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Positive Solutions for Fractional Boundary Value Problems

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Abstract

The paper deala with the existence of positive solutions to the following system of fractional boundary value problem,

$$\begin{cases} D^{\alpha}x(t) + b_1(t)h_1(t,x,y) = 0, & 0 < t < 1, \\ D^{\alpha}y(t) + b_2(t)h_2(t,x,y) = 0, & 0 < t < 1, \ \alpha \in (2,3) \end{cases}$$

with boundary conditions,

$$\begin{cases} x(0) = 0, \ x'(0) = 0, \ x'(1) - \delta_1 x'(\xi_1) = 0, \\ y(0) = 0, \ y'(0) = 0, \ y'(1) - \delta_2 y'(\xi_2) = 0, \end{cases}$$

where D^{α} represents the $\alpha - th$ order R-L type differential operator. We obtained the existence of at least three positive solutions for nonlinear fractional boundary value problem by using Avery-Peterson fixed point theorem.

MSC: 26A33, 34A08, 47H10

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I. INTRODUCTION

Fractional calculus is an extension of ordinary calculus since it defines derivatives and integrals for any arbitrary real order. Fractional calculus is important for many phenomena in the real world as well as for pure mathematics. Fractional calculus has drawn a lot of attention as a result of its numerous applications in a variety of fields, including physics, chemistry, biophysics, control theory, capacitor theory, signal processing, population dynamics, mechanics, and electromagnetics, among many others. Fractional differential equations are more efficient at reflecting the dynamics of such a wide range of systems, according to recent study. [2,3,7-11,16,17] has more details.

The focus of this paper is to deal with the positive solutions to a nonlinear fractional boundary value problem,

$$\begin{cases} D^{\alpha}x(t) + b_1(t)h_1(t,x(t),y(t)) = 0, & 0 < t < 1, \\ D^{\alpha}y(t) + b_2(t)h_2(t,x(t),y(t)) = 0, & 0 < t < 1, \ 2 < \alpha < 3 \end{cases} \tag{1}$$

with boundary conditions,

$$\begin{cases} x(0) = 0, \ x'(0) = 0, \ x'(1) - \delta_1 x'(\xi_1) = 0, \\ y(0) = 0, \ y'(0) = 0, \ y'(1) - \delta_2 y'(\xi_2) = 0, \end{cases} \tag{2}$$

where D^α represents the α-th order R-L type differential operator. There are two arbitrary constants $\xi_i \in (0,1)$ and $\delta_i \in [0, \frac{1}{\xi_i^{\alpha-2}})$. The functions $b_i \in C((0,1);[0,\infty)), h_i : [0,1] \times [0,\infty) \to [0,\infty)$, $i = 1,2$ where $b_1(t), h_1(t,0,0)$ or $b_2(t), h_2(t,0,0)$ does not vanish identically on (0,1).

There are still many undiscovered areas in the theory of boundary-value problems for nonlinear fractional differential equations, which makes it suitable for future study. The motivation of our work are [5], [10] and [15].

In [5], E. R. Kaufmann and E. Mboumi used the Krasnosel'skii and Liggette-William fixed point theorem to obtain the at least one or at least three positive solutions of the nonlinear fractional boundary value problem,

$$\begin{cases} D^\alpha(x) + a(t)f(x(t)) = 0, & 0 < t < 1,\ 1 < \alpha \le 2 \\ x(0) = 0,\ x'(1) = 0, \end{cases}$$

In [10] Nemat Nyamoradi used the Leggett-William fixed point theorem to study the existence of positive solution to the fractional boundary value problem,

$$\begin{cases} D^\alpha x(t) + a_1(t)f_1(t,x(t),y(t)) = 0, & 0 < t < 1, \\ D^\alpha y(t) + a_2(t)f_2(t,x(t),y(t)) = 0, & 0 < t < 1,\ 2 < \alpha < 3 \end{cases}$$

with boundary conditions,

$$\begin{cases} x(0) = 0,\ x'(0) = 0,\ x'(1) - \mu_1 x'(\eta_1) = 0, \\ y(0) = 0,\ y'(0) = 0,\ y'(1) - \mu_2 y'(\eta_2) = 0, \end{cases}$$

where D^α represents the $\alpha - th$ order R-L type differential operator.

