4 f - (1 + \xi)^2 (w_{xx} + \xi w_{yy}) &= - \frac{1}{2} H (1 + \xi) S (w + \epsilon \bar{w}, w)
\end{align*}
\]

(14)

(15)

\[
w = w_{xx} = \tilde{f} = \tilde{f}_{xx} = 0 \text{ at } x = 0, \pi
\]

The classical buckling load \( \lambda_c \) is sought by letting

\[
(w, \tilde{f}) = (a_{mk}, b_{mk}) \sin mx \sin (ky + \phi_{mk})
\]

(16)

where \( \phi_{mk} \) is an inconsequential phase and where \( (a_{mk}, b_{mk}) \neq (0, 0) \). On substituting (16) in (15), yields

\[
b_{mk} = \frac{- (1 + \xi)^2 m^2 a_{mk}}{(m^2 + \xi k^2)^2 + (1 + \xi)^2 \xi k^2}
\]

(17a)

On substituting (17a) into (15) and simplifying, yields

\[
(m^2 + \xi k^2)^2 - \lambda \left( \frac{am^2}{2} + \xi k^2 \left( 1 - \frac{\alpha}{2} \right) \right) - \frac{K(\xi) \left( m^2 + \xi k^2 \right)(1 + \xi)^2}{(m^2 + \xi k^2)^2 + (1 + \xi)^2 \xi k^2} = 0
\]

The classical buckling load \( \lambda_c \) is determined from the maximization (assuming that \( k \) varies continuously) \( \frac{d\lambda}{dk} = 0 \). Thus, if \( k = n \) is the value of \( k \) at classical buckling, we get

\[
\lambda_c = \frac{(m^2 + \xi n^2)^2 - \frac{K(\xi)(1 + \xi)^2 \left( m^2 + \xi n^2 \right)}{(m^2 + \xi n^2)^2 + (1 + \xi)^2 \xi n^2} \left( \frac{a m^2}{2} + \left( 1 - \frac{\alpha}{2} \right) \xi n^2 \right)}{ \frac{a m^2}{2} + \left( 1 - \frac{\alpha}{2} \right) \xi n^2}\]

(17b)
On substituting for \(K(\xi)\) from (7) into (17b) and letting \(m = 1, \xi = \xi n^2\), we get

\[
\lambda_c = \left(1 + \zeta\right)^2 + \frac{\alpha^2(1+\xi)}{(1+\xi)^2(1+\zeta^2)\zeta r}
\]

(18a)

It follows that

\[
(w,f) = \left(1, \frac{-(1+\xi)^2}{(1+\zeta)^2(1+\zeta^2)\zeta r}\right) a_{in} \sin x \sin \left(ny + \phi_{in}\right)
\]

(18b)

VI. Perturbation Solution of the Dynamic Problem

Let

\[\tau = \epsilon^2 t, \quad w(x,y,t,\epsilon) = U(x,y,t,\tau,\epsilon), \quad f(x,y,t,\epsilon) = f(x,y,t,\tau,\epsilon)\]

\[w_\tau = U_{,t} + \epsilon^2 U_{,t\tau}, \quad w_{,tt} = U_{,tt} + 2\epsilon^2 U_{,t\tau} + \epsilon^4 U_{,t\tau\tau}\]

Further let

\[
\begin{align*}
(U(x,y,t,\tau,\epsilon), f(x,y,t,\tau,\epsilon)) &= \sum_{i=1}^{\infty} (U^{(i)}(x), f^{(i)}(x)) \epsilon^i \\

(19a)
\end{align*}
\]

(19b)

Substituting (19a, b) and (20) into (10) and (11), yields

\[
\begin{align*}
O(\epsilon) &= \left\{
\begin{array}{ll}
U^{(1)}_{,tt} & - K(\xi)(f^{(1)}_{,xx} + \xi f^{(1)}_{,yy}) = \frac{\alpha}{2} (U^{(1)} + \bar{w})_{,xx} + \xi \left(1 - \frac{\alpha}{2}\right) (U^{(1)} + \bar{w})_{,yy} \\
\end{array}
\right.
\end{align*}
\]

(21)

\[
\begin{align*}
\nabla^4 f^{(1)} - (1 + \xi)^2 (U^{(1)}_{,xx} + \xi U^{(1)}_{,yy}) &= 0 \\
\nabla^4 f^{(1)} - (1 + \xi)^2 (U^{(1)}_{,xx} + \xi U^{(1)}_{,yy}) &= 0
\end{align*}
\]

(22)

\[
\begin{align*}
O(\epsilon^2) &= \left\{
\begin{array}{ll}
U^{(2)}_{,tt} & - K(\xi)(f^{(2)}_{,xx} + \xi f^{(2)}_{,yy}) + \frac{\alpha}{2} (U^{(2)} + \bar{w})_{,xx} + \xi \left(1 - \frac{\alpha}{2}\right) (U^{(2)} + \bar{w})_{,yy} \\
\end{array}
\right.
\end{align*}
\]

(23)

\[
\begin{align*}
\nabla^4 f^{(2)} - (1 + \xi)^2 (w^{(2)}_{,xx} + \xi r w^{(2)}_{,yy}) &= - \frac{1}{2} \left(1 + \xi\right) \left[s(w^{(1)}_{,xx} + \xi r w^{(2)}_{,yy})]\right.
\end{align*}
\]

(24)

\[
\begin{align*}
O(\epsilon^3) &= \left\{
\begin{array}{ll}
U^{(3)}_{,tt} & - K(\xi)(f^{(3)}_{,xx} + \xi f^{(3)}_{,yy}) + \frac{\alpha}{2} (U^{(3)} + \bar{w})_{,xx} + \xi \left(1 - \frac{\alpha}{2}\right) (U^{(3)} + \bar{w})_{,yy} \\
\end{array}
\right.
\end{align*}
\]

(25)

\[
\begin{align*}
\nabla^4 f^{(3)} - (1 + \xi)^2 (U^{(3)}_{,xx} + \xi U^{(3)}_{,yy}) &= - \frac{1}{2} \left(1 + \xi\right) \left[s(U^{(1)}_{,xx} + \xi r U^{(3)}_{,yy})\right]
\end{align*}
\]

(26)

etc.

However, \((U^{(1)}, U^{(2)}) = (U^{(2)}, U^{(1)})\).

\[
\begin{align*}
U^{(i)}(x, y, 0, 0) &= f^{(i)}(x, y, 0, 0) = 0, \quad i = 1, 2, 3, \ldots \\
U^{(i)}_x(x, y, 0, 0) &= f^{(i)}_x(x, y, 0, 0) = 0, \quad k = 1, 2, \ldots \\
U^{(i)}(x, y, 0, 0) &= U^{(i-2)}(x, y, 0, 0) = 0, \quad r = 3, 4, \ldots
\end{align*}
\]

(27)

\[
U^{(i)} = U^{(i)}_{,xx} = f^{(i)}_{,xx} = 0 \quad \text{at} \quad x = 0, \quad \pi i, \quad i = 1, 2, 3, \ldots
\]

(28)

In line with the boundary conditions, we shall let

\[
\bar{w} = \left(a \cos ny + b \sin ny\right) \sin mx
\]

(29)

Generally, the solution of equations of any order of perturbation will be of the form

\[
(U^{(i)}_{,f^{(i)}}) = \sum_{p=1,q=1}^{\infty} \left[\begin{array}{c}
(U^{(i)}_{,f^{(i)}})_{1} \cos qy + (U^{(i)}_{,f^{(i)}})_{2} \sin qy
\end{array}\right] \sin px
\]

(30)

Hence, using (30), the following will be of general applicability.
\[ U^{(i)}_{1t} + \bar{V}^4 U^{(i)} - K(\xi) \left( f_x^{(i)} + \xi f_{xy}^{(i)} \right) + \lambda \left[ \frac{\alpha}{2} U_{xx}^{(i)} + \xi \left( 1 - \frac{\alpha}{2} \right) U_{yy}^{(i)} \right] \]

\[ \equiv \sum_{p=1, q=1}^{m, n} \left\{ \left( \frac{p^2 + \xi q^2}{2} u_1^{(i)} + \frac{(p^2 + \xi q^2) u_2^{(i)}}{2} \right) \sin p \sin q \right\} \]

\[ + \left\{ \left( \frac{p^2 + \xi q^2}{2} u_1^{(i)} + \frac{(p^2 + \xi q^2) u_2^{(i)}}{2} \right) \sin p \sin q \right\} \]

and

\[ \bar{V}^4 f^{(i)} - (1 + \xi)^2 \left( U_{xx}^{(i)} + \xi U_{yy}^{(i)} \right) \]

\[ \equiv \sum_{p=1, q=1}^{m, n} \left\{ \left( \frac{(p^2 + \xi q^2) f_1^{(i)}}{2} + (1 + \xi)^2 (q^2 r \xi - p^2) U_{1}^{(i)} \right) \sin p \sin q \right\} \]

Any integration with respect to \( x \) will have 0 and \( \pi \) as the lower and upper limits respectively while a similar integration with respect to \( y \) has 0 and 2\( \pi \) as the lower and upper limits respectively.

