Moving Space Curve in Minkowski 3-Space and Soliton Equations

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Abstract:
In this paper we introduce a relationship between curve evolution and the soliton equations in Minkowski 3-space in case of space-like curve with space-like principal normal.
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I. Introduction

The study of possible links between intrinsic kinematics of space curves [23] and integrable soliton bearing equations [1] deserves attention because of a wide variety of applications of moving curves such as vortex filament motion in fluids [13]. dynamics of continuum spin chain [17], interface dynamics [11], etc. The pioneering work by Hasimoto [13] on the motion of a vortex filament in a fluid was the first to suggest such a link [3].

After the work of Hasimoto, several authors [4,5,10,15,16, 17,18, 20,21, 22, 25] studied the connection between the integrable nonlinear Schrodinger equation and non-stretching vortex filament equation. Ding and Inoguchi also presented this connection in Minkowski 3-space [6,7,8,12].

The present work is aimed to study the relationship between moving space curve in Minkowski 3-space in case of space-like curve and soliton equations. The paper is organized as follows: In section 2 we discuss the basic geometry of a curve in Minkowski 3-space and introduce the Frenet-Serret equations which describe it. In section 3 we derive the relationship between curve evolution and soliton equations in Minkowski 3-space in case of space-like curve with space-like principal normal.

II. Preliminaries

The Minkowski 3-space $E_1^3$ is the Euclidean 3-space $E^3$ provided with the standard flat metric given by

$$ds^2 = -dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $E_1^3$. Since $ds^2$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian causal characters; it can be space-like vector if $ds^2(u,v) > 0$ or $v = 0$, timelike vector if $ds^2(u,v) < 0$ and null (light-like) vector if $ds^2(u,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $E_1^3$ can locally be space-like curve, time-like curve or null (light-like) curve, if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null (light-like) vectors. Denote by $\{t, n, b\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space $E_1^3$ [9, 14, 24].

III. Moving space-like curves with a space-like normal

A space curve embedded in three-dimensions may be described using the usual Frenet-Serret equations. In the case of space-like curve with a space-like principal normal, the usual Frenet equations read as the following [14]:

$$t_s = \kappa n, \quad n_s = -\kappa t + \tau b, \quad b_s = \tau n$$

where $\kappa$, $\tau$ are curvature, torsion and arc-length of $\alpha(s)$ and

$$\langle t, t \rangle = \langle n, n \rangle = \langle b, b \rangle = 1, \quad \langle t, n \rangle = \langle n, b \rangle = \langle b, t \rangle = 0.$$

To describe the time evolution of the triad $\{t, n, b\}$, from above relations we have the following set of equations:

$$t_u = g n + h b, \quad n_u = -g t + f b, \quad b_u = h t + f n,$$

where the functions $g = g(s, u), h = h(s, u)$ and $f = f(s, u)$ determine the motion of the curve with respect to time $u$. In the case of non-stretching curves, requiring that the unit triad satisfy the compatibility conditions:

$$t_{uu} = t_{us}, \quad n_{uu} = n_{us}, \quad b_{uu} = b_{us},$$

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By substituting in the compatibility conditions (3.3) from (3.1) and (3.2), we obtain the following relations:

\[ κ_s = g_s + hr, \quad τ_s = km + f_s, \quad h_s = κf - gh. \]  

(3.4)

Now, we will study three formulations to the relation between moving space-like curve with space-like normal and soliton equations as follows:

**Formulation (3.1)**

The second and third equations of the set (3.1) are combined to yield

\[ (n + b)_t - \tau(n + b) = -κt, \quad (n - b)_s + τ(n - b) = -κt. \]  

(3.5)

This immediately suggests the definition of certain complex vectors

\[ N_1 = \frac{1}{\sqrt{2}} (n + b) \exp \left( -\int t \, ds \right) N_2 = \frac{1}{\sqrt{2}} (n - b) \exp \left( \int t \, ds \right) \]  

(3.6)

Then we have a new frame \( t, N_1, N_2 \) satisfying the following conditions:

\[ (t, t) = (N_1, N_2) = 1, \quad (t, N_1) = (t, N_2) = (N_1, N_1) = (N_2, N_2) = 0 \]