In [15], Saadi and Benbachir considered the following boundary value problem

$$\begin{cases} D^\alpha x(t) + a(t)f(x(t)) = 0, & 0 < t < 1, 2 < \alpha < 3 \\ x(0) = 0,\ x'(0) = 0,\ x'(1) - \mu x'(\eta) = \lambda. \end{cases}$$

In order to establish various conclusions on the existence, nonexistence, and uniqueness of positive solutions, they employed the Guo-Krasnosel'skii fixed point theorem and Schauder's fixed point theorem.

Motivated by the works mentioned above, our purpose in this paper is to show the existence of positive solutions to the problem (1)-(2) by using Avery-Peterson fixed point theorem as a main tool. Also see [13,14,18].

The remainder of the paper is divided into three parts. Introduction is in Section 1. We review some fundamental definitions, concepts and lemmas regarding fractional boundary value problem in Section 2, as well as we defined a well-known fixed point theorem. In Section 3, we establish the main result and provide necessary condition to ensure the presence of at least three solutions to (1)-(2).

II. Preliminaries

Definition 2.1[6]. Let n be a positive integer, and α be a real number satisfying $n-1 < \alpha \le n$ .The α-th R-L fractional integral of the function $h \in L^1(R^+)$, denoted by $I^\alpha\ h(t)$, defined as

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

where $\Gamma(.)$ denoted the Euler gamma function.

Definition 2.2[6]. Let n be a positive integer, and α be a real number satisfying $n-1 < \alpha \leq n$ .The α-th R-L fractional derivative of the function $h \in L^1(R^+)$, denoted by $D^{\alpha}_{0+}h(t)$, defined as

$$D^{\alpha}_{0+}h(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}h(s)ds,$$

where the function $h(t)$ have absolutely continuous derivatives up to order $(n-1)$.

Definition 2.3. Let $\mathbf{B}$ be a Banach space over R. A closed nonempty subset $\mathcal{P}$ of $\mathbf{B}$ is said to be a cone provided the following hold,

(i) $\mu x + \eta y \in \mathcal{P}$, $\mu, \eta \geq 0$ for all $x, y \in \mathcal{P}$.

(ii) $x \in \mathcal{P}, -x \in \mathcal{P} \implies x = 0$.

Lemma 2.4[10]. For $h \in L(0,1)$, $D^{\alpha}I^{\alpha}h(t) = h(t)$, $\alpha > 0$ holds.

Lemma 2.5[10]. The general solution to $D^{\alpha}x(t) = 0$ with $\alpha \in (n-1, n]$ and $n > 1$ is the function

$$x(t) = a_1t^{\alpha-1} + a_2t^{\alpha-2} + ... + a_nt^{\alpha-n}, a_i \in R, i = 1, 2, ..., n$$

Lemma 2.6[10]. Let $\alpha > 0$.Then the following equality holds for $x(t)$:

$$I^{\alpha}D^{\alpha}x(t) = x(t) + a_1t^{\alpha-1} + a_2t^{\alpha-2} + ... + a_nt^{\alpha-n},$$

where $a_i \in R, i = 1, 2, ..., n$.

Lemma 2.7[15]. If $u \in C[0,1]$, for $i = 1, 2$, the fractional boundary value problem,

$$\begin{cases} D^{\alpha}x(t) + u(t) = 0, \ 0 < t < 1, \\ x(0) = x'(1) = 0, \ x'(1) - \delta_i x'(\xi_i) = 0, \end{cases} \tag{3}$$

then

$$x(t) = \int_0^1 G(t,s)u(s)ds + \frac{\delta_i t^{\alpha-1}}{(1-\delta_i\xi_i^{\alpha-2})}\int_0^1 G_{1i}(\xi_i, s)u(s)ds, \tag{4}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{5}$$

$$G_{1i}(\xi_i, s) = \frac{1}{\Gamma(\alpha)}\begin{cases} \xi_i^{\alpha-2}(1-s)^{\alpha-2} - (\xi_i-s)^{\alpha-2}, & 0 \leq s \leq \xi_i \leq 1, \\ \xi_i^{\alpha-2}(1-s)^{\alpha-2}, & 0 \leq \xi_i \leq s \leq 1. \end{cases} \tag{6}$$

