On Topology of epi \ hypo-graphical operations in a sense of Mosco's epi \ hypo graphical

Mohamed Soueycatt$^1$, Nisreen Alkhamir $^2$

Abstract: In this paper, we generalize the topological results of the convergence of convex – sequences in the epigraphical sense to the convex–concave sequences in Mosco – epi \ hypo graphical sense. We actually prove that if two convex – concave sequences are convergent in Mosco – epi \ hypo graphical sense, then the sequence of epi \ hypo graphical – sum of the two sequences is convergent in Mosco – epi \ hypo graphical sense. Also, we use our result to study the convergence of a sequence of Moreau – Yosida functions for convex – concave functions.

Keywords and Phrases: convex-concave function, epi-graph, epihypo-graph, epihypo-sum, epihypo-multiplication, parent convex function, parent concave function, Mosco's epi \ hypo graphical convergence.

I. Introduction

Epigraphical analysis studies minimization problems by using the epigraph concept:

\[ \text{epi } f = \{(x, r) \in X \times R \mid f(x) \leq r\} \]

Hypographical analysis studies maximization problems by using the hypograph concept:

\[ \text{hypo } f = \{(x, r) \in X \times R \mid f(x) \geq r\}, \]
where \( X \) is a vector space.

whereas epi \ hypo graphical analysis studies maximization-maximization problems, sometimes called the saddle points problems. This led to creation of new concepts such as: epi \ hypo graphical convergence - epi \ hypo graphical derivation - epi \ hypo graphical integration - epi \ hypo graphical sum - epi \ hypo graphical multiplication ….. etc.

Many mathematicians have adopted these concepts in the study of saddle points problems. For more details, see [1,11,12,13].

II. Preliminaries

We recall some basic definitions and concepts that will be needed through the paper.

\( X \) will be a vector space unless Otherwise is stated.

Definition2.1 (the epigraphical operation):

Let \( f, g : X \rightarrow \mathbb{R} \). Then the epi-sum of \( f \) and \( g \) is defined by the relation

\[ \left( f + g \right)(x) = \inf_{u \in X} \{ f(u) + g(x-u) \} \quad \forall x \in X \]

The epi-multiplication of \( f : X \rightarrow \mathbb{R} \) by \( \lambda > 0 \) is defined by the relation

\[ \left( \lambda \ast f \right)(x) = \lambda f \left( \frac{x}{\lambda} \right) \quad \forall x \in X \]

In [1] Attouch and Wets proved that

\[ \text{epi} \left( \lambda \ast f \right) = \text{epi} \left( f \right) + \text{epi} \left( g \right), \quad \text{epi} \left( \lambda \ast f \right) = \lambda \text{epi} \left( f \right) \]

where \( \text{epi} \left( f \right) = \{(x, r) \in X \times R \mid f(x) < r\} \)

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$^1$Professor of Pure Mathematics, Tishreen University, Syria.

$^2$Master of Pure Mathematics, Tishreen University, Syria.

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Definition 2.2 (the hypographical operation)
Let \( f, g : X \rightarrow \mathbb{R} \). Then the hypo-sum of \( f \) and \( g \) is defined by the relation
\[
(f + g)(x) = \sup_{u \in X} \{ f(u) + g(x - u) \} \quad \forall x \in X
\]
\[
= -\left( (-f)_e + (-g)_e \right)
\]
The hypo-multiplication of \( f : X \rightarrow \mathbb{R} \) by \( \lambda > 0 \) is defined by the relation
\[
(\lambda \cdot f)(x) = \lambda f\left( \lambda^{-1}x \right) \quad \forall x \in X
\]

Definition 2.3 \( \{ f_n : X \rightarrow \mathbb{R} ; \ n \in \mathbb{N} \} \) is equi–coercive function if there exists
\[
\theta : \mathbb{R}^+ \rightarrow [0, +\infty \] where \( \lim_{t \rightarrow \infty} \theta(t) = +\infty \), such that:
\[
\forall n \in \mathbb{N} \ , \ \forall x \in X \ ; \ f_n(x) \geq \theta(\|x\|)
\]
We also recall some basic definitions and notions from epi–hypo graphical analysis.
For more information see [2,3,5,7,14].

Let \( L : X \times Y \rightarrow \mathbb{R} \). Then, we have

Definition 2.4 \( L \) is a convex - concave function if it is convex with respect to the first variable and concave with respect to the second variable, i.e., \( \forall x \in X \; \forall y \in Y \; L(x,,y) \) is a convex function, and
\[
\forall y \in Y \; L(x,,)\) is a concave function.