**Solution of Equations of order \( \epsilon \)**

Substituting (30) into (22) and first multiplying through by \( \cos n \phi x \) and next, by \( \sin n \phi x \) and for \( p = m \) and \( q = n \) in each case, we get

\[ f_1^{(i)} = \frac{-(1 + \xi)^2 (n^2 \xi - m^2) w_1^{(i)}}{(m^2 + \xi n^2)^2}, \quad f_2^{(i)} = \frac{-(1 + \xi)^2 (n^2 \xi - m^2) w_2^{(i)}}{(m^2 + \xi n^2)^2} \]

Next, substituting (30) into (21), noting (31) and multiplying, first by \( \cos n \phi x \) and next, by \( \sin n \phi x \), it is noted that for \( p = m, q = n \), the following result equations are obtained (after substituting for \( f_1^{(i)} \) and \( f_2^{(i)} \) from (33))

\[ U_{1,tt}^{(i)} + \varphi q^2 U_1^{(i)} = \bar{a} B^{(i)}; \quad U_{2,tt}^{(i)} + \varphi q^2 U_2^{(i)} = \bar{b} B^{(i)} \]

\[ B^{(i)} = \lambda \left( \frac{a m^2}{2} + \frac{1 - \alpha}{2} \right) \xi n^2 \]

\[ U_1^{(i)}(0,0) = 0, \quad U_1^{(i)}(0,0) = 0; \quad U_2^{(i)}(0,0) = 0, \quad U_2^{(i)}(0,0) = 0 \]

\[ \varphi = \left[ \left( m^2 + \xi n^2 \right)^2 + \left( \frac{m A}{1 + \xi} \right) + n^2 \xi \right] \left( 1 + \xi \right)^2 \frac{(n^2 \xi - m^2)}{(m^2 + \xi n^2)^2} - \lambda \left( \frac{a m^2}{2} + \frac{1 - \alpha}{2} \right) \xi n^2 \]

\[ \varphi > 0 \vee m, n. \quad (35) \]

The solutions of (34a) – (35) are

\[ U_1^{(i)} = \delta_1^{(i)} (\tau) \cos \phi t + \beta_1^{(i)} (\tau) \sin \phi t + \bar{a} B^{(i)} \]

\[ B = \frac{\varphi q^2}{\varphi} \]

\[ \delta_1^{(i)} (0) = -\bar{a} B, \quad \beta_1^{(i)} (0) = 0 \]

\[ U_2^{(i)} = \delta_2^{(i)} (\tau) \cos \phi t + \beta_2^{(i)} (\tau) \sin \phi t + \bar{b} B \]

\[ \delta_2^{(i)} (0) = -\bar{b} B, \quad \beta_2^{(i)} (0) = 0 \]

So far, it is clear that in the final analysis, we shall have

\[ \left( U^{(i)} \right) = \left( \frac{1}{\varphi q_0} \right) \left( U_1^{(i)} \cos \phi y + U_2^{(i)} \sin y \right) \sin \phi y \sin \phi y \]

where,

\[ \varphi_0 = (1 + \xi)^2 \frac{n^2 \xi - m^2}{(m^2 + n^2 \xi)^2} \]

\[ \text{(38)} \]

**Solution of equations of Order \( \epsilon^2 \)**

On simplifying the right hand sides of (23) and (24) and equally simplifying their left hand sides using (31) and (32) and noting (30), we have, for \( i = 2 \).
\[
\sum_{p=1}^{\infty} \left\{ \left[ (p^2 + \xi q^2)U_1^{(p)} + (p^2K(\xi) - q^2r\xi)U_1^{(2)} - \lambda \left( \frac{ap^2}{2} + (1 - \frac{a}{2})\xi q^2 \right)U_1^{(2)} \right] \sin px \cos qy \right. \\
+ \left. \left\{ (p^2 + \xi q^2)U_2^{(p)} + (p^2K(\xi) - q^2r\xi)U_2^{(2)} - \lambda \left( \frac{ap^2}{2} + (1 - \frac{a}{2})\xi q^2 \right)U_2^{(2)} \right\} \sin px \sin qy \right\}
\]

For the solution of (40), we multiply both sides, first by \( \cos 2mn x \) and afterwards by \( \sin 2nysinxm \). In the first multiplication, we get, for \( p = m, q = 2n \)

\[
f_1^{(2)} = \frac{-2H(1 + \xi)^2mn^2}{\pi} \left( (U_1^{(1)2} - U_2^{(1)2} - \bar{a}U_1^{(1)} - \bar{b}U_2^{(1)}) \right)
\]

This can further be written as

\[
f_1^{(2)} = -\varphi_1 U_1^{(2)} + \varphi_2 \left( U_1^{(1)2} - U_2^{(1)2} + \bar{a}U_1^{(1)} + \bar{b}U_2^{(1)} \right)
\]

where,

\[
\varphi_1 = \frac{(1 + \xi)^2(4n^2r\xi - m^2)}{(m^2 + 4n^2\xi)^2}, \quad \varphi_2 = \frac{2H(1 + \xi)^2mn^2}{\pi(m^2 + 4n^2\xi)^2}
\]

The second multiplication gives, for \( m, q = 2n \),

\[
f_2^{(2)} = \frac{-2H(1 + \xi)^2mn^2}{\pi} \left( (U_1^{(1)2} - U_2^{(1)2} - \bar{a}U_1^{(1)} + \bar{b}U_2^{(1)}) \right)
\]

This can further be written as

\[
f_2^{(2)} = -\varphi_3 U_1^{(2)} + \varphi_4 \left( U_1^{(1)2} - U_2^{(1)2} - \bar{a}U_1^{(1)} + \bar{b}U_2^{(1)} \right)
\]

Next, multiplying (39), first by \( \cos 2mn x \) and after, by \( \sin 2nysinxm \) and for \( p = m, q = 2n \), the result (in the first multiplication) is

\[
U_{1,tt}^{(2)} + \Omega^2 U_1^{(2)} = \varphi_3 \left( U_1^{(1)2} - U_2^{(1)2} + \bar{a}U_1^{(1)} - \bar{b}U_2^{(1)} \right)
\]

\[
+ \varphi_4 \left( U_1^{(1)2} - U_2^{(1)2} - \bar{a}U_1^{(1)} - \bar{b}U_2^{(1)} \right)
\]

\[
U_1^{(2)}(0, 0) = 0, \quad U_1^{(2)}(0, 0) = 0
\]

where,

\[
\Omega^2 = \left( m^2 + 4\xi n^2 \right)^2 \left[ \frac{\left( \frac{am}{1+t} \right)^2 + 4n^2r\xi \xi q^2 - m^2}{(m^2 + 4n^2\xi)^2} - \lambda \left( \frac{am^2}{2} + (1 - \frac{1}{2})4\xi q^2 \right) \right]
\]

\[
> 0 \ \forall m, n.
\]

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\[ \varphi_3 = \frac{-2H (1 + \xi)^2 m n^2}{\pi (m^2 + 4n^2 \xi^2)}, \quad \varphi_4 = \frac{4H m n^2 \varphi_3 A^2}{(1 + \xi)^2} \] 

Equation (44a) can be further written as

\[ U_{1st}^{(2)} + \Omega^2 U_{11}^{(2)} = (\varphi_3 - \varphi_4) \left( U_1^{(1)} - U_2^{(1)} + (\bar{a}U_1^{(1)} - bU_2^{(1)}) \right) \]  

(45)

The second multiplication by \( \sin 2\pi n \sin m x \) in (39) yields

\[ U_{1st}^{(2)} + \Omega^2 U_{12}^{(2)} = (\varphi_3 + \varphi_4) \left( 2U_1^{(1)} + \bar{a}U_1^{(1)} U_1^{(1)} \right) \]  

\[ U_2^{(2)}(0, 0) = 0, \quad U_{12}^{(2)}(0, 0) = 0 \]  

(46a)

(46b)

The following simplifications are necessary in the solutions of (45) and (46a) that soon follow.

\[ U_1^{(1)} = \left[ \frac{1}{2} (\delta_1^{(1)} + \beta_2^{(1)}) + \alpha^2 B^2 \right] + 2ab \left( \delta_1^{(1)} \cos \varphi t + \beta_1^{(1)} \sin \varphi t \right) \]  

\[ + \frac{1}{2} (\delta_1^{(1)} - \beta_1^{(1)}) \cos 2\varphi t \]  

(47a)

\[ U_2^{(1)} = \left[ \frac{1}{2} (\delta_2^{(1)} + \beta_2^{(1)}) + \alpha^2 B^2 \right] + 2b \left( \delta_2^{(1)} \cos \varphi t + \beta_2^{(1)} \sin \varphi t \right) \]  

\[ + \frac{1}{2} (\delta_2^{(1)} - \beta_2^{(1)}) \cos 2\varphi t \]  

(47b)

\[ 2U_1^{(1)} U_2^{(1)} = 2 \left[ \frac{1}{2} (\delta_1^{(1)} \delta_2^{(1)} + \beta_1^{(1)} \beta_2^{(1)}) + B (\alpha \delta_1^{(1)} + \bar{a} \alpha \delta_2^{(1)}) \cos \varphi t + B (\bar{a} \beta_1^{(1)} + \alpha \beta_2^{(1)}) \sin \varphi t \right] \]  

\[ + \frac{1}{2} (\delta_1^{(1)} \beta_2^{(1)} - \beta_1^{(1)} \delta_2^{(1)}) \cos 2\varphi t + \frac{1}{2} (\delta_2^{(1)} \beta_1^{(1)} + \beta_2^{(1)} \delta_2^{(1)}) \sin 2\varphi t \]  

(47c)

Substituting (47a, b) in (45) and simplifying, yields

\[ U_{1st}^{(2)} + \Omega^2 U_{11}^{(2)} = (\varphi_3 - \varphi_4) (r_1 + r_2 \cos \varphi t + r_3 \sin \varphi t + r_4 \cos 2\varphi t) \]  

(48)

where,

\[ r_0 = \frac{1}{2} (\delta_1^{(1)} + \beta_1^{(1)}) + \alpha^2 B^2 \]  

\[ - \frac{1}{2} (\delta_2^{(1)} + \beta_2^{(1)}) + \alpha^2 B^2 \]  

\[ + B (\bar{a} - \bar{b}) \]  

(49a)

\[ r_1 = 2b (\bar{a} + \bar{b}) (\delta_1^{(1)} - \delta_2^{(1)}) \]  

\[ + (\bar{a} \delta_1^{(1)} - \bar{b} \delta_2^{(1)}) \]  

(49b)

\[ r_2 = 2b (\bar{a} + \bar{b}) (\beta_1^{(1)} - \beta_2^{(1)}) \]  

\[ + (\bar{a} \beta_1^{(1)} - \bar{b} \beta_2^{(1)}) \]  

(49c)

\[ r_3 = (\delta_1^{(1)} \beta_1^{(1)} - \delta_2^{(1)} \beta_2^{(1)}) \]  

\[ r_4 = \frac{1}{2} \left( \left( \delta_1^{(1)} - \beta_1^{(1)} \right) - \left( \delta_2^{(1)} - \beta_2^{(1)} \right) \right) \]  

\[ r_0(0) = (\bar{a} - \bar{b}) \left( \frac{3B^2}{2} + B \right), \quad r_1(0) = (\bar{a} - \bar{b}) \left( 2B^2 + B \right) \]  