Lemma 2.8 [10]. For all $(t,s) \in [0,1] \times [0,1]$,

(i) $0 \leq G_{1i}(\xi_i, s) \leq \frac{1}{\Gamma(\alpha)}\xi_i^{\alpha-2}(1-s)^{\alpha-2}, G(t,s) \geq 0$;
$(ii) \mu G(1,s) \leq G(t,s) \leq G(1,s), (t,s) \in [\eta, 1] \times [0,1]$,

where $\mu = \eta^{\alpha-1}$, for $i = 1, 2$, η satisfies

$$\int_0^1 s(1-s)^{\alpha-2} b_i(s) ds > 0, \tag{7}$$

and $G(1,s) = \frac{1}{\Gamma(\alpha)} S(1-s)^{\alpha-2}$.

We now assume the system (1)-(2). It is clear that $(x,y) \in C^2(0,1) \times C^2(0,1)$ is the solution of the system (1)-(2) if and only if $(x,y) \in C[0,1] \times C[0,1]$ is a solution of the following integral systems:

$$\left.\begin{aligned} x(t) &= \int_0^1 G(t,s) b_1(s) h_1(s,x,y) ds \\ &\quad + \frac{\delta_1 t^{\alpha-1}}{(1-\delta_1 \xi_1^{\alpha-2})} \int_0^1 G_{11}(\eta_1, s) b_1(s) h_1(s,x,y) ds \\ y(t) &= \int_0^1 G(t,s) b_2(s) h_2(s,x,y) ds \\ &\quad + \frac{\delta_2 t^{\alpha-1}}{(1-\delta_2 \xi_2^{\alpha-2})} \int_0^1 G_{12}(\xi_2, s) b_2(s) h_2(s,x,y) ds. \end{aligned}\right\} \tag{8}$$

Definition 2.9[13]. A map Φ is said to be a nonnegative continuous concave functional on a cone $\mathcal{P}$ of a real Banach space **B** if $\Phi : \mathcal{P} \to R_+$ is continuous and

$$\Phi(tx + (1-t)y) \geq t\ \Phi(x) + (1-t)\Phi(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0,1]$.

Similary, we say the map ϕ is a nonnegative convex functional on a cone $\mathcal{P}$ of a real Banach space **B** if $\phi : \mathcal{P} \to R_+$ is continuous and

$$\phi(tx + (1-t)y) \leq t\ \phi(x) + (1-t)\phi(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0,1]$.

An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

We will use the following notations as introduced by Avery and peterson [1]. Let ϕ and Θ be nonnegative convex functionals on $\mathcal{P}$, let Φ be a nonnegative continuous concave functional on $\mathcal{P}$, and let ψ be a nonnegative continuous functional on $\mathcal{P}$.Then, for positive numbers c_1, c_2, c_3, and c_4, we define the following sets:

$$\mathrm{P}(\phi, c_4) = \{x \in \mathcal{P} : \phi(x) < c_4\};$$

$$\overline{\mathcal{P}(\phi, c_4)} = \{x \in \mathcal{P} : \phi(x) \leq c_4\};$$

$$\mathcal{P}(\phi, \Phi, c_2, c_4) = \{x \in \mathcal{P} : c_2 \leq \Phi(x),\ \phi(x) \leq c_4\};$$

$$\mathcal{P}(\phi, \Theta, \Phi, c_2, c_3, c_4) = \{x \in \mathcal{P} : c_2 \leq \Phi(x),\ \Theta(x) \leq c_3,\ \phi(x) \leq c_4\};$$

$$R(\phi, \psi, c_1, c_4) = \{x \in \mathcal{P} : c_1 \leq \psi(x),\ \phi(x) \leq c_4\};$$

To establish the existence of three positive solutions of sysytem (1)-(2), we will employ the following Avery-Peterson fixed point theorem to (1)-(2).