Definition 2.5 Let \( L \) be a convex - concave function.
The parent convex \( F_L : X \times Y \rightarrow \mathbb{R} \) of \( L \) is defined by the relation
\[
F_L(x,, y^*) = \sup_{y \in Y} \{ L(x,, y) + \langle y,, y^* \rangle \}
\]
The parent concave \( G_L : X^* \times Y \rightarrow \mathbb{R} \) of \( L \) is defined by the relation
\[
G_L(x^*, y) = \inf_{x \in X} \{ L(x,, y) - \langle x,, x^* \rangle \}
\]
\( L \) is closed if \( F_L = -G_L^* \ , \ F_L^* = -G_L \) whereas \( F_L^* , G_L^* \) the conjugate functions for \( F \ , \ G \) respectively.

Definition 2.6 (epi–hypo – graphical operators)
Let \( L, K : X \times Y \rightarrow \mathbb{R} \). The epi–hypo–sum of \( L \) and \( K \) is defined by the relation
\[
\left( L + K \right)(x,, y) = \inf_{u \in X} \sup_{v \in Y} \{ L(u,, v) + K(x - u,, y - v) \}
\]
\[
= \inf_{u_1,u_2 \in X} \sup_{v_1,v_2 \in Y} \{ L(u_1,, v_1) + K(u_2,, v_2) \}
\]
The epi–hypo–multiplication of \( L : X \times Y \rightarrow \mathbb{R} \) by \( \lambda > 0 \) is defined by the relation
\[
\left( \lambda \cdot L \right)(x,, y) = \lambda L\left( \lambda^{-1}x,, \lambda^{-1}y \right)
\]
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**Theorem 2.1** [14]: Let $L, K : X \times Y \to \overline{R}$ be convex-concave functions and $\lambda > 0$ then $L + K$ and $\lambda \cdot L$ are convex-concave functions.

**Theorem 2.2** [14]: Let $L, K : X \times Y \to \overline{R}$ be convex-concave functions. Then

$$F_{L+K}(x, y) = \left[ F_L\left( x, y^* \right) + F_K\left( x, y^* \right) \right](x)$$

$$G_{L+K}(x^*, y) = \left[ G_L\left( x^*, y^* \right) + G_K\left( x^*, y^* \right) \right](y)$$

where:

- $F_{L+K}, F_L, F_K$ are the parent convex functions for $L + K, L, K$ respectively.
- $G_{L+K}, G_L, G_K$ are the parent concave functions for $L + K, L, K$ respectively.

**Definition 2.7**. (The converges in a sense of Mosco's epi \- hypothetical)

Let $X, Y$ be Banach reflexive spaces and let the set

$$\left\{ K_n, K : X \times Y \to \overline{R}, \ n \in N \right\}$$

be a sequence of closed and convex - concave functions. The upper limit (Limsup) of the sequence $(K_n)_{n \in \mathbb{N}}$ in a sense of Mosco's epi \- hypothetical is defined by the relation:

$$\left( \frac{e_s}{h_w} - ls K_n \right)(x, y) = \sup_{y_n \to y, x_n \to x} \limsup_{n} K_n(x, y_n)$$

and denoted by $\left( \frac{e_s}{h_w} - ls K_n \right)$. Also, the limitin of the sequence $(K_n)_{n \in \mathbb{N}}$ in a sense of Mosco's epi \- hypothetical is defined by the relation:

$$\left( \frac{h_s}{e_w} - li K_n \right)(x, y) = \inf_{x_n \to x, y_n \to y} \liminf_{n} K_n(x, y_n), \ \forall (x, y) \in X \times Y.$$ 

Where $\left( \frac{w}{s} \right)$ refers to the weak topology on $X \times Y$.

We say that the sequence $(K_n)_{n \in \mathbb{N}}$ converges to $K$ in a sense of Mosco's epi \- hypothetical, denoted by $K_n \xrightarrow{M - e / h} K$ or $K = M - e / h - \lim K_n$, if the following holds true:

$$\frac{e_s}{h_w} - li K_n \leq K \leq \frac{h_s}{e_w} - li K_n$$

Note that when the functions $(K_n)_{n \in \mathbb{N}}$ do not depend on the variable $Y$, the definition of the convergence in a sense of Mosco's epi \- hypothetical is identical to that of Mosco's epi \- hypothethical with respect to the first variable. Also, when the functions $(K_n)_{n \in \mathbb{N}}$ do not depend on the variable $X$, the definition of the convergence in a sense of Mosco's epi \- hypothetical is identical to that of Mosco's epi \- hypothethical with respect to the second variable. See [5] for more details.

