Equality of Internal Angles and Vertex Points in Conformal Spherical Triangles

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Abstract
In this study, by using the conformal structure in Euclidean space, the conformal structures in spherical space and the equality of the internal angles and vertex points of conformal triangles in spherical space are given. Especially in these special conformal triangles, the conformal spherical equilateral triangle and the conformal spherical isoceles triangle, the internal angles and vertices are shown.

Keywords: Conformal spherical triangle, Conformal spherical isoceles triangle, Conformal spherical equilateral triangle

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I. Introduction
The set \( S^n = \{ x \in \mathbb{R}^{n+1} : < x, x > = 1 \} \) is also called the n-dimensional unit pseudo-spherical space. Standard model for n-dimensional spherical geometry, \( S^n = \{ x \in \mathbb{R}^{n+1} : \| x \| = 1 \} \) defined as \( S^n \) is the unit sphere of \( R^{n+1} \).

Euclidean metric over \( S^n \) as follows \( d_E (x, y) = \| x - y \| [1, 2, 8] \).

Firstly, we remember the concepts of lines and triangles in the spherical plane. As for \( \alpha : \mathbb{R} \rightarrow S^n \), \( x, y \in S^n \),

\[ \alpha(t) = \cos t + \sin t \left( \frac{y - (x,y)x}{\| y - (x,y)x \|} \right), \quad t \in [0, t_1] \]

curve segment is called the line segment of \( S^n \) limited to \( x, y \) [9].

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For \( x, y, z \), three of which are three points are not on the same spherical line;

\[ \alpha(t) = \cos t + \sin t \left( \frac{y - \cos t_1 x}{\sin t_1} \right), \quad t \in [0, t_1] \]

\[ \beta(s) = \cos s y + \sin s \left( \frac{z - \cos s_1 y}{\sin s_1} \right), \quad s \in [0, s_1] \]

\[ \gamma(u) = \cos u z + \sin u \left( \frac{x - \cos u_1 z}{\sin u_1} \right), \quad u \in [0, u_1] \]

The combination of the \( \alpha(t_1) = \beta(0), \beta(s_1) = \gamma(0) \) ve \( \gamma(u_1) = \alpha(0) \) segmented line segments is called the spherical triangle, and the spherical zone bounded by the triangle is called the spherical triangular zone [9].

\( \Omega \) ispherical triangle with \( P_1, P_2, P_3 \) vertex points;

\[ M = \begin{pmatrix} 1 & \cos \varphi_{12} & \cos \varphi_{13} \\ \cos \varphi_{12} & 1 & \cos \varphi_{23} \\ \cos \varphi_{13} & \cos \varphi_{23} & 1 \end{pmatrix} \]

matrix is called egde matrix of \( \Omega \) [4]. \( P_1, P_2, \Omega \) 's two vertices;

\[ \cos \varphi_v = (P_i, P_j) \]

the real number \( \varphi_v \) in the property \( \cos \varphi_v = (P_i, P_j) \) is called \( \Omega \) 's edge length limited by \( P_i, P_j \) [4].

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If the edges of the $P_i, P_j, P_k$-pointed $\Omega$ spherical triangle through $P_k$ point are also

$\alpha : \mathbb{IR} \to S^2$,

$\beta : \mathbb{IR} \to S^2$;

the $\theta_y$ angle, which is to be $\langle \alpha'(t), \beta'(s) \rangle = \cos \theta_y$, is called the internal angle of $\Omega$ at point $P_k$ [9].

\[
\theta_y = \cos \left( \phi_{ij} \right)
\]

Figure 1. Triangle with internal angle in Spherical Space

II. Conformal Triangles in Spherical Space

Definition 2.1. The set \( \{ P \in S^2 : \langle m, P \rangle = \cos r \} \), as \( m \in S^2 \) and \( r \in \mathbb{IR}^+ \), is called the m-centered r spherical circle in $S^2$ [9].

Definition 2.2. Let $\Omega$ be the spherical triangle with $P_1, P_2, P_3$ vertex points. If there are real numbers $r_1, r_2, r_3 \in \mathbb{IR}^+$ as $0 < \varphi_y = r_i + r_j \leq \frac{\pi}{2}$ with an edge length $\varphi_y$ limited to $P_i, P_j; \Omega$ is called conformal spherical triangle [9].

