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# Hypercyclicity Of Basic Elementary Operator

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***Abstract:*** *Hypercyclicity is the study of linear and continuous operators that possess a dense orbit. Hypercyclicity of linear operators is one of the most studied properties of linear dynamics and has become an active area of research. Hypercyclicity of various types of elementary operators including the generalised derivations, left and right multiplication operators among others has been studied and various results obtained. In this paper we present some results on hypercyclicity of the basic elementary operator.*

***Key words:*** *Hypercyclicity , Basic elementary operator and Orbit*

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## I. Introduction

Linear dynamics has been an active area of study in the recent past. In particular the study of hypercyclicity as a linear dynamic property has yielded interesting results on different operators. Hypercyclicity is the study of linear and continous operators that possess a dense orbit. It is an infinite dimensional property when the operator generates a dense set. For an operator to be hypercyclic it suffices to require that it has an orbit with a single non-zero limit point. Different researchers have shown results of hypercyclicity of different operators including differentiation operators, weighted backward shift, generalized derivations and left and right multiplication operators among others.

Maclane [7] proved the hypercyclicity of differentiation operator while Rolewicz [9] discussed the hypercyclicity on Banach spaces by showing that $\lambda B$ is hypercyclic whenever $B$ is unilateral backward shift. Ansari [1] showed that every non-zero power $T^m$ of hypercyclic linear operator T is hypercyclic.

Salas [10] showed that every perturbation of identity by unilateral weighted backward shift with non-zero bounded weight is hypercyclic. Toukmati [11] proved that if a generator A does not have a single valued extension property, then there exists a closed subspace such that the $C_o$ semigroup is hypercyclic.

De la Rosa [4] gave a characterization when an extended eigenoperator is hypercyclic. Farrukh [5] showed that there does not exist any hypercyclic operator on finite dimensional space.

In this paper we determine hypercyclicity of basic elementary operator in a complex Hilbert space endowed with the Strong Operator Topology .

## II. Hypercyclicityof Elementary Operators

An operator $E : B(H) \to B(H)$ is called an *elementary operator* if $E$ can be expressed in the form; $EX = \Sigma T_i X S_i$ with $T_i$ and $S_i (1 \leq i \leq l)$ in $B(H)$ and $l \in$ □.

If $T, S \in B(H)$, we define a *basic elementary operator;*

$$M_{T,S}: B(H) \to B(H) by M_{T,S}X = TXS$$

Hypercyclicity requires separability of the underlying space. We note that separability in the *strong operator topology implies separability in the strong operator topology.

The space $B(H)$ which has many topologies including; operator norm topology,strong operator topology and * strong operator topology is usually non-separable under the operator norm-topology when $X$ is a classical Banach space. To overcome this obstacle Chan [2] gave an option of considering spaces of operators which are separable when endowed with weaker topologies. This makes $B(H)$ separable in the strong operator topology. Using this approach, Chan [2] investigated the hypercyclicity of left multiplier $L_A$ on the space $B(H)$, under strong operator topology, where H is a separable Hilbert space.

Yousefi [12] obtained results for $B(H)$ when endowed with the * strong operator topology. Rezaei [8] gave a simple result on hypercyclicity of the perturbation of identity operators by weighted backward shift.

According to Clifford [3] examples of hypercyclic generalized derivations can be obtained by taking either $A \equiv 0$ or $B \equiv 0$ in the generalized derivation $T_{A,B}$ which takes us to the case of the multipliers $L_A$ *and* $R_B$. He analysed the types of hypercyclic generalized derivations that exist by considering the most fundamental classes of hypercyclic operators of the form $I + B_w$ where I is the identity map and $B_w$ is a weighted backward shift.