(49d)

\[ r_2(0) = r_3(0) = 0; \quad r_4(0) = \frac{B^2}{2} \]  

(49f)

The solution of (48) and (49a) - (f), using (44b), is

\[ U_{1}^{(2)}(t, r) = \delta_1^{(2)}(r) \cos \Omega t + \beta_1^{(2)}(r) \sin \Omega t \]  

\[ + (\varphi_3 - \varphi_4) \left[ \frac{r_0}{\Omega^2} + \frac{1}{\Omega^2 - \varphi^2} \left( r_1 \cos \varphi t + r_2 \sin \varphi t \right) \right] \]  

\[ + \left( \frac{1}{\Omega^2 - \varphi^2} \right) \left( r_3 \sin 2\varphi t + r_4 \cos 2\varphi t \right) \]  

(50)

\[ \delta_1^{(2)}(0) = - (\varphi_3 - \varphi_4) (\bar{a} - \bar{b}) \left( \frac{3B^2 + 2B^2 + B^2}{2\Omega^2 + 2\Omega^2 - 4\varphi^2} \right) \]  

\[ \beta_1^{(2)}(0) = 0 \]  

(51a)

\[ \beta_1^{(2)}(0) = 0 \]  

(51b)

Substituting (47c) into (46a) and simplifying, yields

\[ U_{22}^{(2)} + \Omega^2 U_{22}^{(2)} = (\varphi_3 + \varphi_4) (r_5 + r_6 \cos \varphi t + r_7 \sin \varphi t + r_8 \cos 2\varphi t + r_9 \sin 2\varphi t) \]  

(52)

where,

\[ r_5 = \left( \delta_1^{(1)} \delta_2^{(1)} + \beta_1^{(1)} \beta_2^{(1)} + 2\bar{a} \bar{b} B \right) \]  

\[ r_6 = \left( 2B (\bar{a} \delta_1^{(1)} + \bar{a} \delta_2^{(1)}) + \bar{a} \beta_1^{(1)} + \bar{b} \delta_1^{(1)} \right) \]  

\[ r_7 = \left( 2B (\bar{a} \beta_1^{(1)} + \bar{b} \beta_1^{(1)}) + \bar{a} \beta_2^{(1)} + \bar{b} \beta_2^{(1)} \right) \]  

\[ r_8 = \left( \delta_1^{(1)} \beta_2^{(1)} - \beta_1^{(1)} \beta_2^{(1)} \right) \]  

\[ r_9 = \left( \delta_1^{(1)} \beta_1^{(1)} - \delta_2^{(1)} \beta_2^{(1)} \right) \]  

\[ r_5(0) = \bar{a} \bar{b} B^2 + 2\bar{a} \bar{b} B, \quad r_6(0) = 2\bar{a} \bar{b} (B^2 + B) \]  

\[ r_7(0) = 0, \quad r_8(0) = \bar{a} \bar{b} B^2, \quad r_9(0) = 0 \]  

The solution of (52), using (46b) yields

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\[ U_2(t, \tau) = \delta_2(t) \cos \omega t + \beta_2(t) \sin \omega t \]
\[ + (\varphi_3 + \varphi_4) \left[ \frac{r_5}{\Omega^2} + \frac{1}{\Omega^2 - \varphi^2} \right] (r_6 \cos \varphi t + r_7 \sin \varphi t) \]
\[ + \left( \frac{1}{\Omega^2 - 4\varphi^2} \right) (r_6 \cos 2\varphi t + r_7 \sin 2\varphi t) \]
\[ + r_3 \sin 2\varphi t \] 
\[ (53a) \]

where,
\[ \delta_2(0) = - (\varphi_3 + \varphi_4) \left[ \frac{r_5}{\Omega^2} + \frac{r_6}{\Omega^2 - \varphi^2} + \frac{r_7}{\Omega^2 - 4\varphi^2} \right] \bigg|_{\tau=0} \]
\[ = - (\varphi_3 + \varphi_4) a \beta B^2 \left[ \frac{1}{2\Omega^2} + \frac{1}{\Omega^2 - \varphi^2} + \frac{1}{2(\Omega^2 - 4\varphi^2)} \right] + O(\beta), \quad \beta_2(0) = 0 \]
\[ (53b) \]

**Solution of equations of Order e^3**

After simplifying the right hand sides of (25) and (26), and substituting on their left hand sides, using (31) and (32) for \( i = 3 \), the resultant equations are

\[ \sum_{p=1,q=1} \left\{ U_{1,2t}^{(3)} + (p^2 + \xi q^2) U_{1}^{(3)} + (p^2K(\xi) - q^2r) f_{1}^{(3)} - \lambda \left( \frac{a p^2}{2} + \left( 1 - \frac{a}{2} \right) \xi q^2 \right) u_2^{(3)} \right\} \sin \pi \xi \sin \pi \eta = \]
\[ - K(\xi) H \left( mn \right)^2 \left[ \frac{5}{4} \left( f_{1}^{(1)} U_{1}^{(2)} + f_{2}^{(1)} U_{2}^{(2)} \right) \cos \eta y + \left( f_{1}^{(1)} U_{1}^{(2)} - f_{2}^{(1)} U_{2}^{(2)} \right) \sin \eta y \right] \]
\[ + \left( f_{1}^{(1)} U_{1}^{(2)} + f_{2}^{(1)} U_{2}^{(2)} \right) \sin 3\eta y + \left( f_{1}^{(2)} U_{1}^{(2)} - f_{2}^{(2)} U_{2}^{(2)} \right) \cos 3\eta y \right] (1 - \cos 2\pi \mu x) \]
\[ = \left( f_{1}^{(1)} U_{1}^{(2)} + f_{2}^{(1)} U_{2}^{(2)} \right) \sin \eta y + \left( f_{1}^{(2)} U_{1}^{(2)} - f_{2}^{(2)} U_{2}^{(2)} \right) \cos \eta y \right] (1 + \cos 2\pi \mu x) \]
\[ = \frac{5}{4} \left( \tilde{a} f_{1}^{(2)} + \tilde{b} f_{2}^{(2)} \right) \cos \eta y + \left( \tilde{a} f_{1}^{(2)} - \tilde{b} f_{2}^{(2)} \right) \sin \eta y \]
\[ = \left( \tilde{b} f_{1}^{(2)} + \tilde{a} f_{2}^{(2)} \right) \cos \eta y + \left( \tilde{b} f_{1}^{(2)} - \tilde{a} f_{2}^{(2)} \right) \sin \eta y \right] \] 
\[ (54) \]

\[ U_1^{(3)}(0,0) = 0, \quad U_1^{(3)}(0,0) + U_1^{(1)}(0,0) = 0 \]
\[ (55) \]
\[
\sum_{p=1,q=1}^\infty \left[ \left\{ (p^2 + \xi q^2) f_1^{(3)} + (1 + \xi)(q^2 r \xi - p^2) U_2^{(3)} \right\} \sin p x \cos q y \\
+ \left\{ (p^2 + \xi q^2) f_2^{(3)} + (1 + \xi)(q^2 r \xi - p^2) U_2^{(3)} \right\} \sin p x \sin q y \right] \\
- \frac{1}{4} \left\{ (u_1^{(1)} u_1^{(2)} + u_2^{(1)} u_2^{(2)}) \cos n y + (u_1^{(1)} u_2^{(2)} - u_2^{(1)} u_1^{(2)}) \sin n y \right\} \\
\left\{ \frac{1}{4} \left( a u_1^{(2)} + b u_2^{(2)} \right) \cos n y + \left( a u_2^{(2)} - b u_1^{(2)} \right) \sin n y \right\} \left( 1 - \cos 2m x \right) \right]
\]

\[\begin{align*}
U_2^{(3)}(0, 0) = 0, & \quad U_2^{(3)}(0, 0) + U_2^{(1)}(0, 0) = 0 \quad (57) \\
A \text{ careful observation of (54) and (56) shows that there will be buckling modes in the shapes of } \cos n y \sin m x, \sin n y \sin m x, \cos 3 n y \sin m x \text{ and } \sin 3 n y \sin m x. \text{ To determine the associated Airy stress function in the shape of } \cos n y \sin m x, \text{ we multiply (56) by } \cos n y \sin m x \text{ and for } p = m, q = n, \text{ we get} \\
f_1^{(3)}(m, n) = -\frac{1}{(m^2 + n^2 \xi)^2} \left( 1 + \xi \right) (q^2 r \xi - m^2) U_1^{(3)}(m, n) \\
+ \frac{4H(1 + \xi)2mn}{\pi} \left( u_1^{(1)} u_1^{(2)} + u_2^{(1)} u_2^{(2)} \right) \\
+ \frac{1}{2} \left( \frac{1}{4} \left( a u_1^{(2)} + b u_2^{(2)} \right) \cos n y + \left( a u_2^{(2)} - b u_1^{(2)} \right) \sin n y \right) \left( 1 - \cos 2m x \right) \right] \\
\left( 58 \right) \\
\text{This is valid for } m \text{ odd. Next, multiplying (56) by } \sin n y \sin m x \text{ and for } p = m, q = n, \text{ the Airy stress function is} \\
f_2^{(3)}(m, n) = -\frac{1}{(m^2 + n^2 \xi)^2} \left( 1 + \xi \right) (q^2 r \xi - m^2) U_2^{(3)}(m, n) \\
+ \frac{4H(1 + \xi)2mn}{\pi} \left( u_1^{(1)} u_2^{(2)} - u_2^{(1)} u_1^{(2)} \right) \\
+ \frac{1}{2} \left( \frac{1}{4} \left( a u_2^{(2)} + b u_2^{(2)} \right) \cos n y + \left( a u_1^{(2)} - b u_2^{(2)} \right) \sin n y \right) \left( 1 - \cos 2m x \right) \right] \\
\left( 59 \right) \\
\text{valid for } m \text{ odd. Multiplying (56) by } \cos 3 n y \sin m x \text{ and for } p = m, q = 3n, \text{ we get} \\
f_1^{(3)}(m, 3n) = -\frac{1}{(m^2 + 9n^2 \xi)^2} \left( 1 + \xi \right) (9q^2 r \xi - m^2) U_1^{(3)}(m, 3n) \\
+ \frac{28H(1 + \xi)2mn}{3\pi} \left( u_1^{(1)} u_1^{(2)} - u_2^{(1)} u_2^{(2)} \right) \\
+ \frac{1}{2} \left( \frac{1}{4} \left( a u_2^{(2)} - b u_1^{(2)} \right) \cos n y + \left( a u_1^{(2)} + b u_2^{(2)} \right) \sin n y \right) \left( 1 - \cos 2m x \right) \right] \\
\left( 60 \right) \\
\text{valid for } m \text{ odd. Lastly, we multiply (56) by } \sin 3 n y \sin m x \text{ and for } p = m, q = 3n, \text{ and get}
valid for m odd. Next, multiplying (54) by \(\cos \text{sinsm} x\), we observe that for \(p = m, q = n\), we get (after substituting for \(f_{1(m,n)}^{(3)}\) from (58))

