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Exploring the Application of Finite Cyclic Groups to Musical Notations.

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Abstract:
The purpose of this research is to bridge the disparity between the theory of mathematics and musical practice by investigating the application of finite cyclic groups to musical notations. It also provides novel approaches to long-standing problems in music representation and analysis. This work intends to further computational musicology and music theory through multidisciplinary collaboration and empirical validation, with possible applications to a variety of musical pursuits. Modulo 12, the musical notes form an additive abelian group. Knowing how to identify a cyclic group would allow someone to use the generator to find the basic circuit needed for additional practical applications of pure mathematics. cyclic groups to musical notations the goal of this study is to close the gap between mathematical theory and musical practice, offering innovative solutions to age-old challenges in music representation and analysis. Through interdisciplinary collaboration and empirical validation, this study aims to contribute to the advancement of both music theory and computational musicology, with potential implications for a wide range of musical endeavors.
Key words: *Group theory, Cylic group, Abelian group, Musical NOTES, P-Sylow Subgroup.*

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I. Introduction:

Musical notation has evolved over centuries, aiming to represent the intricate elements of music in a structured and understandable format. Traditional notation systems often rely on linear representations, such as staff notation. However, recent advancements in mathematical theory and computational techniques have opened up new avenues for exploring alternative approaches to musical representation. One such promising avenue is the application of finite cyclic groups, which offer unique properties that can potentially enrich the understanding and analysis of musical structures.

The area of pure mathematics known as group theory is originated from the algebra. Because it was abstract, It appeared to be more of an arts subject than a science subject. was actually viewed as being purely abstract and impractical. Even students of group theory after being introduced to the abstract nature of the subject, the course appears to not believe that it has any real-world applications. (Tsok, 2013). This challenge encourages scholars to examine the various ways that groups can be expressed in concrete terms from a theoretical and practical perspective in order to bring its applicability to real-world situations, especially in the context of musical notes. The purpose of this work is to apply some group theory principles to the analysis and comprehension of musical notes in light of the group's axioms.. The main objective is to see the relationship that exist between these musical notes and cyclic group and their interpretation algebraically. This work focuses on the behavior of musical notes, which is mostly determined by group axioms, theorems like the first theorem by Langrange and Sylow, cyclic groups, and two left cosets.

A group that has an element to which an operation is applied that yields the entire set is called a cyclic group. It is clear that a cyclic groups is a groups of repeated patterns until returning to the beginning. In general, a Groups that have all of their elements as powers of a fixed element are known as cyclic groups. A group G is called cyclic if G = $< a >= \{a^n | n \in \mathbb{Z}\}$. That is, a group G is cyclic if G is generated by one of its elements. Cyclic group has many important applications in Chemistry, Material Sciences to mention but a few. Although the concept of group theory originated with the idea of abstract algebra, it can be applied to many other mathematical fields, other scientific fields, and even the musical arts. Despite the fact that written or printed indications that indicate vocal or instrumental sound constitute music. Broadly speaking, The arrangement of

sounds to create a particular shape, harmony, melody, rhythm, or other expressive element or thought of feeling is refers to as music. Without musical music is not complete. and music NOTES are oval-shaped symbols that are place on the lines and in the spaces of staff. They represent musical sound called PITCHES.
Hence these are $C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B.$ When logically combine give out pleasant sound to the ear.

Definition of Terms:

We wish to present some definitions to help understand the relationship between the concepts of music and abstract algebra.

Music:

Thus, the study of the essential components—notes, scales, chords, rhythm, harmony, and form—that define and control the language of music refers to as Musical theory. It serves as a set of rules and guidelines that musicians use to create, analyse, and interpret music. And these notes are given in the table below:

Notes	C	$C\#$	D	$D\#$	E	F	$F\#$	G	$G\#$	A	$A\#$	B
Nam	1st note	2nd Note	3rd note	4th note	5th note	6th note	7th note	8th note	9th note	10th note	11th note	12th note

Musical Flat b

The FLAT sign (b) before a note lower the pitch of that note. A musical flat is the transition of sound from one pitch to a lower note. For instance transition to any other note on the left from F.

Musical Sharp #

The sharp sign (#) before a note raises the pitch of the note. Hence can be represents the transition of sound from one pitch (note) to a higher pitch, indicated by #. For instance, movement from F to any other note to the right on the musical notes.

Musical Tone:

Musical tone is the term used to describe any change in pitch from one musical note to the next. this pair of steps in any direction relative to the musical notes. As an illustration, movement from $F\ to\ G$ or to $D\#$.