It should be noted that if $(K_n)_{n \in \mathbb{N}}$ is a sequence of convex - concave and closed functions, then

- $\frac{e_s}{h_w} - ls K^n(., y)$ is convex and semi continuous from below with respect to $X$ and
- $\frac{h_s}{e_w} - li K^n(x, .)$ is concave and semi continuous from above with respect to $Y$.

We can give an equivalent definition to the previous one as the following:

**Definition 2.8**[5]. We say that the sequence $(K_n)_{n \in \mathbb{N}}$ converges to $K$ in a sense of Mosco's epi \- hypothetical if the following two conditions hold true:
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i) $\forall (x, y) \in X \times Y$, $\forall y_n \xrightarrow{w} y$, $\exists x_n \xrightarrow{s} x$ \quad \limsup_{n} K_n(x_n, y_n) \leq K(x, y)

ii) $\forall (x, y) \in X \times Y$, $\forall x_n \xrightarrow{w} x$, $\exists y_n \xrightarrow{s} y$ \quad \liminf_{n} K_n(x_n, y_n) \geq K(x, y)

**Theorem 2.3** [5]. Let $X, Y$ be Banach reflexive spaces and let the set

$$\{ F_n, F : X \times Y^* \rightarrow \overline{R}, \ n \in N \}$$

be a sequence of parent, convex and closed functions depending on convex - concave and closed functions. Then, $i \iff ii$.

where, $i) \ F_n \xrightarrow{M} F$, $\ ii) \ K_n \xrightarrow{M - e/h} K$

**Definition 2.9 (Moreau-Yosida function):**
Let $L : X \times Y \rightarrow \overline{R}$ be a convex – concave function. Moreau-Yosida function with the two indices

$\lambda > 0, \mu > 0$ of the function $L \in \overline{R}^{X \times Y}$, is defined by the relation:

$$L_{\lambda, \mu}(x, y) := \inf_{x, y} \left\{ \frac{1}{2\lambda} \| x - u \|^2 - \frac{1}{2\mu} \| y - v \|^2 : u \in X, v \in Y \right\}.$$ 

This function is usually denoted by $L_{\lambda, \mu}$. It is well known that $L_{\lambda, \mu}$ is a locally Lipschitz function. In Hilbert spaces, $L_{\lambda, \mu}$ admits a Saddle point denoted by $(x_{\lambda, \mu}, y_{\lambda, \mu})$. For more details, see [5].

It was proved by Autoch and Wets that the convergence of Mosco’s epi/hypo graphical of a sequence of convex-concave functions is equivalent to the simple convergence of the sequence of related Moreau-Yosida functions.

### III. The Main Result:

In this section, we study the Convergence of epi/hypo-sum of two sequences of convex - concave functions by using Mosco’s epi/hypo convergence as the following:

**Theorem 3.1:** Let $X, Y$ be Banach reflexive spaces and let the set

$$\{ L_n, K_n, L : X \times Y \rightarrow \overline{R}, \ n \in N \}$$

be a sequence of convex - concave and closed functions in which each term of the sequence is equi-coercive on $X$. If the following holds true:

$$L_n \xrightarrow{M - e/h} L \quad K_n \xrightarrow{M - e/h} K$$

Then,

$$L_n + K_n \xrightarrow{M - e/h} L + K.$$ 

Proof. Using Theorem 1.1, it is enough to prove that $F_n \xrightarrow{e/h} F$, where $F_n \xrightarrow{e/h}$, $F \xrightarrow{e/h}$ are the parent and convex functions of the functions $L_n + K_n$, $L + K$ respectively for all $n \in N$. Hence, we have to prove the following two conditions:

1) $\forall (x, y^*) \in X \times Y^*$, $\forall (x_n, y_n^*) \xrightarrow{w} (x, y^*)$ \quad $\liminf_{n \rightarrow \infty} F_n(e/h)(x_n, y_n^*) \geq F(e/h)(x, y^*)$

2) $\forall (x, y^*) \in X \times Y^*$, $\exists (x_n, y_n^*) \xrightarrow{s} (x, y^*)$ \quad $\limsup_{n \rightarrow \infty} F_n(e/h)(x_n, y_n^*) \leq F(e/h)(x, y^*)$

According to Theorem 2.2, we have:

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Using (3.1) and substituting in (3.3) we obtain the following:

$$\lim_{n \to \infty} F_{e,l}^n\left(x_n, y_n^*\right) = \lim_{n \to \infty} F_{e,l}\left(x, y^*\right)$$

Where, $F_{L_n}, F_{K_n}, F_K$ are the parent convex functions of the functions $L_n, K_n, L, K$ respectively for all $n \in N$.