\[
U_{1,1}^{(3)} + \varphi^2 U_{1}^{(3)} = -\varphi_3 \left[ U_1^{(1)} U_2^{(1)} + U_2^{(1)} U_1^{(2)} + \frac{1}{2} (\bar{a} U_1^{(2)} + \bar{b} U_2^{(2)}) \right] \\
- \varphi_6 \left[ (f_1^{(1)} U_1^{(2)} + f_2^{(1)} U_2^{(2)}) + (\bar{a} f_2^{(2)} + \bar{b} f_2^{(2)}) + (U_2^{(1)} f_2^{(2)} + U_1^{(1)} f_1^{(2)}) \right] \\
- 2(U_1^{(1)} + U_1^{(1)})
\]

(62)

\[
U_{1,1}^{(3)}(0, 0) = 0, \quad U_{1,1}^{(3)}(0, 0) + U_{1,1}^{(1)}(0, 0) = 0
\]

(63)

\[
\varphi_3 = \frac{4H(1 + \xi)^2 m n^2}{\pi (m^2 + n^2 \xi)^2}, \quad \varphi_6 = \frac{4HK(\xi)n^2}{\pi}
\]

(64)

Multiplying (54) by \(\sin \text{sinsm} x\) and for \(p = m, q = n\), we get

\[
U_{2,2}^{(3)} + \varphi^2 U_{2}^{(3)} = [\varphi_3 (U_1^{(1)} U_2^{(1)} - U_1^{(1)} U_2^{(2)}) - \varphi_6 (-\varphi_3 U_1^{(1)} U_2^{(2)})] + \varphi_3 U_2^{(1)} U_2^{(1)} + U_1^{(1)} f_2^{(2)} - U_1^{(1)} f_1^{(2)} \\
- 2(U_1^{(1)} + U_2^{(1)})
\]

(65)

\[
U_{2,2}^{(3)}(0, 0) = 0, \quad U_{2,2}^{(3)}(0, 0) + U_{2,2}^{(1)}(0, 0) = 0
\]

(66)

In (65), we have retained only the terms that are cubic in displacement on expansion. The following simplifications are necessary

\[
U_{1,1}^{(2)} f_1^{(1)} = -\varphi_0 U_1^{(1)} U_1^{(2)}, \quad U_{2,2}^{(2)} f_1^{(1)} = -\varphi_0 U_2^{(1)} U_2^{(2)}, \quad U_2^{(1)} f_2^{(1)} = -\varphi_0 U_2^{(1)} U_2^{(2)} + 2\varphi_0 U_1^{(1)} U_2^{(2)} \\
U_{1,1}^{(1)} f_1^{(2)} = -\varphi_7 U_1^{(1)} U_1^{(2)} + \varphi_0 (U_1^{(1)} U_2^{(2)} - U_1^{(1)} U_2^{(2)}) \\
U_{1,1}^{(2)} = (\delta_1^1 \cos \varphi t + \beta_1^1 \sin \varphi t + \bar{a} B)^3
\]

(67a)

where,

\[
r_{55} = \left( \frac{3 \delta_1^1 B}{2} - \frac{3 \beta_1^1 B}{2} + \bar{a} B \right), \quad r_{56} = \left( \frac{3 \delta_1^1}{4} + \frac{3 \beta_1^1}{4} + \delta_1^1 B \right)
\]

(67b)

\[
r_{57} = \frac{3}{4} \left( \delta_1^1 \beta_1^1 + \delta_1^2 \beta_1^1 \right) + 3 \beta_1^1 B, \quad r_{58} = \frac{3}{2} \left( \delta_1^1 B - \beta_1^1 B \right)
\]

(67c)

\[
r_{59} = 3 \delta_1^1 \beta_1^1 B, \quad r_{60} = \frac{1}{4} \left( \delta_1^1 \beta_1^2 - \delta_1^1 \beta_1^2 \right), \quad r_{61} = \frac{1}{4} \left( \delta_1^1 \beta_1^2 - \delta_1^1 \beta_1^2 \right)
\]

(67d)

\[
r_{55}(0) = B^3 \left( \frac{3 \bar{a}^2}{2} + 1 \right), \quad r_{56}(0) = 3 B^3 \left( \frac{3 \bar{a}^3}{4} - \bar{a} B \right), \quad r_{57}(0) = 0
\]

(67e)

\[
r_{59}(0) = \frac{3 \bar{a}^3 B^3}{2}, \quad r_{55}(0) = 0, \quad r_{56}(0) = - \frac{3 \bar{a}^3 B^3}{4}, \quad r_{61}(0) = 0
\]

(67f)

We also have

\[
U_{1,1}^{(1)} U_2^{(1)} = r_{37} + r_{38} \cos \varphi t + r_{39} \sin \varphi t + r_{41} \cos 2\varphi t + r_{42} \sin 2\varphi t + r_{43} \cos 3\varphi t + r_{44} \sin 3\varphi t
\]

(68a)

\[
r_{37} = \delta_1^1 \left( 3 \tilde{B} + \tilde{B} \right) + \bar{a} B \beta_1^1 \beta_1^1 + \frac{1}{2} (\tilde{B}^2 + \beta_1^1 \beta_1^1)
\]

(68b)

\[
r_{38} = (\delta_1^1 \left( \frac{1}{2} \delta_1^2 \beta_1^2 + \delta_1^2 \beta_1^2 \right) + (\bar{a} B)^2) + \left( \delta_1^1 \left( \frac{1}{2} \beta_1^1 \beta_1^1 + \delta_1^2 \beta_1^2 \right) + \frac{1}{2} \delta_1^1 \beta_1^1 \beta_1^1 \right)
\]

(68c)

\[
r_{39} = \frac{1}{2} \delta_1^1 \left( \frac{1}{2} \delta_1^2 \beta_1^2 + \frac{1}{2} \beta_1^1 \delta_1^2 + \beta_1^1 \beta_1^1 \right) + (\bar{a} B)^2 \beta_1^1 \beta_1^1
\]

(68d)
We also simplify the following
Similarly, we have
where,

\[ r_{11} = (\varphi_3 - \varphi_4) \left[ \frac{1}{\Omega^2 - \varphi^2} \left( \frac{r_1 \delta_1^{(1)}}{2} + \frac{r_1 \beta_1^{(1)}}{2} \right) + \frac{\bar{a}B \varphi_3}{\Omega^2} \right] \]  

\[ r_{12} = (\varphi_3 - \varphi_4) \left[ \frac{r_2 \delta_1^{(1)}}{\Omega^2} + \frac{r_2 \beta_1^{(1)}}{2} + \frac{\bar{a}B \varphi_4}{\Omega^2 - \varphi^2} \right] \]  

\[ r_{13} = (\varphi_3 - \varphi_4) \left[ \frac{r_3 \delta_1^{(1)}}{\Omega^2 - 4\varphi^2} + \frac{r_3 \beta_1^{(1)}}{2} + \frac{\bar{a}B \varphi_4}{\Omega^2 - \varphi^2} \right] \]  

\[ r_{14} = (\varphi_3 - \varphi_4) \left[ \frac{r_4 \delta_1^{(1)}}{\Omega^2 - \varphi^2} - \frac{r_4 \beta_1^{(1)}}{2} + \frac{\bar{a}B \varphi_4}{\Omega^2 - 4\varphi^2} \right] \]  

\[ r_{15} = (\varphi_3 - \varphi_4) \left[ \frac{r_5 \delta_1^{(1)}}{\Omega^2 - 4\varphi^2} + \frac{r_5 \beta_1^{(1)}}{2} + \frac{\bar{a}B \varphi_4}{\Omega^2 - \varphi^2} \right] \]  

\[ r_{16} = (\varphi_3 - \varphi_4) \left[ \frac{r_6 \delta_1^{(1)}}{2(\Omega^2 - 4\varphi^2)} - \frac{r_6 \beta_1^{(1)}}{2} + \frac{\bar{a}B \varphi_4}{2(\Omega^2 - \varphi^2)} \right] \]  

\[ r_{17} = \frac{2(\Omega^2 - 4\varphi^2)}{\Omega^2 - \varphi^2} \left( r_3 \delta_1^{(1)} + r_3 \beta_1^{(1)} \right), \quad r_{18} = \bar{a}B \varphi_4, \quad r_{19} = \bar{a}B \varphi_4 \]  

\[ r_{20} = \frac{1}{2} (\delta_1^{(1)} \beta_1^{(2)} + \beta_1^{(1)} \beta_1^{(2)}), \quad r_{21} = \frac{1}{2} (\delta_1^{(2)} \beta_1^{(1)} - \delta_1^{(1)} \beta_1^{(2)}) \]  

\[ r_{22} = \frac{1}{2} (\delta_1^{(1)} \beta_1^{(2)} - \delta_1^{(2)} \beta_1^{(1)}) \]  

\[ r_{11}(0) = R_{11} B^3 + O(B^2), \quad r_{12}(0) = R_{12} B^3 + O(B^2) \]  

\[ r_{14}(0) = R_{14} B^3 + O(B^2), \quad r_{15}(0) = 0 \]  

\[ r_{17}(0) = 0, \quad r_{16}(0) = R_{16} B^3 + O(B^2), \quad r_{19}(0) = 0 \]  

\[ r_{20}(0) = \frac{1}{2} (\delta_1^{(1)} \beta_1^{(2)} - \delta_1^{(2)} \beta_1^{(1)}), \quad r_{22}(0) = 0 \]  