Octaves. The difference in pitch between two musical notes is called an octave.

Semitone;

half-tone When moving from one musical note to the next, is it a step forward or step backward on the scale?. For example, movement from $F\ to\ F\#\ or\ F\ to\ E$.

3.4 Chord

When two, three, or more notes are played simultaneously, a chord is formed.

3.5 Abstract Group

Let (G,$*$) be an algebraic structure, where $*$ is a binary operation. Then (G,$*$) is called a group under this condition if the following axioms are satisfied

i. closure law:

G is closed under the operation $*$, that is, to each ordered pair $a, b \in G$, there exists a unique element $a * b \in G$

ii. Associativity law:

* is an associative binary operation, that is,

$(a * b) * \text{c} = a * (b * \text{c}), \forall$ a, b, c $\in$ G.

iii The reality of identity element:

There exists an element $e \in G$ (called identity element of G) s.t.

$a * e = e * a = a, \forall$ a $\in$ G.

iv. Existence of inverse element for each element in G:To each a $\in$ G,

There is a component $a^{-1} \in$ G (referred to as the opposite of a with respect to $*$) s.t.

$a * a^{-1} = a^{-1} * \text{a} = \text{e}$

v. In addition a group is said to be abelian (or commutative) if

$a * b = b * a\ \forall\ a, b \in G.$ i.e. the operation does not change the result.

For example; $3 + 5 = 5 + 3$. For this paper addition operation will be equivalent to: $\underbrace{a * a * a * a * \ldots * a}_{n-times} = a^n$.

For instance Z, the set of integers.

P- Group

Assume p is a fixed prime. A limited group H group is referred to as a p-group if every element within it has an order that is a power of p..

P- Subgroup

If $H \leq G$ and $|H| = p^r$ for some $r \geq 0$ then H is called p - subgroup of G.

Definition (P-Sylow Subgroup):
Suppose G is a finite group and $o(G) = P^m n$ where P is prime number and P is not a divisor of n. Then a subgroup $H$of G is said to be a $P - Sylow$ subgroup of $G\ iff\ o(H) = P^m$
Or
Let p be a prime number and G be a finite group. If $P^m / o(G)$ and $P^{m+1} \nmid o(G)$, then a subgroup of G of order P^m is called a $P - Sylow$ subgroup of G.
Cyclic group:
When an element $a \in G$ exist and is such that $\mathrm{G} = \{a^n | n \in \mathbb{Z}\}$. The generator of G is an element of this type, designated as $G = < a >$.
Generator of a group: An element whose powers comprise the element of the group is called a generator of the group..
Integers $\boldsymbol{Modulo\ m}$
The set of integer modulo m is denoted by $\mathbb{Z}_m$. And this is a finite group that is called the additive group of the residue class of integers m.
Number Theory: There are cyclic groups in nature, patterns, and other mathematical domains. Number theory is one area where cyclic groups are frequently used. When studying cyclic groups, one essential tool is the division algorithm.
Definition: A set's permutation A is a one to one and onto function defined as Φ:A → A. Permutations are different configurations for how a set can be put together or organized.

2.1: Historical background of the relationship between mathematics and music

Music and mathematics have a long history of collaboration. Pythagoras (428–347), who is credited with founding the first mathematical school that focused only on logical reasoning, is also credited with founding the first school of theoretical music. Pythagoras was not only a mathematician but also a composer and a music theorist. when the ear detects the ratio between the amount of vibrations produced in the same amount of time; that when the ear fails to detect this ratio, their effect is unpleasant. (Taken from the Overview of Given that Euler's music theory addresses a number of naturally occurring prime and prime factorization concerns, it is highly likely that he worked on other combinatorial and number theory projects at the same time. Listening to composers discuss mathematics is equally fascinating. Since music is a science, there must be established guidelines. These guidelines must stem from an obvious concept, which can only be ascertained with the aid of mathematics. The most liberated and passionately individualistic artists are mathematicians. There are no material or instrument restrictions on them. Their preferences and intellectual curiosity ultimately dictate where they end up at any given time. In actuality, their research focuses on the human mind. Mathematically inclined teaching of music. "Statable as a connected set of axioms, definitions, and theorems, the proofs of which are derived by means of an appropriate logic" is the ideal description for a musical theory. This is in the line of the classic musical treatises of composers such as Rameau [94], Mersenne [78], and others. This, as we have already discussed, is an exposition presented in the form of a lemma, theory, etc. Before Babbitt, a number of serial composers were aware that they were working with groups. Fokker was greatly impacted by Huygens's music theory writings. Halsey and Hewitt's article is a significant contribution to the methodical application of group theory in music. Seeing how mathematics can benefit music and vice versa is always beneficial.