Theorem 2.1. Let $\Omega$ be spherical triangle with $P_1, P_2, P_3$ vertex points. $\Omega$ to be conformal if and only if

If $r_1 \in \left( 0, \frac{\pi}{4} \right)$, $r_2 \in (0, r_1)$ and $r_3 \in \left( 0, \frac{\pi - 2r_1}{2} \right)$ (2.1)

or

If $r_1 \in \left[ \frac{\pi}{4}, \frac{\pi}{2} \right]$, $r_2, r_3 \in \left( 0, \frac{\pi - 2r_1}{2} \right)$

where $r_1, r_2, r_3 \in \mathbb{IR}^+$ [9].

Now, we give egde matrices for conformal spherical triangles. These matrices play very important roles throughout the paper for calculations.

Lemma 2.1. Edge matrix of conformal spherical triangles, edge matrix of conformal spherical equilateral triangles and edge matrix of conformal spherical isosceles triangles as follows

\[
M = \begin{bmatrix}
1 & \cos(r_1 + r_2) & \cos(r_1 + r_3) \\
\cos(r_1 + r_2) & 1 & \cos(r_2 + r_3) \\
\cos(r_1 + r_3) & \cos(r_2 + r_3) & 1
\end{bmatrix}
\]

(2.2)

\[
\tilde{M} = \begin{bmatrix}
1 & \cos(r_1 + r_2) & \cos(r_1 + r_3) \\
\cos(r_1 + r_2) & 1 & \cos(r_2 + r_3) \\
\cos(r_1 + r_3) & \cos(r_2 + r_3) & 1
\end{bmatrix}
\]

(2.3)

\[
\check{M} = \begin{bmatrix}
1 & \cos(r_1 + r_2) & \cos(r_1 + r_3) \\
\cos(r_1 + r_2) & 1 & \cos(r_2 + r_3) \\
\cos(r_1 + r_3) & \cos(r_2 + r_3) & 1
\end{bmatrix}
\]

(2.4)

respectively [9].

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From[4]

\[
\cos \theta_y = \frac{-M_{ii} M_{jj}}{\sqrt{M_{ii} M_{jj}}}, \quad i \neq j; \quad i, j = 1, 2, 3 \quad (2.5)
\]

and from equation (14) in [5], we can define

\[
\sin \theta_y = \frac{\sqrt{M}}{\sqrt{M_{ii} M_{jj}}}, \quad i \neq j; \quad i, j = 1, 2, 3 .
\]

(2.6)

III. Equality of Internal Angles and Vertex Points in Conformal Spherical Triangles

In this section, using the expressions of the internal angles and vertex points we have defined earlier, the equations of internal angles to vertex points of the conformal spherical triangle and special conformal spherical triangles will be shown.

Now, in Eq. 2.5

\[
\cos \theta_y = \frac{-M_{ii} M_{jj}}{\sqrt{M_{ii} M_{jj}}}, \quad i \neq j; \quad i, j = 1, 2, 3
\]

was given.

As sin \( P_3 = \frac{\sqrt{M}}{\sqrt{(M_{ii})(M_{jj})}} \), \( i \neq j, i \neq k, j \neq k; \quad i, j, k = 1, 2, 3 \quad (3.1) \)

It is

\[
\cos \theta_{12} = \frac{-M_{12}}{\sqrt{M_{11} M_{22}}}
\]

If \( M_{11}, M_{12} \) and \( M_{22} \) from Eq. 2.2 are calculated and replaced,

\[
\cos \theta_{12} = \frac{\cos(r_1 + r_2) \cos(r_1 + r_3) - \cos(r_2 + r_3)}{\sqrt{\sin^2(r_2 + r_3) \sin^2(r_1 + r_3)}}
\]

is obtained.

Similarly, if \( M_{11}, M_{22} \) and \( |M| \) are used at Eq 3.1, calculated from Eq 2.2,

\[
\sin P_3 = \frac{\sqrt{M}}{\sqrt{M_{11} M_{22}}}
\]

\[
\sin P_3 = \frac{4 \sin r_1 \sin r_2 \sin r_3 \sin(r_1 + r_3)}{\sqrt{\sin^2(r_2 + r_3) \sin^2(r_1 + r_3)}}
\]

would be. From here

\[
\theta_{12} = \arccos \left( \frac{\cos(r_1 + r_2) \cos(r_1 + r_3) - \cos(r_2 + r_3)}{\sqrt{\sin(r_2 + r_3) \sin(r_1 + r_3)}} \right),
\]

\[
P_3 = \arcsin \left( \frac{4 \sin r_1 \sin r_2 \sin r_3 \sin(r_1 + r_3)}{\sqrt{\sin(r_2 + r_3) \sin(r_1 + r_3)}} \right) \quad (3.2)
\]

are obtained.