We also simplify the following

\[ U_1^{(1)} U_1^{(2)} \]
\[ U_2^{(1)} U_2^{(2)} = r_{24} + r_{25} \cos \omega t + r_{26} \sin \omega t + r_{27} \cos 2\omega t + r_{28} \sin 2\omega t + r_{29} \cos 3\omega t + r_{30} \sin 3\omega t + r_{31} \cos \Delta t + r_{32} \sin \Delta t + r_{33} \cos (\varphi - \Omega) t + r_{34} \sin (\varphi - \Omega) t + r_{35} \cos (\varphi + \Omega) t + r_{36} \sin (\varphi + \Omega) t \]  

(70a)

where,

\[ r_{24} = \frac{\delta_2^{(1)} r_6}{2(\Omega^2 - \varphi^2)} + \frac{\beta_2^{(1)} r_7}{2(\Omega^2 - \varphi^2)} + \frac{BB_r_6}{\Omega^2} \]  

(70b)

\[ r_{25} = \frac{\delta_2^{(1)} r_5}{\Omega^2} + \frac{\delta_2^{(1)} r_9}{2(\Omega^2 - \varphi^2)} + \frac{\beta_2^{(1)} r_9}{2(\Omega^2 - \varphi^2)} + \frac{BB_r_6}{\Omega^2 - 4\varphi^2} + \frac{BB_r_7}{\Omega^2 - 4\varphi^2} \]  

(70c)

\[ r_{26} = \frac{r_9 \phi_2^{(1)}}{2(\Omega^2 - 4\varphi^2)} + \frac{r_9 \beta_2^{(1)}}{2(\Omega^2 - 4\varphi^2)} - \frac{r_9 \phi_2^{(1)}}{\Omega^2} + \frac{BB_r_7}{\Omega^2 - 4\varphi^2} \]  

(70d)

\[ r_{27} = \frac{r_9 \delta_2^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{r_9 \beta_2^{(1)}}{2(\Omega^2 - \varphi^2)} - \frac{BB_r_6}{\Omega^2 - 4\varphi^2} \]  

(70e)

\[ r_{28} = \frac{r_9 \delta_2^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{r_9 \beta_2^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{BB_r_6}{\Omega^2 - 4\varphi^2} \]  

(70f)

\[ r_{29} = \frac{\delta_2^{(1)} r_2}{2(\Omega^2 - \varphi^2)} + \frac{r_9 \delta_2^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{r_9 \beta_2^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{BB_r_6}{\Omega^2 - 4\varphi^2} \]  

(70g)

\[ r_{30} = \frac{r_9 \delta_2^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{r_9 \beta_2^{(1)}}{2(\Omega^2 - \varphi^2)} \]  

(70h)

\[ r_{31} = \frac{BB_2^{(2)}}, \quad r_{32} = \frac{BB_2^{(2)}}, \quad r_{33} = \frac{1}{2}(\delta_2^{(2)} \delta_2^{(2)} + \beta_2^{(2)} \beta_2^{(2)}) \]  

(70i)

\[ r_{34} = \frac{1}{2}(\delta_2^{(2)} \beta_2^{(2)} + \delta_2^{(2)} \beta_2^{(2)}), \quad r_{35} = \frac{1}{2}(\delta_2^{(2)} \beta_2^{(2)} + \beta_2^{(2)} \beta_2^{(2)}) \]  

(70j)

\[ r_{24}(0) = R_{24} B^3 + \nu B^2, \quad R_{24} = \frac{\alpha b^2(\varphi_3 + \varphi_4)}{\Omega^2} - \frac{2a b^2(\varphi_3 + \varphi_4)}{2(\Omega^2 - \varphi^2)} \]  

(71a)

\[ r_{25}(0) = R_{25} B^3 + \nu B^2, \quad R_{25} = \frac{a b^2(\varphi_3 + \varphi_4)}{\Omega^2 + 2(\Omega^2 - 2\varphi^2)} \]  

(71b)

\[ r_{26}(0) = 0 \quad r_{27}(0) = R_{27} B^3 + \nu B^2, \quad R_{27} = \frac{-a b^2(\varphi_3 + \varphi_4)}{2(\Omega^2 - 8\varphi^2)} \]  

(71c)

\[ r_{28}(0) = 0 \quad r_{29}(0) = R_{29} B^3 + \nu B^2, \quad R_{29} = \frac{-a b^2(\varphi_3 + \varphi_4)}{2(\Omega^2 - 2\varphi^2)} \]  

(71d)

\[ r_{30}(0) = 0 \quad r_{31}(0) = R_{31} B^3 + \nu B^2, \quad R_{31} = -a b^2(\varphi_3 + \varphi_4) \left[ \frac{1}{\Omega^2 + \frac{2}{\Omega^2 - \varphi^2} + \frac{1}{\Omega^2 - 4\varphi^2}} \right] \]  

(71e)

\[ r_{32}(0) = 0 \quad r_{33}(0) = R_{33} B^3 + \nu B^2, \quad R_{33} = \frac{1}{2} a b^2(\varphi_3 + \varphi_4) \left[ \frac{1}{\Omega^2 + \frac{2}{\Omega^2 - \varphi^2} + \frac{1}{\Omega^2 - 4\varphi^2}} \right] \]  

(71f)

\[ r_{34}(0) = 0 \quad r_{35}(0) = R_{35}(0) = R_{35} B^3 + \nu B^2, \quad r_{35}(0) = 0 \]  

(71g)

Now going back to (63) and retaining only the terms that are cubic in displacement components and making use of the simplifications earlier initiated, we have

\[ U_{1,tt}^{(3)} + \nu^2 U_1^{(3)} = \varphi_0 U_1^{(1)} U_2^{(2)} + \varphi_1 U_1^{(1)} U_2^{(2)} + \varphi_1 U_1^{(1)} U_1^{(1)} - 2(U_{1,tt}^{(1)} + U_{1,tt}^{(1)}) \]  

(72a)

where,

\[ \varphi_0 = \varphi_0 \varphi_6 - \varphi_6 \varphi_7 - \varphi_5 - \varphi_6 \varphi_8 \]  

(72b)

\[ \varphi_1 = \varphi_0 \varphi_6 + \varphi_6 \varphi_7 - \varphi_5 - \varphi_6 \varphi_8 \]  

(72c)

Substituting into (62) gives, after simplification

\[ U_{1,tt}^{(3)} + \nu^2 U_1^{(3)} = r_{14} + \varphi_1 r_{12} + \varphi_1 r_{25} + \varphi_1 r_{30} \cos \omega t + (\varphi_2 r_{13} + \varphi_1 r_{26} + \varphi_1 r_{30}) \sin \omega t + r_{45} \cos 2\omega t + r_{46} \sin 2\omega t + r_{47} \cos 3\omega t + r_{48} \sin 3\omega t + r_{49} \cos \Delta t + r_{50} \sin \Delta t + r_{51} \cos (\varphi - \Omega) t + r_{52} \sin (\varphi - \Omega) t + r_{53} \cos (\varphi + \Omega) t + r_{54} \sin (\varphi + \Omega) t + 2 \varphi_1 (\delta_1^{(1)} + \delta_1^{(1)}) \sin \omega t \]  

(73a)

\[ U_1^{(3)}(0, 0) = 0, \quad U_1^{(3)}(0, 0) + U_1^{(1)}(0, 0) = 0 \]  

(73b)

where, \( \frac{d}{dx}(...) \) and \( r_{44} = \varphi_0 r_{11} + \varphi_1 r_{25} + \varphi_1 r_{37}, \quad r_{45} = \varphi_0 r_{14} + \varphi_1 r_{27} + \varphi_1 r_{37}, \) and

\[ r_{46} = \varphi_0 r_{15} + \varphi_1 r_{28} + \varphi_1 r_{31}, \quad r_{47} = \varphi_0 r_{16} + \varphi_1 r_{29} + \varphi_1 r_{32}, \]  

(74a)

\[ r_{48} = \varphi_0 r_{17} + \varphi_1 r_{30} + \varphi_1 r_{33}, \quad r_{49} = \varphi_0 r_{15} + \varphi_1 r_{31}, \quad r_{50} = \varphi_0 r_{19} + \varphi_1 r_{32} \]  