Group theory in music and mathematics

Few authors have explored the area of music, although many have focused on applying group theory to various fields in the sciences, games, and other domains. The father of theoretical music is Pythagoras (428–347 B.C.), who is credited with founding the first school of mathematics as a fully deductive discipline. He used to claim that musical notes are not unusual and that "everything is number.", that is C, C#, D, D#,E, F, F#, G, G#, A, A#, B. But why "all is number"? The Pythagoreans gave numbers symbolic meanings and interpretations. They classified even numbers as belonging to women and odd numbers to men. One represents reason, two represents opinion, three symbolizes harmony, four represents justice, five represents marriage, six represents creation, seven represents awe, and ten represents the universe, according to the Pythagoreans. There were two potential explanations offered. The Eastern influence comes first. Pythagoras may have been affected by numerology, which deals with numbers and mystical relationships between them, as a result of his travels to Egypt and Babylon. Offering a substitute is a potential second justification A vibrating string could produce two distinct kinds of musical notes if it was divided into two lengths by a moveable device. The length ratios of the vibrating string's component sections then describe these notes. The Pythagoreans may have concluded that numbers might be used to explain other phenomena after using them to explain musical notes and to describe constellations. Heath (1965) Thomas M. Flore (1993). He referred to C, C#, D, D#, E, F, F#, G, G#, A, A#, B. As the Z_{12} Model of pitch class. Hence the musical clock and the cyclic group of order 12 can be constructed as below:

3.1: Methodology:
We collect some theorems, Lemmas and proposition including examples that are used in proving our main results

Theorem 3.1:
Let $H \leq G$ be groups and $g \in G$. then;
(i) $g \in gH$ (ii) Two left cosets of H in G are either identical or disjoint;
(iii) The quantity of components in gH is $|H|$.

Proof;
(i). Since $1 \in H$, we have $g = g1 \in gH$.
(ii). Take two left cosets aH and bH such that $aH \cap bH \neq \emptyset$.
Let $c \in aH \cap bH$. Then $c \in aH$ and $c \in bH$.
This imply that $c = ah_1$ and $c = ah_2$ for some $h_1, h_2 \in H$.
But $ah_1 = bh_2$ imply that $a = bh_2{h_1}^{-1}$ and $b = ah_1{h_2}^{-1}$.
So for any $h \in H$
$ah = \left(bh_2{h_1}^{-1}\right)h = b(h_2{h_1}^{-1}h) \in bH$ by associativity
That is, $aH \subseteq bH \ldots\ldots\ldots\ldots (*)$
And $bh = \left(ah_1{h_2}^{-1}\right)h = a(h_1{h_2}^{-1}h) \in aH$ by associativity
That is, $bH \subseteq aH \ldots\ldots\ldots\ldots (**)$
Thus from $(*)$ and $(**)$ we have that $aH = bH$.
It thus follows that if $aH \cap bH \neq \emptyset$,
Then $aH = bH$ and as such, distinct left cosets are disjoint.
(iii). The map $H \longrightarrow gH$ defined by $h \longrightarrow gh$ is readily seen to be bijective.
Thus $|H| = |gH|$.

Theorem 3.2: (Lagrenge's theorem**)**
A finite group's subgroup order is determined by the group's order..

Proof;
Let $H \leq G$ and $|G| = n$, $|H| = m$
Now, $H's\ pairwise\ disjoint\ cosets\ are\ united\ to\ form\ G.$
Let there be j distincts cosets of $H\ in\ G$.
We are aware that for any $a \in G$
$\mid aH \mid = \mid H \mid = m$
Consequently, the overall count of elements in G $is\ mj$.
$s.t. n = mj;$
Hence m divides n as asserted.

Note:
Observe that $n = mj$ as in the proof of the theorem 3.2 imply that
$|G| = |G : H|/|H|$

Theorem 3.3:
A cyclic group's subgroups are all cyclic.