Theorem 2.10.(Avery and Peterson [1]) Let $\mathcal{P}$ be a cone in a real Banach space $\mathbf{B}$. Let ϕ and Θ be nonnegative continuous convex functionals on $\mathcal{P}$, let Φ be a nonnegative continuous concave functional on $\mathcal{P}$ satisfying $\psi(kx) \le k\psi(x)$ for $0 \le k \le 1$, such that for some positive numbers $\bar{M}$ and c_4

$$\Phi(x) \le \psi(x) \text{ and } ||x|| \le \bar{M}\phi(x)$$

for all $x \in \overline{\mathcal{P}(\phi, c_4)}$. Suppose

$$T : \overline{\mathcal{P}(\phi, c_4)} \to \overline{\mathcal{P}(\phi, c_4)}$$

is a completely continuous operator and there exist constants c_1, c_2, and c_3 with $c_1 < c_2$ such that

(B1): $\{x \in \mathcal{P}(\phi, \Theta, \Phi, c_2, c_3, c_4) : \Phi(x) > c_2\}$ is nonempty and $\Phi(Tx) > c_2$ for $x \in \mathcal{P}(\phi, \Theta, \Phi, c_2, c_3, c_4)$;

(B2): $\Phi(Tx) > c_2$ for $x \in \mathcal{P}(\phi, \Phi, c_2, c_4)$ with $\Theta(Tx) > c_3$;

(B3): $0 \notin R(\phi, \psi, c_1, c_4)$ and $\psi(Tx) < c_1$ for $x \in R(\phi, \psi, c_1, c_4)$ with $\psi(x) = c_1$.

Then T has atleast three fixed points $x_1, x_2, x_3 \in \overline{\mathcal{P}(\phi, c_4)}$, such that $\phi(x_i) \le c_4, i = 1, 2, 3,\ c_2 < \Phi(x_1),\ c_1 < \psi(x_2), \Phi(x_2) < c_2$, and $\psi(x_3) < c_1$.

III. Main Result

We define $\mathbf{B} = C([0,1], R) \times C([0,1], R)$ with the norm$||(x,y)|| := ||x|| + ||y||$, where $||x|| = \max_{0 \le t \le 1} |x(t)|$. Then $\mathbf{B}$ is a Banach space and define a cone cone $\mathcal{P} \subset \mathbf{B}$ by

$$\mathcal{P} = \{(x,y) \in \mathbf{B} : x(t) \ge 0, y(t) \ge 0,\ min_{\eta \le t \le 1}(x(t) + y(t)) \ge\ \mu\ ||(x,y)||\}, \quad (9)$$

and the operator $T : \mathbf{B} \to \mathbf{B}$ by

$$T(x,y)(t) = (A(x,y)(t), B(x,y)(t)),\ \forall\ \ 0 < t < 1, \quad (10)$$

where

$$\begin{cases} A(x,y)(t) = \int_0^1 G(t,s)b_1(s)h_1(s,x,y)ds \\ \qquad\qquad + \frac{\delta_1 t^{\alpha-1}}{(1-\delta_1\xi_1^{\alpha-2})} \int_0^1 G_{11}(\xi_1, s)b_1(s)h_1(s,x,y)ds \\ B(x,y)(t) = \int_0^1 G(t,s)b_2(s)h_2(s,x,y)ds \\ \qquad\qquad + \frac{\delta_2 t^{\alpha-1}}{(1-\delta_2\xi_2^{\alpha-2})} \int_0^1 G_{12}(\xi_2, s)b_2(s)h_2(s,x,y)ds. \end{cases} \quad (11)$$

The following notations are provided for the convenience to prove the result. Assuming

$Z_i = max_{t\in[0,1]}\ [\int_0^1 G(t,s)b_i(s)ds + \frac{\delta_i t^{\alpha-1}}{(1-\delta_i\xi_i^{\alpha-2})} \int_0^1 G_{1i}(\xi_i, s)b_i(s)ds]$,

$z_i = min_{t\in[\eta,1]}\ [\int_0^1 G(t,s)b_i(s)ds + \frac{\delta_i \eta^{\alpha-1}}{(1-\delta_i\xi_i^{\alpha-2})} \int_\eta^1 G_{1i}(\xi_i, s)b_i(s)ds], \quad i = 1, 2.$

Then $0 < z_i < Z_i,\ i = 1, 2.$

Lemma 3.2[10]. The operator T defined in (10) is completely continuous and T: $\mathcal{P} \to \mathcal{P}$.