(74b)
\[ r_{51} = \varphi_9 r_{20} + \varphi_{10} r_{33}, \quad r_{52} = \varphi_9 r_{22} + \varphi_{10} r_{35} \]  
\[ r_{53} = \varphi_9 r_{20} + \varphi_{10} r_{33}, \quad r_{54} = \varphi_9 r_{23} + \varphi_{10} r_{36} \]  
\[ r_{44}(0) = R_{44} B^3 + O(B^2), \quad R_{44} \]
\[ = \varphi_9 \left( 2a(\varphi_3 - \varphi_4) \left( \frac{\bar{a}^2 - \bar{b}^2}{2(\Omega^2 - \varphi^2)} + \frac{(\varphi_3 + \varphi_4)}{\bar{\Omega}^2} \right) \right) + \varphi_{10}(\varphi_3 + \varphi_4)a\bar{b}^2 \left( \frac{1}{\Omega^2} + \frac{1}{\bar{\Omega}^2 - \varphi^2} \right) + 3\varphi_{11}(3\bar{a}\bar{b}^2 + 2\bar{a}^3) \]  
\[ r_{45}(0) = R_{45} B^3 + O(B^2), \quad R_{45} \]
\[ = \varphi_9(\varphi_3 - \varphi_4)(\bar{a}^2 - \bar{b}^2)\bar{a} \left( \frac{1}{2(\Omega^2 - 4\varphi^2)} - \frac{2}{\Omega^2 - \varphi^2} \right) + \frac{\varphi_{10}(\varphi_3 + \varphi_4)a\bar{b}^2}{2(\Omega^2 - 4\varphi^2)} \]  
\[ r_{46}(0) = 0, \quad r_{47}(0) = R_{47} B^3 + O(B^2), \quad R_{47} \]
\[ = - \left[ \varphi_9(\varphi_3 - \varphi_4)(\bar{a}^2 - \bar{b}^2) + \varphi_{10}(\varphi_3 + \varphi_4)a\bar{b}^2 + \frac{\varphi_{11}a\bar{b}^2}{4} \right] \]  
\[ r_{49}(0) = 0, \quad r_{49}(0) = R_{49} B^3 + O(B^2), \quad R_{49} \]
\[ = \varphi_9(\varphi_3 - \varphi_4)(\bar{a}^2 - \bar{b}^2) \left( \frac{3}{2\Omega^2} + \frac{2}{\Omega^2 - \varphi^2} + \frac{1}{2(\Omega^2 - 4\varphi^2)} \right) - \varphi_{10}(\varphi_3 + \varphi_4)a\bar{b}^2 \left( \frac{1}{\Omega^2} + \frac{2}{\Omega^2 - \varphi^2} \right) + \frac{1}{\Omega^2 - 4\varphi^2} \]  
\[ r_{50}(0) = 0, \quad r_{51}(0) = R_{51} B^3 + O(B^2), \quad R_{51} \]
\[ = - \varphi_9\bar{a}(\bar{a}^2 - \bar{b}^2) \left( \frac{3}{2\Omega^2} + \frac{2}{\Omega^2 - \varphi^2} + \frac{1}{2(\Omega^2 - 4\varphi^2)} \right) + \varphi_{10}(\varphi_3 - \varphi_4)a\bar{b}^2 \left( \frac{1}{\Omega^2} + \frac{2}{\Omega^2 - \varphi^2} \right) + \frac{1}{\Omega^2 - 4\varphi^2} \]  
\[ r_{52}(0) = 0, \quad r_{53}(0) = R_{53} B^3 + O(B^2), \quad R_{53} = \varphi_9 r_{20} + \varphi_{10} r_{33}, \quad r_{52}(0) = 0 \]

To ensure a uniformly valid solution in terms of \( t \), we equate to zero in (73a) the coefficients of \( \cos \varphi t \) and \( \sin \varphi t \). This yields, separately
\[ \beta_1^{(1)} + \beta_1^{(1)} = \frac{1}{2\varphi} \left[ \varphi_9 r_{12} + \varphi_{10} r_{25} + \varphi_{11} r_{38} \right] \]
\[ \delta_1^{(1)} + \delta_1^{(1)} = -\frac{1}{2\varphi} \left[ \varphi_9 r_{13} + \varphi_{10} r_{26} + \varphi_{11} r_{39} \right] \]

Equations (75a, b) are coupled but their explicit solutions are not needed. We only need
\[ \delta_1^{(1)}(0) = -\delta_1^{(1)} = aB, \quad \beta_1^{(1)}(0) = \frac{1}{2\varphi} \left[ \varphi_9 r_{12}(0) + \varphi_{10} r_{25}(0) + \varphi_{11} r_{38}(0) \right] \]

The solution of the remaining equation in (73a) is
\[ U_1^{(3)}(t, \gamma) = \delta_1^{(3)}(t) \cos \varphi t + \beta_1^{(3)}(t) \sin \varphi t + \frac{r_{44}}{\varphi^2} - \frac{1}{3\varphi^2} (r_{45} \cos 2\varphi t + r_{46} \sin 2\varphi t) \]
\[ - \frac{1}{8\varphi} (r_{47} \cos 3\varphi t + r_{48} \sin 3\varphi t) + \frac{1}{\varphi^2 - \Omega^2} (r_{49} \cos \varphi t + r_{50} \sin \varphi t) \]
\[ + \frac{1}{\Omega(2\varphi - \Omega)} (r_{51} \cos(\varphi - \Omega) t + r_{52} \sin(\varphi - \Omega) t) \]
\[ - \frac{1}{\Omega(2\varphi + \Omega)} (r_{53} \cos(\varphi - \Omega) t + r_{54} \sin(\varphi - \Omega) t) \]  

where,
\[ \delta^{(3)}_1(0) = - \left[ \frac{r_{44}}{\varphi^3} - \frac{r_{45}}{3\varphi^2} - \frac{r_{56}}{8\varphi} + \frac{r_{76}}{\varphi^2 - \Omega^2} + \frac{r_{89}}{2(\varphi^2 - \Omega^2) - \frac{r_{53}}{\Omega(2\varphi - \Omega)}} \right]_{r=0}, \beta^{(3)}_1(0) = -\frac{\bar{a}B}{\varphi} \quad (77b) \]

After simplifying (65) and retaining only the terms that are cubic in displacement, we get

\[ U_{z,t}^{(2)} + \varphi^2 U_{z,t}^{(3)} = \varphi_{12} U_1^{(1)} U_2^{(2)} - U_2^{(1)} U_1^{(2)} - \varphi_0 U_2^{(1)} U_1^{(2)} + \varphi_0 U_2^{(1)} U_1^{(2)} + \varphi_0 U_2^{(1)} U_1^{(2)} - U_2^{(1)} U_1^{(2)} - 2U_{z,t}^{(1)} + 2U_{z,t}^{(1)} \quad (78a) \]

\[ U_2^{(3)}(0,0) = 0, \ U_2^{(3)}(0,0) + U_2^{(1)}(0,0) = 0 \quad (78b) \]

where,

\[ \varphi_{12} = \varphi_5 + \varphi_0 + \varphi_6 \]

\[ \varphi_{13} = (\varphi_7 - \varphi_5 - \varphi_6) \quad (78c) \]

The simplification of terms on the right hand side of (78a) is as follows:

\[ U_1^{(1)} U_2^{(2)} = r_{69} + r_{70} \cos \varphi t + r_{71} \sin \varphi t + r_{72} \cos 2\varphi t + r_{73} \sin 2\varphi t + r_{74} \cos 3\varphi t + r_{75} \sin 3\varphi t \]

\[ + r_{76} \cos(\varphi + \Omega) t + r_{77} \sin(\varphi + \Omega) t + r_{78} \cos(\varphi - \Omega) t + r_{79} \sin(\varphi - \Omega) t \quad (79a) \]

where,

\[ r_{69} = \frac{\Omega(\varphi_0)}{2(\varphi^2 - \varphi^2)} + \frac{\varphi_0 \beta_1^{(1)}}{2(\varphi^2 - \varphi^2)} + \frac{\bar{a}B}{\varphi^2} \quad (79b) \]

\[ r_{70} = \frac{\Omega(\varphi_0)}{2(\varphi^2 - 4\varphi^2)} + \frac{\varphi_0 \beta_1^{(1)}}{2(\varphi^2 - 4\varphi^2)} + \frac{\bar{a}B}{\varphi^2} \quad (79c) \]

\[ r_{71} = \frac{\varphi_0 \beta_1^{(1)}}{2(\varphi^2 - 4\varphi^2)} + \frac{\bar{a}B}{\varphi^2} \quad (79d) \]

\[ r_{72} = \frac{\varphi_0 \beta_1^{(1)}}{2(\varphi^2 - 4\varphi^2)} + \frac{\varphi_0 \beta_1^{(1)}}{2(\varphi^2 - 4\varphi^2)} \quad (79e) \]

\[ r_{73} = \frac{\varphi_0 \beta_1^{(1)}}{2(\varphi^2 - 4\varphi^2)} + \frac{\varphi_0 \beta_1^{(1)}}{2(\varphi^2 - 4\varphi^2)} \quad (79f) \]

\[ r_{74} = \frac{\varphi_0 \beta_1^{(1)}}{2(\varphi^2 - 4\varphi^2)} + \frac{\varphi_0 \beta_1^{(1)}}{2(\varphi^2 - 4\varphi^2)} \quad (79g) \]

\[ r_{75} = \frac{\varphi_0 \beta_1^{(1)}}{2(\varphi^2 - 4\varphi^2)} + \frac{\varphi_0 \beta_1^{(1)}}{2(\varphi^2 - 4\varphi^2)} \quad (79h) \]

\[ r_{76} = \frac{1}{2}(\delta_1^{(1)} \beta_2^{(2)} - \beta_1^{(1)} \beta_2^{(2)}) \quad r_{77} = \frac{1}{2}(\delta_1^{(1)} \beta_2^{(2)} + \beta_1^{(1)} \beta_2^{(2)}) \quad (79i) \]

\[ r_{76} = \frac{1}{2}(\delta_1^{(1)} \beta_2^{(2)} + \beta_1^{(1)} \beta_2^{(2)}) \quad r_{79} = \frac{1}{2}(\delta_1^{(1)} \beta_2^{(2)} - \beta_1^{(1)} \beta_2^{(2)}) \quad (79j) \]

\[ r_{69}(0) = R_{69} B^3 + O(B^2), \quad R_{69} = -\frac{\varphi_3 + \varphi_4}{\varphi^2} \quad (80a) \]

\[ r_{70}(0) = R_{70} B^3 + O(B^2), \quad R_{70} = \frac{\varphi_3 + \varphi_4}{\varphi^2} \quad (80b) \]

\[ r_{71}(0) = 0, \quad r_{72}(0) = R_{72} B^3 + O(B^2), \quad R_{72} = \frac{1}{\varphi^2} - \frac{1}{\varphi^2} \quad (80c) \]

\[ r_{73}(0) = 0, \quad r_{74}(0) = R_{74} B^3 + O(B^2), \quad R_{74} = \frac{1}{2(\varphi^2 - 4\varphi^2)} \quad (80d) \]

\[ r_{75}(0) = 0, \quad r_{76}(0) = R_{76} B^3 + O(B^2), \quad R_{76} = \frac{1}{\varphi^2} + \frac{1}{\varphi^2} \quad (80e) \]

\[ r_{77}(0) = 0, \quad r_{78}(0) = R_{78} B^3 + O(B^2), \quad R_{78} = \frac{1}{2(\varphi^2 - 4\varphi^2)} \quad (80f) \]