Proof:
Let $G = \langle g \rangle$ be a cyclic group, where $g \in G$.
Let $H \subseteq G$. If $H = \{e\}$, then $H = \{g^o\}$ is trivially cyclic with generator e.
So assume $H \neq \{e\}$. And chose $h \in H$.
Then $h = g^s$ for some $s \in \mathbb{Z}$ and $h^{-1} = g^{-1} \in H$.
Consequently, positive integers exist t, such that $g^t \in H$.
Choose the least positive integer among them, and refer to it as l, (Any collection of positive integers has a smallest number inside it according to the well-ordering principle of natural numbers).
By Division Algorithm we may write
$s = ql + r$ where $0 \leq r < l$.
Then $h = g^s = g^{ql+r} = (g^l)^q \cdot g^r$.
So that $g^r = (g^l)^{-q} \cdot h \in H$.
If $r \neq 0$, then $r < l$
Which contradict the choice of l.
Thus $r = 0$ and so $h = (g^l)^q$.
Hence $H \subseteq \langle g^l \rangle$
Now $g^l \in H$ and so, $\langle g^l \rangle \subseteq H$.
Accordingly, $H = \langle g^l \rangle$ and the result follows that g^l generate H.
So H is cyclic.

Theorem 3.4:
Let G a cyclic group having a generator. If the order of G is infinite, then G is isomorphic to $\langle \mathbb{Z}, + \rangle$. If G has $|G| < \infty$, then G is isomorphic to $\langle \mathbb{Z}_n, + \rangle$
Proof. Case 1: For all positive integers m, $a^m \neq e$. Here, we assert that there are never two different exponents $h\ and\ k$ can give equal elements $a^h = a^k$ and say $h > k$. Then $a^h a^{-k} = a^{h-k} = e$, in opposition to our case one presumption. Thus, each component of G may be written as a^m for a unique $m \in Z$. The map $\Phi: G \to Z$ given by $\Phi(a^i) = 1$is thus well defined, one to one, and onto $\mathbb{Z}$. Also, $\Phi(a^i a^j) = (a^{i+j}) = i + j = \Phi(a^i) + \Phi(a^j)$
therefore the homomorphism property is satisfied and Φ is an isomorphism.
Case 2: $a^m = e$ for a certain positive integer m. Let n be the smallest positive integer such that $a^n = e$. If $s \in Z$ and $s = nq + r$ for $0 \leq r < n$, then
$a^s = a^{nq+r} = (a^n)^q a^r = e^q a^r = a^r$. As in case 1 if $0 < k < h < n$ and $a^h = a^k$ then, $a^{h-k} = e$ and $0 < h - k < n$, contradicting our choice of n. Thus, the elements $a^0 = e, a, a^2, a^3, \dots, a^{n-1}$ are unique and make up every component of G. The map $\Phi(a^i) = i$ for $i = 0, 1, 2, \dots, n - 1$ is therefore well defined, onto and one to one $\mathbb{Z}$.
Because $a^n = e$, we see that $a^{i+j} = a^k$ where $k = +_n j$. $(a^{i+j}) = i +_n j = \Phi(a^i) +_n \Phi(a^j)$.
Thus, $\Phi(a^{i+j}) = i +_n j = \Phi(a^i) +_n \Phi(a^j)$.
Thus, Φ is an isomorphism and the homomorphism property is satisfied.
Theorem 3.5: (Sylow's first theorem):
Assume P is a prime number and G is a group of finite order.. If $P^m / o(G)$ and $P^{m+1} \nmid o(G)$, then G has a subgroup of order P^m.
Proof;
We shall prove the theorem by induction on $o(G)$. We see that the theorem is obviously true of $o(G) = 1$
Let $o(G) = P^m n$ where P is not a divisor of n. If $m = 0$, Clearly, the theorem is correct. If $m = 1$ the theorem is true by cauchy's theorem. So let $m > 1$. Then G is a group of composite order and so G must possess a group H such that $H \neq G$.
If P is not a divisor of $\frac{o(G)}{o(H)}$, then $P^m / o(H)$ Because $o(G) = P^n n = o(H) \cdot \frac{o(G)}{o(H)}$
Also P^{m+1} cannot be a divisor of $o(H)$ because then P^{m+1} will be a divisor of $o(G)$ of which $o(H)$ is a divisor. Further $o(H) < o(G)$. Therefore by our induction hypothesis, the theorem is true for H.
Therefore H has a subgroup of G. So let us assume that for every subgroup H of G where $H \neq G$, P is a divisor of $\frac{o(G)}{o(H)}$.
Consider the class equation, $o(G) = o(Z) + \sum_{a \notin Z} \frac{o(G)}{o[N(a)]} \dots \dots \dots \quad \dots \dots (1)$
Since $a \notin Z \Longrightarrow N(a) \neq G$, therefore according to our assumption P is a divisor of ∑_(a∉Z)▒o(G)/o[N(a)] Also $P / o(G)$ Therefore from (1), Thus, we deduce that P is a divisor. of $o(\mathbb{Z})$. Cauchy's Theorem thus states that $\mathbb{Z}$ has an element b of order P. $\mathbb{Z}$ is the centre of G. Also $N = \{b\}$ is a cycle subgroup of $\mathbb{Z}$ of order P. Therefore N is a cyclic subgroup of G or order P. Since $b \in \mathbb{Z}$, therefore N is a normal subgroup of G of order P. Now consider the quotient group $G' = G/N$. We have $o(G) = o(G)/o(N) = P^m n / P = P^{m-1} n$
Thus $o(G') < o(G)$. Also $\frac{P^{m-1}}{o(G')}$ But P^G is not a divisor of $o(G')$. Therefore Through our induction hypothesis m' has a subgroup, say s' of order P^{m-1} we know that the natural mapping $\Phi: G \to G/N$ define by $\Phi(x) = Nx \; \forall \; x \in G$ is a homomorphism of G onto G/N with kernel N. Let $S = \{x \in G: \Phi(x) \in S'\}$.
Then S is a subgroup of G and $S' \cong S/N$
$\therefore o(S') = o(S/N) = \frac{o(S)}{o(N)}$
Therefore S is a subgroup of G of order P^m. The theorem's proof is now complete.
Lemma 3.1:
Let $H \leq G$ and let G be a group. Consequently, each right cost of H in G has the same cardinality as H.
Proof:
Let $Hg\ be\ a\ \ right\ coset\ of\ H\ in\ G$ and define $\varphi: H \longrightarrow Hg$by $\varphi(h) = hg$ then $\varphi(h_1) = \varphi(h_2)$,
$\Longrightarrow h_1 g = h_2 g$
$(by\ cancellation\ law)$
$h_1 = h_2$
$hence\ \varphi\ is\ 1 - 1$
Now we have to show that φ is onto.
Take $y \in Hg$ then $y = hg$ for some $h \in H$.
Then $\varphi(h) = hg = y$.
Consequently, φ is onto.