We calculate the cosine of the right side of Eq 3.2. It would be
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\[
\cos \left( \arcsin \left( \frac{4 \sin r_1 \sin r_2 \sin (r_1 + r_2 + r_3)}{\sin (r_1 + r_2 + r_3)} \right) \right)
\]

\[
= \sqrt{1 - \sin^2 \left( \arcsin \left( \frac{4 \sin r_1 \sin r_2 \sin (r_1 + r_2 + r_3)}{\sin (r_1 + r_2 + r_3)} \right) \right)}
\]

\[
= \sqrt{1 - \left( \frac{4 \sin r_1 \sin r_2 \sin (r_1 + r_2 + r_3)}{\sin (r_1 + r_2 + r_3)} \right)^2}
\]

\[
= \frac{\sin (r_1 + r_2 + r_3)}{\sin (r_1 + r_2 + r_3)} - 4 \sin r_1 \sin r_2 \sin (r_1 + r_2 + r_3)
\]

When necessary calculations are made, we get

\[
\sin (r_1 + r_2) \sin (r_1 + r_2) - 4 \sin r_1 \sin r_2 \sin (r_1 + r_2 + r_3) = \left( \cos (r_1 + r_2) \cos (r_2 + r_3) - \cos (r_1 + r_2) \right)^2.
\]

Thus, \( \theta_{12} = \theta_1 \)

equation is obtained. By using similar method

\( \theta_{13} = \theta_1 \)

and

\( \theta_{23} = \theta_2 \)

are obtained [6].

III.1. Equality of Internal Angles and Vertex Points in the Conformal Spherical Equilateral Triangle

Let \( \Omega \) be a spherical triangle with \( P_1, P_2, P_3 \) vertex points, dihedral angles and \( \phi_{12}, \phi_{13}, \phi_{23} \) edge lengths. Let \( \Omega \in S^2 \); if \( \theta_{12} = \theta_{13} = \theta_{23} \), \( \phi_{12} = \phi_{13} = \phi_{23} \) and \( \theta_{12} > \frac{\pi}{3} \), \( \Omega \) is called equilateral spherical triangle [7].

Now, in Eq. 2.5

\[
\cos \theta_{ij} = \frac{-\tilde{M}_{ij}}{\sqrt{\tilde{M}_{ii} \tilde{M}_{jj}}} \quad , i \neq j \; ; \; i, j = 1, 2, 3
\]

was given.

Including

\[
\sin P_i = \frac{\sqrt{|M|}}{\sqrt{(M_{ii})(M_{jj})}} \quad , i \neq j, i \neq k, j \neq k \; ; \; i, j, k = 1, 2, 3 \quad (3.3)
\]

If \( \tilde{M}_{11}, \tilde{M}_{12} \) and \( \tilde{M}_{22} \) are calculated and replaced from Eq. 2.3;

\[
\cos \theta_{12} = \frac{\cos (r_1 + r_2)(\cos (r_1 + r_2) - 1)}{\sin (r_1 + r_2)}
\]

is obtained.

Similarly, if \( \tilde{M}_{11}, \tilde{M}_{22} \) and \( |\tilde{M}| \) calculated from Eq. 2.3 used in Eq. 3.3, it becomes as

\[
\sin P_i = \frac{\sqrt{|M|}}{\sqrt{M_{11} M_{22}}}
\]

\[
\sin P_i = \frac{\sqrt{[\cos (r_1 + r_2) - 1] \cos (r_1 + r_2) + 1}}{\sin (r_1 + r_2)}.
\]
Here,

\[ \theta_{12} = \arccos \left( \frac{\cos(r_i + r_j) \cos(r_i + r_k) - 1}{\sin(r_i + r_j)} \right) \]

\[ P_1 = \arcsin \left( \sqrt{\frac{(\cos(r_i + r_j) - 1)^2 \left( \cos(r_i + r_k) + 1 \right)}{\sin(r_i + r_j)}} \right) \]

are obtained. We calculate the cosine of the right side of Eq. 3.4. It is

\[ \cos \left( \arcsin \left( \sqrt{\frac{(\cos(r_i + r_j) - 1)^2 \left( \cos(r_i + r_k) + 1 \right)}{\sin(r_i + r_j)}} \right) \right) = 1 - \sin^2 \left( \arcsin \left( \sqrt{\frac{(\cos(r_i + r_j) - 1)^2 \left( \cos(r_i + r_k) + 1 \right)}{\sin(r_i + r_j)}} \right) \right) \]

\[ = \sqrt{1 - \left( \sqrt{\frac{(\cos(r_i + r_j) - 1)^2 \left( \cos(r_i + r_k) + 1 \right)}{\sin(r_i + r_j)}} \right)^2} \]

