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On Intertwining and Quasi-Affine Sets of Operators

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Abstract
In this paper, we investigate some intertwining sets and quasi-affine sets of some classes of operators in Hilbert spaces. We are interested in the intertwining relation of the form $WX = XR$, where W, R are some bounded linear operators and X is an arbitrary bounded linear operator which we will endow some special properties.
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I. Introduction

Let $\mathcal{H}$ denote a Hilbert space and $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators. If $T \in B(\mathcal{H})$, then T^* denotes the adjoint of T, while $Ker(T)$, $Ran(T)$, $\overline{\mathcal{M}}$ and $\mathcal{M}^\perp$ stands for the kernel of T, range of T, closure of $\mathcal{M}$ and orthogonal complement of a closed subspace $\mathcal{M}$ of $\mathcal{H}$, respectively. We denote by $\sigma(T)$, $\|T\|$ and $W(T)$, the spectrum, norm and numerical range of T, respectively. Recall that an operator $T \in B(\mathcal{H})$ is
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normal if $T^*T = TT^*$.

self-adjoint (or *hermitian*) if $T^* = T$.

skew-adjoint if $T^* = -T$.

unitary if $T^*T = TT^* = I$.

quasinormal if $T(T^*T) = (T^*T)T$.

binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$.

hyponormal if $T^*T \geq TT^*$.

θ-operator if T^*T and $T + T^*$ commute.

a *projection* if $T^2 = T$ and $T^* = T$.

an *involution* if $T^2 = I$.

a *symmetry* if $T = T^* = T^{-1}$. That is, T is self-adjoint unitary.

isometric if $T^*T = I$.

a contraction if $\|T\| \leq 1$.

Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. We say that $X \in B(\mathcal{H}, \mathcal{K})$ *intertwines* A and B if $XA = BX$. We denote by $I(A, B) = \{X \in B(\mathcal{H}, \mathcal{K}) : XA = BX\}$ the *intertwining set* of A and B. In this case we call X the *intertwining operator*. If X has dense range, then we say that A and B are *densely intertwined* by X.
If X intertwines both the pairs (A, B) and (B, A), then we say that X *doubly intertwines* A and B.
The set $I[A, B] = \{X \in B(\mathcal{H}, \mathcal{K}) : XA = BX \text{ and } XB = AX\}$ is called the *double intertwining set* of A and B.

The commutator of $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ is defined as $C(A, B) = [A, B] = AB - BA$. The self-commutator of $A \in B(\mathcal{H})$ is defined as $C(A^*, A) = [A^*, A] = A^*A - AA^*$. Let Ω be a class or subset of $B(\mathcal{H})$. The commutator set of the class Ω is defined as $\mathcal{C}(\Omega) = \{AB - BA : A, B \in \Omega\}$. Clearly, $\mathcal{C}(\Omega) = \{C(A, B) : A, B \in \Omega\}$.

The *commutant* of T denoted by $\{T\}'$ is the set of all operators that commute with T. That is $\{T\}' = \{S \in B(\mathcal{H}) : ST = TS\}$. The *bicommutant* or *double commutant* of $T \in B(\mathcal{H})$ denoted by $\{T\}''$ is defined by

$$\{T\}'' = \{A \in B(\mathcal{H}) : AS = SA, \ S \in \{T\}'\} = \{p(T) : T \in B(\mathcal{H}), \ p \, a \, polynomial\} = \bigcap_{S \in \{T\}'} \{S\}'.$$

Note that the lattices *Lat(T)* and *Hyperlat(T)* have set-theoretic set inclusion ordering $(\subseteq)$ of the power set $\mathcal{P}(\mathcal{H})$ as a partial order $\leq$ on them. With this partial order each of *Lat(T)* or *Hyperlat(T)* is a complete lattice with $\mathcal{H}$ as the greatest element and zero $\{0\}$ as the least element. If L_1 and L_2 are complete lattices, we write $L_1 \equiv L_2$ to signify that there is a (complete) lattice isomorphism of one onto the other.

A quasiaffinity X is said to have the *hereditary property* with respect to an operator $T \in B(\mathcal{H})$ if $X \in \{T\}'$ and $\overline{X(\mathcal{M})} = \mathcal{M}$ for every $\mathcal{M} \in Hyperlat(T)$. If T_1 and T_2 are quasisimilar and there exists an implementing pair (X, Y) of quasiaffinities such that XY has the hereditary property with respect to T_1 and YX has the hereditary property with respect to T_2, then we say that T_1 is *hyper-quasisimilar* to T_2, and we denote this by $T_1 \diamond T_2$. The notion of hyper-quasisimilarity was introduced by C. Foias etal[7].

Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be *similar* (denoted $A \sim B$) if there exists an invertible operator $N \in B(\mathcal{H}, \mathcal{K})$ such that $NA = BN$ or equivalently $A = N^{-1}BN$, and are unitarily equivalent (denoted by $A \cong B$)if there exists a unitary operator $U \in B_+(\mathcal{H}, \mathcal{K})$ (Banach algebra of all invertible operators in $B(\mathcal{H})$) such that $UA = BU$ (i.e. $A = U^*BU$, equivalently, $A = U^{-1}BU$). Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be metrically equivalent (denoted by $A \sim_m B$) if $\|Ax\| = \|Bx\|$, (equivalently, $|\langle Ax, Ax\rangle|^{\frac{1}{2}} = |\langle Bx, Bx\rangle|^{\frac{1}{2}}$ for all $x \in \mathcal{H}$)(see [10]). Clearly similarity, unitary equivalence and metric equivalence are equivalence relations on $B(\mathcal{H})$.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. $X \in B(\mathcal{H}, \mathcal{K})$ is called a quasiaffinity or quasiinvertible it has trivial kernel and dense range(that is $Ker(X) = \{0\}$ and $\overline{Ran(X)} = K$). An operator $S \in B(\mathcal{H})$ is said to be a quasiaffine transform of $T \in B(\mathcal{K})$ (denoted by $S \prec T$) if there

exists a quasiaffinity $X \in B(\mathcal{H}, \mathcal{K})$ such that $XS = TX$. By

$$\mathcal{Q}_{\dashv}(B) = \{A \in B(\mathcal{K}) : XA = BX, X \ a \ quasiaffinity\}$$

the set of quasiaffine transforms of B also called the *quasiaffine orbit* of B. If X is invertible, then $\mathcal{Q}(B)$ coincides with the *similarity orbit* of B. Operators $S \in \mathcal{H}$ and $T \in \mathcal{K}$ are said to be quasisimilar if there exists quasiaffinities $X \in B(\mathcal{H}, \mathcal{K})$ and $Y \in B(\mathcal{K}, \mathcal{H})$ such that $XT = SX$ and $TY = YS$. The set of all operators quasisimilar to $B \in B(H)$ is called the quasisimilarity orbit of B and is denoted by

$$\mathcal{Q}_f(T) = \{A \in B(\mathcal{K}) : XA = BX, YA = BY, X, Y \ quasiaffinities\}.$$

A subspace (closed linear manifold) $\mathcal{M} \subseteq \mathcal{H}$ is said to be *invariant* under $T \in B(\mathcal{H})$ if $x \in \mathcal{M}$ implies that $Tx \in \mathcal{M}$ or $T\mathcal{M} \subset \mathcal{M}$. A subspace (closed linear manifold) $\mathcal{M} \subseteq \mathcal{H}$ is said to be a *reducing* subspace for $T \in B(\mathcal{H})$ or reduces T if it is invariant under both T and T^*(equivalently, if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for T). A subspace (closed linear manifold) $\mathcal{M} \subseteq \mathcal{H}$ is said to be a *hyperinvariant subspace* for $T \in B(\mathcal{H})$ if $S\mathcal{M} \subset \mathcal{M}$ for each $S \in \{T\}'$. That is, it is invariant under every operator commuting with T. By a *subspace lattice* on $\mathcal{H}$ we mean a family of subspaces of $\mathcal{H}$ which is closed under the formation of arbitrary intersections and and arbitrary linear spans and which contains the zero subspace $\{\overline{0}\}$ and $\mathcal{H}$. The subspace lattice of all invariant, reducing and hyperinvariant subspaces of T is denoted by *Lat(T)*, *Red(T)* and *Hyperlat(T)*, respectively. Note that $Red(T)$ may not be a lattice. The subalgebra of all operators generated by an operator $T \in B(\mathcal{H})$, denoted by $W^*(T)$ will be called the (unital) weakly closed (von Neumann) algebra generated by T. We use this algebra to investigate the structures of invariant and hyperinvariant lattices for various operators.

II. Basic Results

Theorem 2.1 *Let $A, B \in B(\mathcal{H})$. Then the commutator $(A, B) \longrightarrow AB - BA$ is a bilinear operation $\varphi : B(\mathcal{H}) \times B(\mathcal{H}) \longrightarrow B(\mathcal{H})$ with respect to the "variables" A and B.*

Proof. Let α be a scalar. Then

$$\varphi(\alpha A, B) = (\alpha A)B - B(\alpha A) = \alpha(AB - BA) = \alpha\varphi(A, B).$$

$$\varphi(A, \alpha B) = A(\alpha B) - (\alpha B)A = \alpha(AB - BA) = \alpha\varphi(A, B).$$

This shows that φ is linear in the first and second variable and hence bilinear.