Since φ is $1-1$ and onto then φ is a bijection.
Thus $|H| = |Hg|$.

Theorem 3.6:
Assume that G is an order finite group say n, and H a subgroup of G. Then $|H|$ divides $|G|$

Proof:
Let $Ha_1, Ha_2, \dots, Ha_k$ be the right cosets of H in G.
$G = Ha_1 \cup Ha_2 \cup Ha_3 \cup, \dots, \cup Ha_k$
$\textit{and union is disjoint.}$
$\Longrightarrow |G| = |Ha_1 \cup Ha_2 \cup Ha_3 \cup, \dots, \cup Ha_k|$
$= |Ha_1| + |Ha_2| + \cdots + |Ha_k|$
$\because \textit{the union is disjoint}$
$= |H| + |H| + \cdots + |H|$
By cosets lemma
$k|H|$
So $|H| \mid |G|$.
Where k is the quantity of unique rights coset of $H \ in \ G$.

Theorem 3.7:
Every element x of a finite group of order g satisfy the equation $x^g = e$

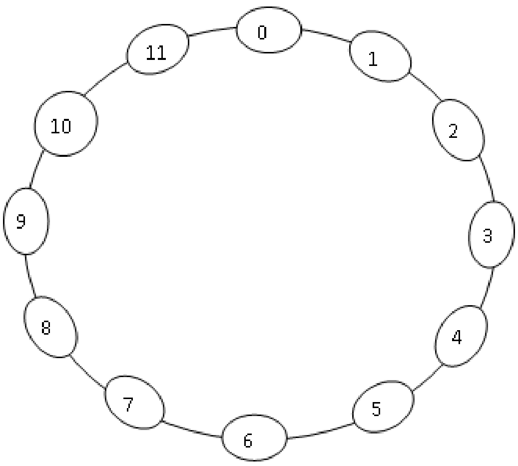
Proof:
The order m of x is a divisor of g say $g = mq$.
This gives $x^g = x^{mq} = (x^m)^q$
$e^q = e$
This complete the proof.

Theorem 3.8:
Every group of cyclics is abelian.

Proof:
Let G be a group of cycles. and a be a generator of G.
Then $G = \langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$.
(By the definition 1.10)
Let $g \ and \ \hat{g}$ be any two elements of G.
Then $g = a^\alpha \ and \ \hat{g} = a^\beta$ for some integer $\alpha \ and \ \beta$
So $g\hat{g} = a^\alpha a^\beta$
$= a^{\alpha+\beta} = a^{\beta+\alpha}$
$= a^\beta a^\alpha = \hat{g}g$
$\therefore g\hat{g} = \hat{g}g$
Therefore G is abelian.