We get

\[ \sin(r_i + r_j) - (\cos(r_i + r_j) - 1)^2 (\cos(r_i + r_k) + 1) = \cos(r_i + r_j) (\cos(r_i + r_k) - 1)^2 \]

when necessary calculations are made. Thus

\[ \theta_{12} = P_1 \]

equality is obtained. By using similar method

\[ \theta_{31} = P_1 \]

and

\[ \theta_{13} = P_2 \]

are obtained [6].

### III.2 Equality of Internal Angles and Vertex Points in the Conformal Spherical Isosceles Triangle

Let \( \Omega \) be a spherical triangle with \( P_1, P_2, P_3 \) vertex points, dihedral angles and \( \phi_{12}, \phi_{23}, \phi_{31} \) edge lengths. Let \( \Omega \in S^2 \); if \( \theta_{12} = \theta_{31} \) and \( 2\theta_{13} > \pi - \theta_{31} \), \( \Omega \) is called isosceles spherical triangle [7].

Now, in Eq. 2.5

\[ \cos \theta_j = \frac{-M_j}{\sqrt{M_i M_j}} \quad \text{, } i \neq j \; ; \; i, j = 1, 2, 3 \]

was given. Including

\[ \sin P_j = \frac{\left| M \right|}{\sqrt{(M_i)(M_j)}} \quad \text{, } i \neq j, i \neq k, j \neq k \; ; \; i, j, k = 1, 2, 3 \quad (3.5) \]

If \( \hat{M}_{11}, \hat{M}_{12} \) and \( \hat{M}_{22} \) are calculated and replaced from Eq. 2.4;

\[ \cos \theta_{12} = \frac{\cos(r_i + r_j) (\cos(r_i + r_k) - 1)}{\sqrt{\sin(r_i + r_j) \sin(r_i + r_k)}} \]

is obtained. Similarly, if \( \hat{M}_{11}, \hat{M}_{22} \) and \( \left| \hat{M} \right| \) calculated from Eq. 2.4 used in Eq. 3.5, it becomes as

\[ \sin P_1 = \frac{\sqrt{M}}{\sqrt{M_{11} \cdot M_{22}}} \]
\[ \sin P_3 = \frac{4 \sin r_i \sin r_j \sin(r_i + r_j)}{\sqrt{\sin(r_i + r_j) \sin(r_i + r_k)}} \]

Here,
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\[
\theta_{12} = \arccos \left( \frac{\cos(r_1 + r_2) - \cos(r_2 + r_1)}{\sin(r_2 + r_1) \sin(r_1 + r_2)} \right),
\]

\[
P_i = \arcsin \left( \frac{4 \sin r_i \sin r_i \sin (r_i + r_2)}{\sin (r_2 + r_1) \sin (r_1 + r_2)} \right) \tag{3.6}
\]

are obtained. We calculate the cosine of the right side of Eq. 3.6. It is

\[
\cos \left( \arcsin \left( \frac{4 \sin r_i \sin r_i \sin (r_i + r_2)}{\sin (r_2 + r_1) \sin (r_1 + r_2)} \right) \right) = \sqrt{1 - \sin^2 \left( \arcsin \left( \frac{4 \sin r_i \sin r_i \sin (r_i + r_2)}{\sin (r_2 + r_1) \sin (r_1 + r_2)} \right) \right)}
\]

\[
= \sqrt{1 - \left( \frac{4 \sin r_i \sin r_i \sin (r_i + r_2)}{\sin (r_2 + r_1) \sin (r_1 + r_2)} \right)^2} = \frac{\sin (r_2 + r_1) \sin (r_1 + r_2) - 4 \sin r_i \sin r_i \sin (r_i + r_2)}{\sin (r_2 + r_1) \sin (r_1 + r_2)}.
\]

We get

\[
\sin (r_2 + r_1) \sin (r_1 + r_2) - 4 \sin r_i \sin r_i \sin (r_i + r_2) = \cos (r_i + r_2) \left( \cos (r_2 + r_1) - 1 \right)^2
\]

when necessary calculations are made. Thus

\[
\theta_{12} = P_i
\]

equality is obtained. By using similar method

\[
\theta_{23} = P_i
\]

and

\[
\theta_{13} = P_i
\]

are obtained [6].

References


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