Let $\left(x_n, y_n^*\right) \xrightarrow{w, n \to \infty} (x, y^*)$ and let $e_n \xrightarrow{n \to \infty} 0$. Then, by definition of epi-graphical summation, there exist two sequences $\left(v_n\right)_{n \in N}, \left(u_n\right)_{n \in N}$ in $X$ where $u_n + v_n = x_n$ such that

$$\left[F_{L_n}\left(x_n, y_n^*\right) + F_{K_n}\left(x_n, y_n^*\right)\right](x_n) \geq F_{L_n}\left(u_n, y_n^*\right) + F_{K_n}\left(v_n, y_n^*\right) - e_n$$

$$\liminf_{n \to \infty} F_{e,l}^n\left(x_n, y_n^*\right) = \liminf_{n \to \infty} \left[F_{L_n}\left(x_n, y_n^*\right) + F_{K_n}\left(x_n, y_n^*\right)\right](x_n)$$

This implies that

$$\geq \liminf_{n \to \infty} F_{L_n}\left(u_n, y_n^*\right) + \liminf_{n \to \infty} F_{K_n}\left(v_n, y_n^*\right)$$

Since $\left(L_n\right)_{n \in N}$ is a sequence of equi-coercive functions on $X$, it follows that $F_{L_n}$ is also equi-coercive functions on $X$ for all $n \in N$.

Using definition 1.14, we find that there exists a function $\theta: \mathbb{R}^+ \to [0, +\infty]$ satisfying the relation

$$\lim_{t \to 0^+} \theta(t) = +\infty$$

such that $F_{L_n}\left(u_n, y_n^*\right) \geq \theta\left(\left|u_n\right|\right)$ for all $y_n^*$ and for all $n \in N$. Thus, the sequence $\left(u_n\right)_{n \in N}$ is bounded (otherwise would imply that $\liminf_{n \to \infty} F_{L_n}\left(u_n, y_n^*\right) = +\infty$). The same argument can be applied to show that the sequence $\left(v_n\right)_{n \in N}$ is bounded. So, there exists a subsequence $\left(n_k\right)_{k \in \mathbb{N}}$ such that

$$\liminf_{n \to \infty} F_{L_n}\left(u_n, y_n^*\right) = \lim_{k \to \infty} F_{L_{n_k}}\left(u_{n_k}, y_{n_k}^*\right)$$

$$\liminf_{n \to \infty} F_{K_n}\left(v_n, y_n^*\right) = \lim_{k \to \infty} F_{K_{n_k}}\left(v_{n_k}, y_{n_k}^*\right)$$

On the other hand, since $\left(u_{n_k}\right)_{k \in \mathbb{N}}$ and $\left(v_{n_k}\right)_{k \in \mathbb{N}}$ are bounded, we can find two subsequences $\left(n_{k'}\right)_{k' \in \mathbb{N}}$ and $\left(n_{k'}\right)_{k \in \mathbb{N}}$ such that $v_{n_{k'}} \xrightarrow{k' \to \infty} v$, $u_{n_{k'}} \xrightarrow{k' \to \infty} u$. Therefore,

$$\lim_{k' \to \infty} F_{L_{n_k}}\left(u_{n_k}, y_{n_k}^*\right) = \lim_{k' \to \infty} F_{L_{n_k}}\left(u_{n_k}, y_{n_k}^*\right)$$

$$\lim_{k' \to \infty} F_{K_{n_k}}\left(v_{n_k}, y_{n_k}^*\right) = \lim_{k' \to \infty} F_{K_{n_k}}\left(v_{n_k}, y_{n_k}^*\right)$$

We have $L_n \xrightarrow{M \to L} L$, $K_n \xrightarrow{M \to K} K$. According to Theorem 2.3 we find that

$$F_{L_n} \xrightarrow{M \to} F_L$$

$$F_{K_n} \xrightarrow{M \to} F_K$$

So, we have

$$\lim_{k' \to \infty} F_{L_{n_k}}\left(u_{n_k}, y_{n_k}^*\right) \geq F_L\left(u, y^*\right)$$

$$\lim_{k' \to \infty} F_{K_{n_k}}\left(v_{n_k}, y_{n_k}^*\right) \geq F_K\left(v, y^*\right)$$

Using (3.2) and (3.3) and substituting in (3.1) we obtain the following:
\[ \liminf_{n \to \infty} F_{e|h}^n(x_n, y_n^*) \geq F_L(u, y^*) + F_K(v, y^*) \]
\[ \geq \inf_{u,v \in X} \left\{ F_L(u, y^*) + F_K(v, y^*) \right\} \]
\[ \geq \left[ F_L(\cdot, y^*) + F_K(\cdot, y^*) \right](x) = F_{e|h}(x, y^*) \]