Similarly, we have
In the same way, we get

\[ U^{(2)}_2 U^{(1)}_1 = r_{67} + r_{89} \cos \varphi t + r_{99} \sin \varphi t + r_{90} \cos 2\varphi t + r_{91} \sin 2\varphi t + r_{92} \cos 3\varphi t + r_{93} \sin 3\varphi t + r_{94} \cos \Omega t + r_{95} \sin \Omega t + r_{96} \cos (\varphi + \Omega) t + r_{97} \sin (\varphi + \Omega) t + r_{98} \cos (\varphi + \Omega) t \]

where,

\[ r_{97} = (\varphi_3 - \varphi_4) \left[ \frac{r_3^2}{2} \right] \left[ \frac{r_3^2}{2} + \frac{\bar{B}B_1}{\Omega^2 - \varphi^2} \right] \]

\[ r_{98} = (\varphi_3 - \varphi_4) \left[ \frac{r_3^2}{2} \right] \left[ \frac{r_3^2}{2} + \frac{\bar{B}B_1}{(\Omega^2 - 4\varphi^2)} \right] \]
To obtain a uniformly valid solution in terms of $t$, we equate to zero in (84a) the coefficients of $\Delta^2$ where, 

$$
\beta_2 = \beta_1, \quad \delta_2 = \delta_1
$$

Substituting all these simplifications in (78a) and maintaining only the terms that are cubic in the displacement, we get

$$
U_2^{(3)} + \phi^2 U_2^{(3)} = r_{100} + r_{101} \cos \phi t + r_{102} \sin \phi t + r_{103} \cos 2\phi t + r_{104} \sin 2\phi t + r_{105} \cos 3\phi t + r_{106} \sin 3\phi t
$$

(84)

where,

$$
n_{100} = \varphi r_{12} r_{69} - \varphi r_{13} r_{87} - \varphi r_{89} + \varphi r_{80}, \quad n_{101} = \varphi r_{12} r_{70} - \varphi r_{13} r_{88} - \varphi r_{89} + \varphi r_{81}, \quad n_{102} = \varphi r_{12} r_{71} - \varphi r_{13} r_{89} - \varphi r_{84} + \varphi r_{82}, \quad n_{103} = \varphi r_{12} r_{72} - \varphi r_{13} r_{92} - \varphi r_{85} + \varphi r_{83}, \quad n_{104} = \varphi r_{12} r_{73} - \varphi r_{13} r_{93} - \varphi r_{86} + \varphi r_{84}, \quad n_{105} = \varphi r_{12} r_{74} - \varphi r_{13} r_{94} - \varphi r_{87} + \varphi r_{85}, \quad n_{106} = \varphi r_{12} r_{75} - \varphi r_{13} r_{95} - \varphi r_{88} + \varphi r_{86}.
$$

(85)

To obtain a uniformly valid solution in terms of $t$, we equate to zero in (84a) the coefficients of $\cos \phi t$ and $\sin \phi t$. This yields

$$
\beta_2^{(1)} + \beta_2^{(1)} = \frac{r_{101}}{2\phi} \quad \delta_2^{(1)} + \delta_2^{(1)} = \frac{-r_{102}}{2\phi}
$$

(86)

We don’t need to express $\beta_2^{(1)}$ and $\delta_2^{(1)}$ but however need
\[
\beta_2^{(1)}(0) = \frac{r_{101}(0)}{2\phi} = \frac{1}{2\phi} \left[ r_{101} = \phi_{12}r_{70} - \phi_{13}r_{88} - \phi_{86}r_{81} + \phi_{86}r_{81} \right]_{\Gamma=0} = B^3 R_{110} \tag{87a}
\]

\[
R_{110} = \frac{1}{2\phi} \left[ r_{101} = \phi_{12}r_{70} - \phi_{13}r_{88} - \phi_{86}r_{83} - \frac{15}{4} \phi \right], \quad \delta_2^{(1)}(0) = -\delta_2^{(1)}(0) = \bar{b}B \tag{87b}
\]

The remaining equation in (84a) is solved to get

\[
U_2^{(3)}(t, \tau) = \delta_2^{(3)}(t\cos\theta + \beta_2^{(3)}(t)\sin\theta) + \frac{r_{100}}{q^2} - \frac{1}{8q^2}(r_{103}\cos2\theta + r_{104}\sin2\theta - \frac{1}{8q^2}(r_{105}\cos3\theta + r_{106}\sin3\theta) + \frac{1}{\Omega(2\phi - \Omega)}(r_{107}\cos(\phi - \Omega)t + r_{108}\sin(\phi - \Omega)t) - \frac{1}{\Omega(2\phi + \Omega)}(r_{109}\cos(\phi + \Omega)t + r_{110}\sin(\phi + \Omega)t) \tag{88a}
\]

where,

\[
\delta_2^{(3)}(0) = \left[ \frac{r_{100}}{q^2} - \frac{r_{103}}{8q^2} + \frac{r_{107}}{\Omega(2\phi - \Omega)} - \frac{r_{109}}{\Omega(2\phi + \Omega)} \right]_{\Gamma=0} \tag{89a}
\]

\[
\beta_2^{(3)}(0) = \frac{-\delta_2^{(3)}(0)}{q} = -\frac{\bar{b}}{q} \tag{89b}
\]

By substituting (60) and (61) into (54) and simplifying, we can determine \( U_{1(3,m,2,n)}^{(3)} \) and \( U_{2(3,m,2,n)}^{(3)} \). This will not be done here so as not to obscure the main focus of the analysis. Nevertheless, the general deformation so far is

\[
(U_f) = \epsilon \left[ \left( U_1^{(1)}(f_1^{(1)}) \right) \cos n\phi + \left( U_2^{(1)}(f_2^{(1)}) \right) \sin n\phi \right] \sin m\pi x + \epsilon^2 \left[ \left( U_1^{(2)}(f_1^{(2)}) \right) \cos 2n\phi + \left( U_2^{(2)}(f_2^{(2)}) \right) \sin 2n\phi \right] \sin m\pi x + \epsilon^3 \left[ \left( U_1^{(3)}(f_1^{(3)}) \right) \cos 3n\phi + \left( U_2^{(3)}(f_2^{(3)}) \right) \sin 3n\phi \right] \sin m\pi x + \ldots \tag{90}
\]

### VII. Maximum Displacement \( U_a \)

In determining the dynamic buckling load \( \lambda_0 \), we shall admit only the buckling modes that are in the shape of imperfection so that we write

\[
U = \epsilon \left[ U_1^{(1)} \cos n\phi + U_2^{(1)} \sin n\phi \right] \sin m\pi x + \epsilon^2 \left[ U_1^{(2)} \cos 2n\phi + U_2^{(2)} \sin 2n\phi \right] \sin m\pi x + \epsilon^3 \left[ U_1^{(3)} \cos 3n\phi + U_2^{(3)} \sin 3n\phi \right] \sin m\pi x + \ldots \tag{91}
\]

The conditions for maximum displacement, which is obtained in space and time, are

\[
U_x = U_y = U_{xx} = U_{yy} = 0; \quad U_{xy} + \epsilon^2 U_{xy} = 0 \tag{92}
\]

Let \( x_0, y_0, t_0, \) and \( \tau_0 \) be the critical values of \( x, y, t \) and \( \tau \) at maximum displacement and let

\[
y_0 = y_0 + \epsilon^2 y_2 + \epsilon^3 y_3 + \ldots, \quad t_0 = t_0 + \epsilon^2 t_2 + \ldots \quad \tau_0 = \epsilon^2 \tau_0 = \epsilon^4 \tau_2 + \ldots \tag{93}
\]

From the first two terms in (92), we get

\[
\epsilon U_1^{(1)} \cos n\phi_0 + U_2^{(1)} \sin n\phi_0 \cos m\pi x + \epsilon^2 \left[ U_1^{(2)} \cos 2n\phi_0 + U_2^{(2)} \sin 2n\phi_0 \right] \cos m\pi x = 0 \tag{94}
\]

and

\[
\epsilon^3 \left[ -U_1^{(3)} \sin 3n\phi_0 + U_2^{(3)} \cos 3n\phi_0 \right] \sin m\pi x + \epsilon^2 \left[ -U_1^{(2)} \sin 2n\phi_0 + U_2^{(2)} \cos 2n\phi_0 \right] \sin m\pi x = 0 \tag{95}
\]

From (94), we require that

\[
\cos m\pi x = 0 \quad i.e. \quad x_0 = \frac{\pi}{2m}, \quad m \text{ odd} \tag{96}
\]

Similarly, by using (96) and expanding (95) and taking terms of order \( \epsilon \), we get

\[
U_1^{(1)} \cos n\phi_0 - U_1^{(1)} \sin n\phi_0 = 0 \tag{97}
\]

which is evaluated at \( t = t_0, \tau = 0, \) to get

\[
y_0 = \frac{1}{n} \tan^{-1} \left( \frac{b}{a} \right) \tag{98}
\]

where we have taken the least nontrivial values in each case. By expanding the last term of (92), using (96) and equating terms of \( \epsilon \), we get

\[
U_1^{(1)} \cos n\phi_0 + U_1^{(1)} \sin n\phi_0 = 0 \tag{99}
\]

On expansion, it follows from

\[
sin n\phi_0 = 0 \quad i.e. \quad t_0 = \frac{\pi}{\phi} \tag{100}
\]

The maximum displacement \( U_a \) is next determined by evaluating (91) at maximum values of the variables. This yields

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\[
U_a = \epsilon[U_1^{(1)} \cos \gamma_a + U_2^{(1)} \sin \gamma_a \sin \alpha_a + \epsilon^3 [(U_1^{(3)} \cos \gamma_a + U_2^{(3)} \sin \gamma_a) \sin \alpha_a + \ldots]
\]

which is evaluated at \( t_a, \gamma_a \). After simplification, using (98) and (100), we get, from (101),

\[
U_a = \epsilon[U_1^{(1)} \cos \gamma_a + U_2^{(1)} \sin \gamma_a] + \epsilon^3 [(U_1^{(3)} \cos \gamma_a + U_2^{(3)} \sin \gamma_a) \cos \gamma_a]
\]

+ \ldots

(102)