Differentiating equation (3.6) with respect to \( s \) and using equation (3.5), we get:

\[ N_{1s} = -\frac{1}{\sqrt{2}}κ \exp \left( -\int t \, ds \right) t, \quad N_{2s} = -\frac{1}{\sqrt{2}}κ \exp \left( \int t \, ds \right). \]  

(3.7)

Thus the functions \( Ψ_1 \) and \( Ψ_2 \) appear in a natural fashion in the above equations

\[ Ψ_1 = \frac{1}{\sqrt{2}}κ \exp \left( -\int t \, ds \right) Ψ_2 = \frac{1}{\sqrt{2}}κ \exp \left( \int t \, ds \right). \]  

(3.8)

By using the definitions of \( N_1 \) and \( N_2 \) in (3.6), we get the following system

\[ \left\{ \begin{array}{l} t_s = Ψ_1 N_1 + iΨ_2 N_2, \quad N_{1s} = -Ψ_1 t, \quad N_{2s} = -Ψ_2 t, \\ t_u = γ_1 N_1 + γ_2 N_2, \quad N_{1u} = γ_1 t + R_1 N_1, \quad N_{2u} = γ_2 t - R_1 N_2, \end{array} \right. \]  

(3.9)

where

\[ γ_1 = \frac{1}{\sqrt{2}}(g - h) \exp \left( -\int t \, ds \right), \quad γ_2 = \frac{1}{\sqrt{2}}(g - h) \exp \left( \int t \, ds \right), \quad R_1 = f - \int t_s \, ds \]

On imposing the compatibility conditions \( t_{ss} = t_{uu} \), \( N_{1ss} = N_{1uu} \), \( N_{2ss} = N_{2uu} \) and using (3.9), we obtain

\[ Ψ_{1u} - γ_{1s} - R_1 Ψ_1 = 0, \quad Ψ_{2u} - γ_{2s} + R_2 Ψ_2 = 0, \quad R_{1s} = γ_1 Ψ_1 - γ_2 Ψ_2 \]  

(3.10)

Or

\[ Ψ_u - γ_s - R_1 Ψ^* = 0, \quad R_{1s} = -i \frac{1}{\sqrt{2}}(γ Ψ^* - γ Ψ^*). \]  

(3.11)

where \( Ψ^* = Ψ_1 + iΨ_2, \quad γ = γ_1 + iγ_2, \quad Ψ^* = Ψ_1 - iΨ_2 \) and \( γ^* = γ_1 - iγ_2 \). The two equations (3.11) are the system of soliton equations because it is an integrable system (of soliton type) [19] corresponding to moving space-like curve with space-like normal in formulation (4).

It is worth noting that: as noted by Lamb [18], the structure of the two Equations in (3.11) which arose from compatibility conditions on curve evolution suggests a possible relationship with soliton-bearing equations, via the Ablowitz-Kaup-Newell-Segur (AKNS) formalism [1,2]. In other wording: If we put \( Ψ_1 = r, \quad Ψ_2 = q, \quad γ_1 = -τ_s, \quad γ_2 = q_s, \quad \text{and} \quad R_1 = -R \) where \( r = r(s, u) \), \( q = q(s, u) \) and \( R = R(s, u) \) are real functions of the variables \( s \) and \( u \), we get the following soliton equations:

\[ r_s + q_ss + rR = 0, \quad q_s - q_ss - qR = 0, \quad R_s = qr_s + r_s q, \]  

(3.12)

which are introduced by Ding and Inoguchi in [8]. It is worth noting that: the third equation of (3.12) implies that \( R \) has the form \( R(s, u) = rq + R_0(u) \), where \( R_0(u) \) is a function depending only on \( u \). Then under the transformations \( q \rightarrow q \exp \left( \int R_0 du \right), r \rightarrow \exp \left( -\int R_0 du \right), \) we obtain

\[ q_u = q_s + q^2r, \quad r_u = -r_s - r^2q \]  

(3.13)