This proves the first condition. Now, for the second one:

Let \( 0 < \varepsilon \). Then there exist \( \vec{v}, \vec{u} \in X \) in which \( \vec{u} + \vec{v} = x \) such that

\[ F_{e|h}(x, y^*) + \varepsilon \geq F_L(\vec{u}, y^*) + F_K(\vec{v}, y^*) \quad \text{......(3.4)} \]

Since \( F_{L_n} \xrightarrow{M} F_L \), there exists \( \left( \vec{u}_n, y_n^* \right) \xrightarrow{s}{n \to \infty} \left( \vec{u}, y^* \right) \) such that

\[ F_L(\vec{u}, y^*) \geq \limsup_{n \to \infty} F_{L_n}(\vec{u}_n, y_n^*) \quad \text{......(3.5)} \]

Also, since \( F_{K_n} \xrightarrow{M} F_K \), there exists \( \left( \vec{v}_n, y_n^* \right) \xrightarrow{s}{n \to \infty} \left( \vec{v}, y^* \right) \) such that

\[ F_K(\vec{v}, y^*) \geq \limsup_{n \to \infty} F_{K_n}(\vec{v}_n, y_n^*) \quad \text{......(3.6)} \]

Substituting (3.5) and (3.6) in (3.4), we obtain:

\[ F_{e|h}(x, y^*) + \varepsilon \geq \limsup_{n \to \infty} F_{L_n}(\vec{u}_n, y_n^*) + \limsup_{n \to \infty} F_{K_n}(\vec{v}_n, y_n^*) \]
\[ \geq \limsup_{n \to \infty} \left[ F_{L_n}(\vec{u}_n, y_n^*) + F_{K_n}(\vec{v}_n, y_n^*) \right] \]
\[ \geq \limsup_{n \to \infty} \left[ F_{L_n}(\cdot, y_n^*) + F_{K_n}(\cdot, y_n^*) \right](x_n) \]
\[ \geq \limsup_{n \to \infty} F_{e|h}^n(x_n, y_n^*) \]

Since the above inequality holds true for all \( 0 < \varepsilon \), it follows (by letting \( 0 < \varepsilon \) tends to zero) that

\[ F_{e|h}(x, y^*) \geq \limsup_{n \to \infty} F_{e|h}^n(x_n, y_n^*) \]. This proves the second condition and completes the proof of the theorem.

**Theorem 3.2**: Let \( L : X \times Y \to \overline{R} \) be a convex-concave function, where \( X, Y \) are Banach reflexive spaces. Then, the following conditions are equivalent:

i) \( L_n \xrightarrow{M-e|h} L \)

ii) \( \forall (x, y) \in X \times Y \quad \forall \lambda > 0 \quad \forall \mu > 0 \)

\[ \lim_{n \to \infty} \left( L_n \right)_{\lambda, \mu} (x, y) = L_{\lambda, \mu} (x, y) \]

It should be noted that the relation (3.4) can be written in the following:

\[ L_{\lambda, \mu} (x, y) = \left( L + \frac{1}{2\lambda} \left\| \cdot \right\|^2 - \frac{1}{2\mu} \left\| \cdot \right\|^2 \right)(x, y) . \]
This means that the function $L_{\lambda,\mu}$ is a sum of the functions $K = \left( \frac{1}{2\lambda} \lVert \cdot \rVert^2 - \frac{1}{2\mu} \lVert \cdot \rVert^2 \right)$ and the epi/hypo graphical of the function $L$.

By applying Theorem 3.1, we obtain a generalization of the previous theorem as the following:

**Theorem 3.3:** Let $X,Y$ be Banach reflexive spaces and let the set \(\{ L_n, L : X \times Y \to \overline{R}, \ n \in \mathbb{N}\}\) be a sequence of convex–concave, closed and equicoercive functions on $X$. If \(L_n \xrightarrow{\mathcal{M} - \varepsilon |h|} L\) for all $(x,y)$ and for all $\lambda > 0$, $\mu > 0$, then $(L_n)_{\lambda,\mu} \xrightarrow{\mathcal{M} - \varepsilon |h|} L_{\lambda,\mu}$.

**Proof.** The proof can be done by putting

$$K_n = K = \left( \frac{1}{2\lambda} \lVert \cdot \rVert^2 - \frac{1}{2\mu} \lVert \cdot \rVert^2 \right)$$

in Theorem 3.1.

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**References**


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