Results and Discussion:

We present here some results which show clearly the relationship between musical notes and cyclic groups.
1st we consider a group of integer modulo 12 in real numbers.
Let $G = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, hence the $|G| = 12$.



$\boldsymbol{Musical\ clock = cyclic\ group\ of\ order\ 12.(\mathbb{Z}_{12}).}$

This group satisfies all the axioms of group theory. In the other hand, the numbering of the musical notes are also listed in the table below;

Name	C	$C\#$	D	$D\#$	E	F	$F\#$	G	$G\#$	A	$A\#$	B
Number	0	1	2	3	4	5	6	7	8	9	10	11

Note that $B\# = C$.
It also demonstrates how the notes in the song form an integer group with a 12-member modulus.
In other words;
$\mathbb{Z}_{12} = \{\mathrm{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B}\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
Let the operation be $* = \# = +$
Next, the connection between group axioms and musical notes
i. $\forall$ E, F $\in \mathbb{Z}_{12}$ Then $* F = A \in \mathbb{Z}_{12}$: Closure law
ii. $\forall$ E, F, F# $\in \mathbb{Z}_{12}$
Then $(E * F) * \mathrm{F\#} = E * (F * \mathrm{F\#}) = A * \mathrm{F\#} = \mathrm{E} * \mathrm{B} = \mathrm{D\#} \in \mathbb{Z}_{12}$: Associative law
iii. F $\in \mathbb{Z}_{12}$
There exists an element $C \in \mathbb{Z}_{12}$ s.t. $* C = C * F = F \in \mathbb{Z}_{12}$: Identity law
iv. F $\in \mathbb{Z}_{12}$ There exists an element $G \in \mathbb{Z}_{12}$ s.t. $F * G = G * F = C \in \mathbb{Z}_{12}$
: Inverse law
v. F, G $\in \mathbb{Z}_{12}$ s.t. $F * G = G * F = C \in \mathbb{Z}_{12}$: Commutative law.
Hence in addition a group is said to be abelian (or commutative) if
$a * b = b * a \ \forall a, b \in G$. i.e. the operation does not change the result.
For instance; $3 + 5 = 5 + 3$.

Proposition (Dido's theorem)
There must be at least one unique element in G if it is cyclic with its inverse.

Proof;
Suppose G is cyclic $\Rightarrow \forall\, x \in G$, each $x \in G$ can be written in the form $x = g^m$ for some $g \in G$ where $m \in \mathbb{Z}$ there exist $y \in G\ s.t.\ x * y = e \in G$
$\Rightarrow x = y$ where e is the identity element $y = x^{-1} \Rightarrow x = x^{-1}$.
Hence the proof.

Result of Theorem
$\mathbb{Z}_{12} = \{\mathrm{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B}\}$.
$H = \{\mathrm{C, C\#, B}\} \Rightarrow H \leq \mathbb{Z}_{12}$
$DH = \{\mathrm{D * C, D * C\#, D * B}\}$.
$DH = \{\mathrm{D, D\#, C\#}\}$.
Clearly, $D \in DH$ and $|H| = |DH| = 3$.
Furthermore, for some $A, F \in \mathbb{Z}_{12}$ $(A * F)H = DH$
$\Rightarrow$ In this instance, two left cosets are the same for some $A, G \in \mathbb{Z}_{12}$
$A * G \in \mathbb{Z}_{12}$ but $A * G = E \neq D, EH \neq DH$. Here, two left cosets are disjoint.

Results of theorem 3.1; 3.
$|\mathbb{Z}_{12}| = 12$ Since $|\mathbb{Z}_{12}| = 12$; $|H| = 3$
$\Rightarrow |\mathbb{Z}_{12}|/|H| = 12/3 = 4$.
It is true that a group's order is divided by a subgroup's order.