For this entire section, we consider that $l_i, i = 1, 2$, are two positive numbers satisfying $\frac{1}{l_1} + \frac{1}{l_2} \leq 1$.

To prove our result, we suppose that the following conditions are satisfied:

(E1) On any subinterval of (0,1), $b_i(t)$ do not vanish identically, and there exists $t_0 \in (0, 1)$ such that $b_i(t_0) > 0$ and $0 < \int_0^1 b_i(s)G(t, s)ds < \infty$,
$0 < \int_0^1 b_i(s)G_{1i}(t, s)ds < \infty, \ i = 1, 2$.

Theorem 3.3. Suppose (E1) holds. Consider that there exist nonnegative numbers c_1, c_2, c_3 and c_4 such that $0 < c_1 < c_2 < c_3 \leq \{\eta, \frac{z_1}{l_1 Z_1}, \frac{z_2}{l_2 Z_2}\}c_4$ are the constants. Assume that $h_i(t, x, y)$ satisfy the following hypothesis:

(E2): $h_i(t, x, y) < \frac{1}{l_i} \cdot \frac{c_1}{Z_i} \ \forall \ t \in [0, 1], x + y \in [0, c_1], i = 1, 2,$
(E3):
(i)$h_1(t, x, y) > \frac{c_2}{z_1} \ \forall \ t \in [\eta, 1], x + y \in [c_2, \frac{c_2}{\mu}], or$
$(ii) h_2(t, x, y) > \frac{c_2}{z_2} \ \forall \ t \in [\eta, 1], x + y \in [c_2, \frac{c_2}{\mu}],$
(E4): $h_i(t, x, y) < \frac{1}{l_i} \cdot \frac{c_4}{Z_i} \ \forall \ t \ \in [0, 1], x + y \in [0, c_4], i = 1, 2.$

Then the fractional boundary value problem (1)-(2) has atleast three positive solutions $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathcal{P}$ such that $||(x_1, y_1)|| < c_1, c_2 < min_{t\in[\eta,1]}(x_2 + y_2)$, and $||(x_3, y_3)|| > c_1$, with $\min_{t\in[\eta,1]}(x_3 + y_3) < c_2$.

Proof :Consider the cone $\mathcal{P}$ given in (9). We define a nonnegative continuous concave functional Φ on $\mathcal{P}$ by

$$\Phi(x, y) = min_{t\in[\eta,1]} \ |x(t) + y(t)|.$$

so that $\Phi(x, y) \leq ||x, y||$. We consider two nonnegative continuous convex functionals ϕ and Θ on $\mathcal{P}$ given by

$$\Theta(x, y) = \phi(x, y) = ||x, y||,$$

and a nonnegative continuous function ψ on $\mathcal{P}$ given by

$$\psi(x, y) = ||x, y||.$$

Then,

$$\psi(r(x, y)) = ||r(x, y)|| \leq |r| \ ||x, y|| \leq |r| \ \psi(x, y) = r \ \psi(x, y), \ 0 \leq r \leq 1.$$

$$\Phi(x, y) = min_{t\in[\eta,1]} \ |x(t) + y(t)| \ \leq ||x, y|| = \psi(x, y),$$

Also we find $\overline{N} \geq 1$ such that $||x, y|| = \phi(x, y) \leq \overline{N}\phi(x, y) \ \forall \ (x, y) \in \overline{\mathcal{P}(\phi, c_4)}$.

We show that the conditions of Theorem (2.10) are satisfied.
By the proof of Lemma (3.2), we can show that. T: $\overline{\mathcal{P}(\phi, c_4)} \to \overline{\mathcal{P}(\phi, c_4)}$.
Let $x, y \in \overline{\mathcal{P}(\phi, c_4)}$. Then $\phi(x, y) = ||x, y|| \leq c_4$ for $0 \leq (x, y) \leq c_4$ and $0 \leq t \leq 1$. Then by (E4), we have