This system is just the second AKNS hierarchies of real type (2) by a scaling transformation

\[ q \rightarrow \sqrt{2} q \text{ and } r \rightarrow \sqrt{2} r. \]

**Formulation (3.2)**

Combining the first and second equations of the set (3.1), we get

\[ (n + it) - iκ(n + it) = τb. \]  

(3.14)

The above equation suggests the definition of second complex vector

\[ M = \frac{1}{\sqrt{2}}(n + it) \exp \left( -i \int κds \right) \]  

(3.15)

Then we have the new moving curve \( (b, M, M^*) \) satisfy

\[ -(b, b) = (M, M^*) = 1, \quad (b, M) = (b, M^*) = (M, M) = (M^*, M^*) = 0 \]

Differentiating equation (3.15) with respect to \( s \) and using equation (3.14), we get:

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\[ M_s = \frac{1}{\sqrt{2}} \tau \exp \left( -i \int \kappa \, ds \right) b \]

Thus the functions \( \Phi \) appears in a natural fashion in the above equation.

\[ \Phi = \frac{1}{\sqrt{2}} \tau \exp \left( -i \int \kappa \, ds \right) \]

By using the definitions of \( M \), we can get the following system of differential equations

\[
\begin{align*}
M_s &= \Phi^* M + \Phi M^*, \\
M_s &= -iR_M + \beta b, \\
M_s &= -iR_M + \beta b,
\end{align*}
\]

(3.38)

where

\[ \beta = \frac{1}{\sqrt{2}} (f + i \gamma) \exp \left( -i \int \kappa ds \right), \]

(3.19)

Now, from compatibility condition \( M_{ss} = M_{ss} \) and equating the coefficients of \( b, M \) and \( M^* \) we get

\[ \Phi_u - \beta \gamma + iR_M = 0 \quad R_{22} = i (\beta^* \gamma - \beta \gamma^*) \]

The system of equations (3.19) are the AKNS-hierarchy which is known to be a universal model in integrable systems since almost all the famous equations coming from varied physical backgrounds, such as the NLS, KdV, mKdV and so on, belong to this hierarchy [2].

**Formulation (3.3)**

From the first and third equations of (3.1), we get

\[ (t + i b)_s = (\kappa + i \tau) n. \]

(3.20)

This suggests the definition of a third complex vector

\[ P = \frac{1}{\sqrt{2}} (t + i b). \]

(3.21)

Then we have the new moving curve \( \{ n, p, p^* \} \) satisfy

\[ \langle n, n \rangle = (P, P) = (P^*, P^*) = 1 \quad \langle n, P \rangle = (n, P^*) = (P, P^*) = 0. \]

Differentiating equation (3.21) with respect to sand using equation (3.20), we get:

\[ P_s = \frac{1}{\sqrt{2}} (\kappa + i \tau) n \]

(3.22)

If we put \( \chi = \frac{1}{\sqrt{2}} (\kappa - i \tau) \) we can get the following system

\[
\begin{align*}
\chi_s &= -(\chi^* P + \chi P^*), \\
\chi_s &= -\gamma \chi^* n, \\
\chi_s &= -\gamma \chi^* n,
\end{align*}
\]

(3.23)

where \( \gamma = \frac{1}{\sqrt{2}} (g - i f) \) and \( R_3 = h \). Then from equations (3.23), we can obtain

\[ \chi_s - \gamma_3 \chi^* + iR_3 \chi^* = 0, \quad iR_3 \chi^* = (\gamma_3 X - \gamma_3 X^*) \]

(3.24)

as an associated integrable system by the AKNS formulation.

**References**


[4]. Bishop, R.L.; There is more than one way to frame a curve, Amer, Math, Monthly 82, (1975), 246-251.


[7]. Ding, Q.; The gauge equivalence of the NLS equation and the Schrödinger flow of maps in 2+1 dimensions, J. Phys. A 32, (1999), 5087-5096.


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[17]. Lakshmanan, M.; Continuum spin system as an exactly solvable dynamical system, Phys. Lett. A 61 (1977), 53-54.
[23]. Struk, D.J.; Lectures on classical geometry, Addison-Wesley, Reading, MA 1961.