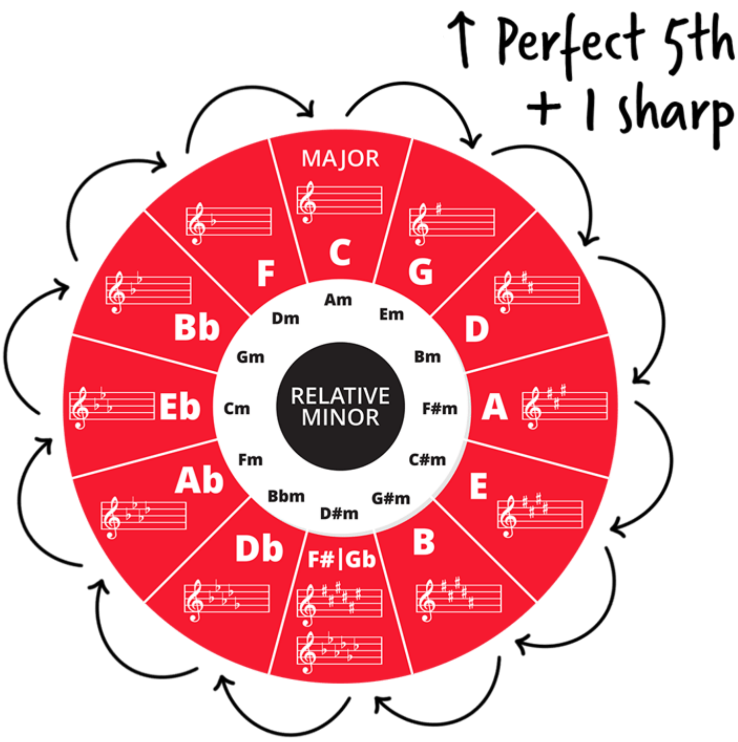
The result of theorem 3.3;8.
From $\mathbb{Z}_{12} = \{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\}$ for $C \in \mathbb{Z}_{12}$ we have
$C^0 = C;\ C^1 = C\#;\ C^2 = D;\ C^3 = D\#; \ldots\ C^{11} = B;\ C^{12} = C$
Musical notes are cyclic.
Hence we can write $\mathbb{Z}_{12} = \langle C \rangle$
From the subgroup $H = \{C, C\#, B\}$; $B \in H \leq \mathbb{Z}_{12}$
We have $B^0 = B;\ B^1 = C;\ B^2 = C\#;\ B^3 = B$.
Clearly, H is cyclic.
We are content with the theorem that says "every subgroup of a cyclic group is also cyclic" when it comes to musical notes.

Proposition: Each note in a song is a source of $\mathbb{Z}_{12}$
The proof of this proposition follows from the theorem 3.3;
$C^0 = C;\ C^1 = C\#;\ C^2 = C * C = D;\ C^3 = C^2 * C = D * C = D\#;$
$C^4 = C^3 * C = D\# * C = E;\ C^5 = C^4 * C = E * C = F;$
$C^6 = C^5 * C = F * C = F\#;\ C^7 = C^6 * C = F\# * C = G;$

$C^8 = C^7 * C = G * C = G\#$; $C^9 = C^8 * C = G * C = A$;
$C^{10} = C^9 * C = A * C = A\#$; $C^{11} = C^{10} * C = A\# * C = B$;
$C^{12} = C^{11} * C = B * C = C$.
Hence $\mathbb{Z}_{12} = \langle C \rangle$. In the same way every other note can behave same.
In the set $\mathbb{Z}_{12} = \{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\}$
$C_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$. The number $0\ to\ 11$ as representing musical intervals in multiple of semitones. The identity 0 may be defined to be any of the set $\{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\}$ the remaining elements correspond to the remaining pitch classes in cyclic order. $\mathbb{Z}_{12}$ The generators that could be used are 1, 5, 7, and 11. By continuously increasing by a fifth, all notes can be obtained from the given notes. The circle of fifths is the unique method for creating all musical intervals. To get back to the starting pitch class, one must possess all twelve tones clockwise, starting at any pitch and climbing by the interval of an equal tempered fifth..

The result of **theorem 3.5:** Recall that $|\mathbb{Z}_{12}| = 12$; $12 = 2 \times 2 \times 3 = 2^2 \times 3$
$\exists\ H \leq \mathbb{Z}_{12}\ s.t. |H| = 2^2$ which is sylow p-subgroup of $\mathbb{Z}_{12}$. It is true that a sylow p-subgroup exists for every finite group. It is consistent with Sylow's first theorem.
Eight-note intervals followed by ones that are half or double as frequent make up the octave. Two notes with frequencies of 2:1 on an octave. A note's octaves occur at a frequency twice that of the note. The ear perceives both notes as being the same when there is octave equivalency. An eight-note interval followed by a half- or double-frequency interval, modulo arithmetic becomes octave equivalency. If there is a complete octave difference between two notes, then they are in the same pitch class. After that, every element of Z_n is represented by a distinct combination of the eight pitch classes.