## III. Main Result

Let $H$ be a separable infinite dimensional complex Hilbert space and $B(H)$ be the set of all bounded linear operators on $H$. Endowed with the strong operator topology , we can show that the basic elementary operator is hypercyclic on $B(H)$.
A set $A \subseteq B$ is said to be dense in B if $\bar{A} = B$
Now let $W: B(H) \to B(H)$ be a bounded linear operator. Then we may consider operator $X \in B(H)$ as a vector. In this case, $X$ is said to be a hypercyclic vector for $W$ if the orbit, $orb(W, X)$ is dense in $B(H)$ with respect to the strong operator topology.
Note that $Orb(W, X) = \{X, WX, W^2X, W^3X, \dots\}$.
If $X \in B(H)$ is a hypercyclic vector for $W$, then $W$ is hypercyclic.
As indicated in section 2 above, $B(H)$ is not separable when endowed with the usual operator norm. However, in the lemma below, we prove that $B(H)$ is strong operator separable.

**Lemma 3.1**

Let $H$ be a separable infinite dimensional complex Hilbert space and $B(H)$ be the set of all bounded linear operators on $H$. Then $B(H)$is Strong Operator separable.

***Proof***

Let $\{e_k: k \geq 1\}$ be an orthonormal basis of $H$, $p_n: H \to H$ be the orthonormal projection onto the span $\{e_k: 1 \leq k \leq n\}$ and $I: H \to H$ be the identity operator. Then for any vector $c \in H$ and for any operator $T \in B(H)$ we have;
$\|P_nTP_nc - Tc\| \leq \|P_nT(P_nc - c)\| + \|T\|\|(P_n - I)Tc\| \leq \|T\|\|(P_n - I)c\| + \|(P_n - I)Tc\|$.
Now, taking the limits as $n \to \infty$ then the right hand side goes to zero. This shows that $P_nTP_n \to T$ as $n \to \infty$. Therefore, $B(H)$ has a countable dense subset in the strong operator topology and this shows that $B(H)$ is separable. ■

In the next theorem we present the main result of this paper.

**Theorem 3.2**

Let $B(H)$ be strong operator topology separable C*-algebra for a complex separable Hilbert space H. Let $M_{T,S}: B(H) \to B(H)$ be the basic elementary operator for some fixed $T, S \in B(H)$ with
$T^n \to I$ and $S^n \to I$ in B(H) as $n \to \infty$. Then $M_{T,S}$ is hypercyclic in the strong operator topology.

**Proof**

We need to show that there exists a vector $X \in B(H)$ which is hypercyclic for $M_{T,S}$. In other words, we must show that the orbit of $X$ is dense in $B(H)$ in the strong operator topology.
Now, the orbit of $X \in B(H)$ with respect to the operator $M_{T,S}$ is the set

$$orb\big(M_{T,S}X\big) = \{X, M_{T,S}X, M^2{}_{T,S}X, M^3{}_{T,S}X, \dots\}$$
$$= \{X, TXS, T^2XS^2, T^3XS^3, \dots\}$$

This orbit is dense in $B(H)$ if every sequence of the form $\{M^n{}_{T,S}(X)\}$ converges to some element of $B(H)$.
Now consider the sequence $\{T^nXS^n\}_{n=0}^{\infty}$ for an operator $X \in B(H)$. We claim that $T^nXS^n \to X$ as $n \to \infty$.
Indeed , for any vector $f \in H$ we have;

$$\|T^nXS^nf - Xf\| = \|T^nXS^nf - T^nXf + T^nXf - Xf\| \leq \|T^nXS^nf - T^nXf\| + \|T^nXf - Xf\|$$
$$= \|T^nX(S^n - I)f\| + \|(T^n - I)Xf\| \leq \|T^n\|\|S^n - I\|\|Xf\| + \|T^n - I\|\|Xf\|$$

Since both $T^n \to I$ and $S^n \to I$ as $n \to \infty$, then the limit of the right hand side is zero. Thus
$\|T^nXS^nf - Xf\| = 0$ or $T^nXS^n \to X$ as $n \to \infty$. This completes the proof. ■

## IV. Conclusion

In this paper we have shown that the basic elementary operator is hypercyclic when endowed with the strong operator topology. One may further investigate if the basic elementary operator satisfies the hypercyclicity criterion.

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