$$\begin{aligned}
||T(x, y)|| &= max_{t\in[0,1]}|A(x, y)(t)| + max_{t\in[0,1]}|B(x, y)(t)| \\
&= max_{t\in[0,1]} \left\{ \int_0^1 G(t, s)b_1(s)h_1(s, x, y)ds + \frac{\delta_1 t^{\alpha-1}}{(1 - \delta_1 \xi_1^{\alpha-2})} \right. \\
&\left. \times \int_0^1 G_{11}(\xi_1, s)b_1(s)h_1(s, x, y)ds \right\} \\
&+ max_{t\in[0,1]} \left\{ \int_0^1 G(t, s)b_2(s)h_2(s, x, y)ds + \frac{\delta_2 t^{\alpha-1}}{(1 - \delta_2 \xi_2^{\alpha-2})} \right. \\
&\left. \times \int_0^1 G_{12}(\xi_2, s)b_2(s)h_2(s, x, y)ds \right\}
\end{aligned}$$

$$\leq \frac{1}{l_1}.\frac{c_4}{Z_1}\ max_{t\in[0,1]}\left\{\int_0^1 G(t,s)b_1(s)ds + \frac{\delta_1 t^{\alpha-1}}{(1-\delta_1\xi_1^{\alpha-2})}\right.$$
$$\left.\times\int_0^1 G_{11}(\xi_1,s)b_1(s)ds\right\}$$
$$+\frac{1}{l_2}.\frac{c_4}{Z_2}\ max_{t\in[0,1]}\left\{\int_0^1 G(t,s)b_2(s)ds + \frac{\delta_2 t^{\alpha-1}}{(1-\delta_2\xi_2^{\alpha-2})}\right.$$
$$\left.\times\int_0^1 G_{12}(\xi_2,s)b_2(s)ds\right\}$$
$$\leq \frac{1}{l_1}.c_4 + \frac{1}{l_2}.c_4 = c_4\left\{\frac{1}{l_1}+\frac{1}{l_2}\right\}$$
$$\leq c_4.$$

Hence, T: $\overline{\mathcal{P}(\phi,c_4)} \to \overline{\mathcal{P}(\phi,c_4)}$. The operator T: $\overline{\mathcal{P}(\phi,c_4)} \to \overline{\mathcal{P}(\phi,c_4)}$ is completely continuous by the Application of Arzela-Ascoli Theorem.

Now, we prove that the condition (B1) of Theorem (2.10) is satisfied.
The constant function $(x+y) = c_3 = \frac{c_2}{\mu} \in \mathcal{P}(\phi,\Phi,\Theta,c_2,c_3,c_4)$ and $\Theta(\frac{c_2}{\mu}) > c_2 \implies (x,y) \in \{\mathcal{P}(\phi,\Phi,\Theta,c_2,c_3,c_4) : \phi(x,y) > c_2\} \neq 0$.
Now, we consider $x,y \in \mathcal{P}(\phi,\Phi,\Theta,c_2,c_3,c_4)$ then $c_2 \leq (x+y) \leq c_3$ for $t \in [\eta,1]$

By assumption (E3)(i),

$$\Phi(T(x,y)(t)) = min_{t\in[\eta,1]}(A(x,y)(t) + B(x,y)(t))$$
$$\geq min_{\eta\leq t\leq 1}\left\{\int_\eta^1 G(t,s)b_1(s)h_1(s,x,y)ds + \frac{\delta_1\eta^{\alpha-1}}{(1-\delta_1\xi_1^{\alpha-2})}\right.$$
$$\left.\times\int_0^1 G_{11}(\xi_1,s)b_1(s)h_1(s,x,y)ds\right\}$$
$$+\,min_{t\in[\eta,1]}\left\{\int_\eta^1 G(t,s)b_2(s)h_2(s,x,y)ds + \frac{\delta_2\eta^{\alpha-1}}{(1-\delta_2\xi_2^{\alpha-2})}\right.$$
$$\left.\times\int_0^1 G_{12}(\xi_2,s)b_2(s)h_2(s,x,y)ds\right\}$$

$$> \frac{c_2}{z_1}min_{t\in[\eta,1]}\left\{\int_\eta^1 G(t,s)b_1(s)ds + \frac{\delta_1\eta^{\alpha-1}}{(1-\delta_1\xi_1^{\alpha-2})}\right.$$
$$\left.\times\int_0^1 G_{11}(\xi_1,s)b_1(s)ds\right\} = \frac{c_2}{z_1}.z_1 = c_2.$$