After simplifying (102), the only nontrivial terms are

\[
U_a = \epsilon[U_1^{(1)} \cos \gamma_a + U_2^{(1)} \sin \gamma_a] + \epsilon^3 [(U_1^{(3)} \cos \gamma_a + U_2^{(3)} \sin \gamma_a) \cos \gamma_a] + \ldots
\]

(103)

The terms in (102) are simplified as follows:

\[
\begin{align*}
U_1^{(1)} \cos \gamma_a + U_2^{(1)} \sin \gamma_a &= 2B(\bar{a} \cos \gamma_a + \bar{b} \sin \gamma_a) \\
U_1^{(3)} \cos \gamma_a &= B^3 \left[ \frac{2R_{44}}{\varphi^2} - \frac{2R_{45}}{3\varphi^2} - \frac{R_{49}(1 + \cos \Omega t_0)}{\varphi^2 - \Omega^2} + \frac{R_{51}(1 - \cos \Omega t_0)}{\Omega(2\varphi - \Omega)} - \frac{R_{53}(1 - \cos \Omega t_0)}{\Omega(2\varphi + \Omega)} \right] \\
U_2^{(1)} \sin \gamma_a &= B^3 \left[ \frac{2R_{100}}{\varphi^2} - \frac{2R_{103}}{3\varphi^2} + \frac{R_{107}(1 - \cos \Omega t_0)}{\Omega(2\varphi - \Omega)} - \frac{R_{109}(1 - \cos \Omega t_0)}{\Omega(2\varphi + \Omega)} \right] \sin \gamma_a
\end{align*}
\]

where,

\[
\begin{align*}
R_{100} &= \Phi_{12} f_{69} - \Phi_{13} f_{97} - \Phi_{14} f_{62} + \Phi_{15} f_{90} \\
R_{103} &= \Phi_{12} f_{79} - \Phi_{13} f_{92} - \Phi_{14} f_{65} + \Phi_{15} f_{93} \\
R_{107} &= \Phi_{12} f_{76} - \Phi_{13} f_{96}, \\
R_{109} &= \Phi_{12} f_{76} - \Phi_{13} f_{98}
\end{align*}
\]

Finally, we have

\[
U_a = 2B\epsilon(\bar{a} \cos \gamma_a + \bar{b} \sin \gamma_a) + \frac{B^3\epsilon^3}{2\varphi^2} \left( (R_{111} \cos \gamma_a + R_{112} \sin \gamma_a) \cos \gamma_a + \frac{2\pi\varphi}{B\epsilon} (\bar{a} \cos \gamma_a + \bar{b} \sin \gamma_a) \right) + \ldots
\]

(107)

where,

\[
\begin{align*}
R_{111} &= \left[ \frac{R_{44}}{3} - \frac{R_{45}}{2} \left( \frac{R_{51}(1 - \cos \Omega t_0)}{\Omega(2\varphi - \Omega)} - \frac{R_{53}(1 - \cos \Omega t_0)}{\Omega(2\varphi + \Omega)} \right) \right] \\
R_{112} &= \left[ \frac{R_{100}}{3} + \frac{R_{103}}{2} \left( \frac{R_{107}(1 - \cos \Omega t_0)}{\Omega(2\varphi - \Omega)} - \frac{R_{109}(1 - \cos \Omega t_0)}{\Omega(2\varphi + \Omega)} \right) \right]
\end{align*}
\]

VIII. Dynamic Buckling Load, \( \lambda_D \)

For the purpose of determining the dynamic buckling load \( \lambda_D \), we write \( U_a \) as

\[
U_a = \epsilon C_1 + \epsilon^3 C_3 + \ldots
\]

where,

\[
C_1 = 2B(\bar{a} \cos \gamma_a + \bar{b} \sin \gamma_a)
\]

(109a)

\[
C_3 = \frac{B^3}{2\varphi^2} \left( (R_{111} \cos \gamma_a + R_{112} \sin \gamma_a) + \frac{2\pi\varphi}{B\epsilon} (\bar{a} \cos \gamma_a + \bar{b} \sin \gamma_a) \right)
\]

(109b)

As is the usual process [5, 6], we first have to reverse the series (109a) such that

\[
\epsilon = d_1 U_a + d_3 U_a^3 + \ldots
\]

(110a)

By substituting in (110a) for \( U_a \) from (109a) and equating the coefficients of powers of \( \epsilon \), we get

\[
d_1 = \frac{1}{C_1}, \quad d_3 = \frac{-C_3}{C_1^3}
\]

(110b)

The maximization (1) easily follows from (110a) to yield

\[
\epsilon = \frac{2}{\sqrt{3}} \frac{C_1}{C_3}
\]

(111)

A simplification of (111) yields

\[
\left( m^2 + n^2 \xi^2 \right) + \left( \frac{mA}{1 + \xi^2} \right)^2 + n^2 \xi \left( 1 + \xi^2 \right) \left( \frac{n^2 \xi - m^2}{(m^2 + n^2 \xi^2)^2} \right) - \lambda_D \left( \frac{\alpha m^2}{2} + n^2 \xi \left( 1 - \frac{\alpha}{2} \right) \right)^{3/2}
\]

\[
= \frac{3\sqrt{3}}{4} \lambda_D \epsilon \left( \frac{\alpha m^2}{2} + n^2 \xi \left( 1 - \frac{\alpha}{2} \right) \right) \frac{A_2}{A_1}
\]

(112a)

where,

\[
A_1 = \bar{a} \cos \gamma_a + \bar{b} \sin \gamma_a
\]

(112b)
\[ A_2 = \frac{B^3}{2\psi} \left[ R_{111} \cos \psi_0 + R_{112} \sin \psi_0 + \frac{2\pi \varphi(\lambda_d)}{B^2(\lambda_d)} (\bar{\alpha} \cos \psi_0 + \bar{b} \sin \psi_0) \right] \]  

(112c)

Equation (112a) is valid for small values of \( \varepsilon \) i.e \( |\varepsilon| < 1 \) and determines the dynamic buckling load \( \lambda_d \) asymptotically. It is implicit in nature and valid for \( m \) odd.

### Analysis of Result

As observed throughout the analyses, the dynamic buckling load \( \lambda_d \), depends, among other things, on the Fourier coefficients \( \bar{a} \) and \( \bar{b} \). As in [6], the static buckling load \( \lambda_s \) can be derived from

\[
\left( m^2 + n^2 \xi \right)^2 + \left( \frac{mA}{(1 + \xi)} \right)^2 \left( 1 + \xi \right)^2 \left( \frac{n^2 \xi r - m^2}{(m^2 + n^2 \xi)^2} \right) - \lambda_s \left( \frac{am^2}{2} + n^2 \xi \left( 1 - \frac{a}{2} \right) \right)^{3/2}
\]

\[ = 3\sqrt{3} \lambda_s e \left( \frac{am^2}{2} + n^2 \xi \left( 1 - \frac{a}{2} \right) \right) \sqrt{Q_1} \]  

(113a)

where,

\[ Q_1 = \frac{\varphi \bar{b}^3 \sin \psi_0 - \varphi_{11} \bar{a}^3 \cos \psi_0 + \bar{a} \bar{b} (\bar{a} \varphi_{10} \sin \psi_0 - \bar{b} \cos \psi_0)}{a \cos \psi_0 + b \sin \psi_0} \]  

(113b)

This is valid for the nomenclature as per the cited publication. We have also been able to obtain the Airy stress function as in equation (89). Using (112) and (113a, b), we can relate the dynamic buckling load \( \lambda_d \) to its equivalent static buckling load \( \lambda_s \) to get

\[
\left( m^2 + n^2 \xi \right)^2 + \left( \frac{mA}{(1 + \xi)} \right)^2 \left( 1 + \xi \right)^2 \left( \frac{n^2 \xi r - m^2}{(m^2 + n^2 \xi)^2} \right) - \lambda_s \left( \frac{am^2}{2} + n^2 \xi \left( 1 - \frac{a}{2} \right) \right)^{3/2}
\]

\[ = \frac{1}{\sqrt{2}} \left( \frac{\lambda_d}{\lambda_s} \right) A_2 \sqrt{A_1 Q_1} \]  

(114)

Certainly, the relationship (114) is independent of the imperfection amplitude \( \varepsilon \). Hence, if either of \( \lambda_d \) or \( \lambda_s \) is given, then the other buckling load can be conveniently evaluated.

With the aid of QBasic codes, we can obtain the numerical values for the relationship between the Dynamic Buckling Loads and the Imperfection parameters for some fixed values of \( r \). Here, we take \( A = 3.5, \xi = 0.3, H = 0.06, K(\xi) = 7, b = 1, m = n = 1, r = 3, 5, 7, 9 \). The results are shown in Table 1 and Figure 1.

The following are easily derived from Table 1 as well as from the graphical plot:

a) The dynamic buckling load decreases with increased imperfection

b) The higher the ratio of the radii of the toroidal shell \( r \), the greater the dynamic buckling load.

### Table 1: Relationship between the Dynamic Buckling Load \( \lambda_d \) and the Imperfection Parameter \( \varepsilon \) for some fixed values of \( r \) and for \( \alpha = 1 \)

<table>
<thead>
<tr>
<th>IMPERFEC TION PARAMETER ( \varepsilon )</th>
<th>DYNAMIC BUCKLING LOAD ( \lambda_d ) FOR ( r = 5 )</th>
<th>DYNAMIC BUCKLING LOAD ( \lambda_d ) FOR ( r = 7 )</th>
<th>DYNAMIC BUCKLING LOAD ( \lambda_d ) FOR ( r = 9 )</th>
<th>DYNAMIC BUCKLING LOAD ( \lambda_d ) FOR ( r = 11 )</th>
</tr>
</thead>
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<tr>
<td>0.01</td>
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<td>128.75078</td>
<td>256.54235</td>
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IX. Conclusion

This investigation has concerned itself with perturbation procedures in determining the deformation and dynamic buckling load of a deterministically imperfect toroidal shell segment trapped by a step load. The results are asymptotic in nature. The analysis is such that we are able to relate the dynamic buckling load to its static equivalent. Hence, if one of these buckling loads is known, the other one can be determined without necessarily carrying out the arduous perturbation process all over.

References


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