Theorem 2.2 *Let $A, B \in B(\mathcal{H})$. Then $I(A, B)$ is a closed subspace of $B(\mathcal{H})$.*

Proof. Let $T, T_1, T_2 \in I(A, B)$ and let $\alpha \in \mathbb{C}$. Then $TA = BT, T_1A = BT_1$ and $T_2A = BT_2$. Thus

$$(T_1 + T_2)A = T_1A + T_2A = BT_1 + BT_2 = B(T_1 + T_2),$$

$$(T_1T_2)A = T_1(T_2A) = T_1(BT_2) = (T_1B)T_2 = B(T_1T_2)$$

and

$$(\alpha T)A = \alpha(TA) = \alpha(AT) = A(\alpha T).$$

This proves that $I(A, B)$ is closed with respect to addition, multiplication and scalar multiplication. Trivially, the zero operator $O \in I(A, B)$. This proves the claim.

Recall that an algebra over a field $\mathbb{F}$ is a vector space with a bilinear product, that is a set together with operations of multiplication, addition and scalar multiplication by elements of a field, satisfying the axioms implied by a vector space. An algebra is unital if it has an identity element with respect to the multiplication operation. A subalgebra is a subset of an algebra, closed under all its operations, and carrying the induced operations.

Theorem 2.3 *Let $A \in B(\mathcal{H})$. Then $\{A\}'$ is a unital subalgebra of $B(\mathcal{H})$.*

Proof. Let $C, C_1, C_2 \in \{A\}'$ and let $\alpha \in \mathbb{C}$. Then by definition $CA = AC, C_1A = AC_1$ and $C_2A = AC_2$. Therefore

$$(C_1 + C_2)A = C_1A + C_2A = AC_1 + AC_2 = A(C_1 + C_2)$$

$$(C_1C_2)A = C_1(C_2A) = C_1(AC_2) = (C_1A)C_2 = A(C_1C_2)$$

and

$$(\alpha C)A = \alpha CA = A(\alpha C).$$

This proves that $C_1 + C_2, C_1C_2$ and αC all belong to $\{A\}'$. That is $\{A\}'$ is closed under addition, multiplication and scalar multiplication. This proves the claim.
Clearly, by definition $I \in \{A\}'$. Hence, $\{A\}'$ is a unital subalgebra of $B(\mathcal{H})$.

Theorem 2.4 *Let $A, B \in B(\mathcal{H})$. Then $I[A, B] \subseteq I(A, B)$.*

Proof. The proof follows from the definition of $I[A, B]$ and $I(A, B)$.

Theorem 2.5 *Let $A, B \in B(\mathcal{H})$. Then the solution to the operator equation $XA = BX$ is I(A,B).*

Theorem 2.6 *Let $A, B \in B(\mathcal{H})$. Then the solution to the operator equations $XA = BX$ and $XB = AX$ is I[A,].*

Theorem 2.7 *Let $A, B \in B(\mathcal{H})$. If $I(A, B)$ contains a unitary operator then A and B are unitarily equivalent.*

Theorem 2.8 *Let $A, B \in B(\mathcal{H})$. If $I(A,B)$ contains an invertible operator then A and B are similar.*

Theorem 2.9 *Let $A, B \in B(\mathcal{H})$. If $I(A,B)$ contains a quasiaffinity then B is a quasiaffine transform of A.*

Theorem 2.10 *Let $A, B \in B(\mathcal{H})$. If $I[A,B]$ contains a quasiaffinity then A and B are quasisimilar.*

If $I[A,B] = \{0\}$, then A and B are called *disjoint operators*.

Corollary 2.11 *Let $A, B \in B(\mathcal{H})$. If $I(A,B)$ and $I(B,A)$ contain quasiaffinities then A and B are quasisimilar.*

Note that if A and B are quasisimilar then they need not have equal spectra(see [9]) but $\sigma(A) \cap \sigma(B) \neq \emptyset$. However, quasisimilar subnormal operators have equal spectra (see [4]).

Theorem 2.12 *Let $A, B \in B(\mathcal{H})$ and $A = B$ then $I(A,A) = I[A,A] = \{A\}'$ and $\{I(A,A)\}' = \{I[A,A]\}' = \{\{A\}'\}' = \{A\}''$.*

Proof. Follows from the definitions.