Similarly, by (E3)(ii), we get
$$\Phi(T(x,y)(t)) > \frac{c_2}{z_2}min_{t\in[\eta,1]}\left\{\int_\eta^1 G(t,s)b_2(s)ds + \frac{\delta_2\eta^{\alpha-1}}{(1-\delta_1\xi_1^{\alpha-2})}\right.$$
$$\left.\times\int_0^1 G_{12}(\xi_1,s)b_2(s)ds\right\} = \frac{c_2}{z_2}.z_2 = c_2.$$

This is for all $(x,y) \in \mathcal{P}(\phi,\Phi,\Theta,c_2,c_3,c_4)$, $\Phi(T(x,y)) > c_2$.
Hence condition (B1) of theorem 2.10 holds.
Next assume that $(x,y) \in \mathcal{P}(\phi,\Phi,c_2,c_4)$ with $\Theta(T(x,y) > c_3$.
Then we have,

$$\Phi(T(x,y)) = min_{t\in[\eta,1]}\ (T(x,y))(t) \quad \geq \mu\ ||T(x,y)|| = \mu\ \Theta(T(x,y)) > \mu\ c_3 = c_2,$$

which proves (B2) of Theorem (2.10) holds.

Clearly $\phi(0,0) = 0 < c_1$ implies that $\phi \in R(\phi,\psi,c_1,c_4)$. Let $(x,y) \in \phi \in R(\phi,\psi,c_1,c_4)$ with $\psi(x,y) = ||x,y|| \leq c_1$ then by (E2), we get

$$
\begin{aligned}
\psi(T(x,y)) &= max_{t\in[0,1]}\ |A(x,y)(t)| + max_{t\in[0,1]}\ |B(x,y)(t)| \\
&< max_{t\in[0,1]}\left\{\int_0^1 G(t,s)b_1(s)h_1(s,x,y)ds + \frac{\delta_1 t^{\alpha-1}}{(1-\delta_1\xi_1^{\alpha-2})}\right. \\
&\left.\times \int_0^1 G_{11}(\xi_1,s)b_1(s)h_1(s,x,y)ds\right\} \\
&+ max_{t\in[0,1]}\left\{\int_0^1 G(t,s)b_2(s)h_2(s,x,y)ds + \frac{\delta_2 t^{\alpha-1}}{(1-\delta_2\xi_2^{\alpha-2})}\right. \\
&\left.\times \int_0^1 G_{12}(\xi_2,s)b_2(s)h_2(s,x,y)ds\right\} \\
&\le \frac{1}{l_1}.\frac{c_1}{Z_1} max_{t\in[0,1]}\left\{\int_0^1 G(t,s)b_1(s)ds + \frac{\delta_1 t^{\alpha-1}}{(1-\delta_1\xi_1^{\alpha-2})}\right. \\
&\left.\times \int_0^1 G_{11}(\xi_1,s)b_1(s)ds\right\} \\
&+ \frac{1}{l_2}.\frac{c_1}{Z_2} max_{t\in[0,1]}\left\{\int_0^1 G(t,s)b_2(s)ds + \frac{\delta_2 t^{\alpha-1}}{(1-\delta_2\xi_2^{\alpha-2})}\right. \\
&\left.\times \int_0^1 G_{12}(\xi_2,s)b_2(s)ds\right\} \\
&\le \frac{1}{l_1}.c_1 + \frac{1}{l_2}.c_1 = c_1\left\{\frac{1}{l_1} + \frac{1}{l_2}\right\} \\
&\le c_1.
\end{aligned}
$$

Hence (B3) of Theorem (2.10) is satisfied.
Therefore, by Theorem (2.10), the fractional boundary value problem (1)-(2) has at least three positive solutions $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in\mathcal{P}$ such that $||(x_1,y_1)|| < c_1, c_2 < min_{\eta\le t\le 1}(x_2+y_2)$, and $||(x_3,y_3)|| > c_1$, with $\min_{\eta\le t\le 1}(x_3+y_3) < c_2$.
This completes the proof of the Theorem (3.3).

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