The twelve chromatic pitches are arranged according to a perfect fifth sequence in music theory using the circle of fifths. (This is accurate in the conventional 12-tone equal temperament system; if one uses an alternative system, one decreased sixth interval must be regarded as a fifth.). If C is chosen as a starting point, the sequence is: C, G, D, A, E, B (=C♭), F♯ (=G♭), C♯ (=D♭), A♭, E♭, B♭, F. Resuming the sequence at point C is the result of continuing the pattern from F. The key signatures that are most closely related to one another are arranged in this sequence. Typically, a circle is used to depict it.



Cycle of fift.
Pitch organization using the circle of fifths results in a series of perfect fifths. The pitches (together with the matching keys) are typically displayed as a circle with a clockwise progression. The circle is a circle of fourths

since it is frequently utilized in a counterclockwise way. Adjacent keys in this system are frequently used in Western music harmonic progressions, making it a helpful tool for harmony and musical composition..

A perfect fifth is made up of two pitches with a frequency ratio of 3:2 using the system of just intonation; nevertheless, producing twelve consecutive perfect fifths in this manner does not return to the pitch class of the initial note. Instruments are typically tuned using the equal temperament system to account for this. A perfect fifth is equal to seven equal-temperament semitones, and twelve equal-temperament fifths result in a note that is precisely seven octaves above the original tone.

The C Major key, without any flats or sharps, is displayed at the top of the circle. The pitches rise by fifths as one moves clockwise. Additionally, the key signatures connected to certain pitches vary: The key of G has one sharp, the key of D has two, and so on. Similar to how the key signatures shift in response to the notes' changing by descending fifths, going counterclockwise from the circle's top: There is one flat in the key of F, two flats in the key of B♭, and so on. Certain keys (located at the bottom of the circle) have the option of being notated in flats or sharps.

A pitch class is made up among all the notes denoted by a particular letter, independent of octave; all "C"s, for example, belong to the same pitch class. Starting at any pitch and climbing by a fifth produces all twelve tones before going back to the beginning pitch class. Pitch descent occurs by a fifth when moving counterclockwise; however, pitch class remains unchanged if one ascends by a perfect fourth, as this will result in an octave higher note. One could think of moving counter-clockwise from C as either ascending by a fourth to F or dropping by a fifth.

Organization and application

Diatonic key signatures

There is a diatonic scale connected with each of the twelve pitches, which can be used as the tonic of a major or minor key. The major key is represented by a capital letter, and the minor key is represented by a lower-case letter, in the circle diagram, which displays the number of sharps or flats in each key signature. Relative major and relative minor of one another are major and minor keys with the same key signature.

Chord progression and modulation

Tonal music frequently shifts to a new tonal center with a key signature that only varies by one flat or sharp. These closely related keys are adjacent in the circle of fifths because they are a fifth apart. The circle of fifths is helpful in depicting the "harmonic distance" between chords since chord progressions frequently shift between chords whose roots are related by perfect fifth.

The harmonic or tonal function of chords is arranged and explained using the circle of fifths. Chords can follow a pattern known as "functional succession" that progresses in ascending perfect fourths (or, alternatively, descending perfect fifths). It's possible to demonstrate "...by the circle of fifths (in which, therefore, scale degree II is closer to the dominant than scale degree IV)". According to this perspective, a chord progression generated from the circle of fifths ends at the tonic, or tonal center."

Relative Minor Keys

What about the inner circle, though? We keep our relative minor keys in this location.

There is a relative minor key for each major key. Because they are related, related minors share the same key signature as their major key sister. The sole distinction is that you use a different note to begin and conclude your scale. Three half-steps separate this note from the relative major key's beginning note. For instance, there are no flats or sharps in C Major. A is three half steps below C. As a result, A Minor is C Major's relative minor key, and it likewise lacks sharps and flats.

SUMMARY

This research has the potential to revolutionize the way musicians conceive, analyze, and communicate musical ideas. By exploring alternative notation systems based on finite cyclic groups, this study aims to offer new perspectives on music theory and composition, foster creativity and experimentation in musical practice, and enhance music education by introducing novel pedagogical tools. And has investigate the theoretical foundations of finite cyclic groups and their applicability to musical notations.

CONCLUSION

By exploring the application of finite cyclic groups to musical notations, this research seeks to bridge the gap between mathematical theory and musical practice, explore potential practical applications of finite cyclic group-based notations in music composition, analysis, and education. Offering innovative solutions to age-old challenges in music representation and analysis. Through interdisciplinary collaboration and empirical validation, this study aims to contribute to the advancement of both music theory and computational musicology, with potential implications for a wide range of musical endeavors.

We propose that musicians may write the best music and satisfy their audiences while also curing people's brains if they have a solid understanding of group theory.

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