Let $\mathcal{S}$ be a subset of $B(\mathcal{H})$. We define

$$\mathcal{S}' := \{T \in B(\mathcal{H}) : TS = ST, \ \forall \ S \in \mathcal{S}\}$$

and

$$\mathcal{S}'' := \{B \in B(\mathcal{H}) : BA = AB, \ \forall \ A \in \mathcal{S}'\}.$$

Note that

$$\mathcal{S}'' = \{\mathcal{S}'\}'.$$

Theorem 2.13 *Let $\mathcal{S}$ be a subset of $B(\mathcal{H})$. Then $\mathcal{S} \subseteq \mathcal{S}''$.*

Proof. By definition, every $S \in \mathcal{S}$ commutes with every $T \in \mathcal{S}'$. Hence $\mathcal{S} \subseteq \mathcal{S}''$.

Corollary 2.14 *Let $\mathcal{S}$ be a subset of $B(\mathcal{H})$. Then $\mathcal{S}' \subseteq \mathcal{S}'''$.*

Proof. The proof follows from Theorem 2.13.

Theorem 2.15 *Let $\mathcal{S}$ and $\mathcal{T}$ be subsets of $B(\mathcal{H})$. If $\mathcal{S} \subseteq \mathcal{T}$ then $\mathcal{T}' \subseteq \mathcal{S}'$.*

Corollary 2.16 *Let $\mathcal{S}$ be a subset of $B(\mathcal{H})$. Then $\mathcal{S}''' \subseteq \mathcal{S}'$.*

Proof. The proof follows from Theorem 2.15.

Proposition 2.17 *Let $\mathcal{S}$ be a subset of $B(\mathcal{H})$. Then $\mathcal{S}' = \mathcal{S}'''$.*

Proof. The proof follows from Corollary 2.14 and Corollary 2.16.

Theorem 2.18 *Let $\mathcal{S}$ and $\mathcal{T}$ be subsets of $B(\mathcal{H})$. Then*

(i). $(\mathcal{S} \cup \mathcal{T})' = \mathcal{S}' \cap \mathcal{T}'$.

(ii). $(\mathcal{S}' \cup \mathcal{T}')'' = (\mathcal{S}'' \cap \mathcal{T}'')' = (\mathcal{S} \cap \mathcal{T})'$ *if we assume that* $\mathcal{S} = \mathcal{S}''$ *and* $\mathcal{T} = \mathcal{T}''$.

Recall that A and B are similar if there exists an invertible operator X such that $B = XAX^{-1}$.

Theorem 2.19 *Suppose A and B are similar. Define the mapping*

$$\varphi : \{A\}' \longmapsto \{B\}'$$

by

$$\varphi(T) = XTX^{-1}$$

for all $T \in \{A\}'$. Then φ is an isomorphism from $\{A\}'$ onto $\{B\}'$.

Proof. It suffices to prove that φ is linear, injective, surjective and φ^{-1} is linear.
Let $T, T_1, T_2 \in \{A\}'$ and $\alpha \in \mathbb{C}$. Then

$$\varphi(T_1+T_2) = X(T_1+T_2)X^{-1} = X(T_1X^{-1}+T_2X^{-1}) = XT_1X^{-1}+XT_2X^{-1} = \varphi(T_1)+\varphi(T_2)$$

and

$$\varphi(\alpha T) = X(\alpha T)X^{-1} = \alpha XTX^{-1} = \alpha\varphi(T).$$

This shows that φ is linear.
Now suppose $T \in \{A\}'$. Then $\varphi(T) = 0$ implies that $XTX^{-1} = 0$ which implies that $T = 0$. Thus φ is injective.
Now suppose $B \in \{B\}'$. We show that there exists a $T \in \{A\}'$ such that $B = \varphi(T)$. But $B = XAX^{-1} = \varphi(T)$. This shows that φ in onto.

III. Main Results

Recall that $T \in B(\mathcal{H})$ is normal if $T^*T = TT^*$. We denote the class of normal operators by $\mathcal{N}$, the class of quasinormal operators by $\mathcal{Q}$, the class of binormal operators by $\mathcal{B}$ and the class of θ-operators by θ. Note that $\mathcal{Q} = \{T : [T, T^*T] = 0\}$, $\mathcal{B} = \{T : [T^*T, TT^*] = 0\}$ and $\theta = \{T : [T^*T, T + T^*] = 0\}$.

Theorem 3.1 *The class* $\mathcal{N} = \{T : [T^*, T] = 0\}$.

Proof. $\mathcal{N} = \{T : T^*T = TT^*\} = \{T : [T^*, T] = 0\}$.

Theorem 3.2 *Let $T \in B(\mathcal{H})$. The class $\mathcal{N} = \{T : [T^*, T] = 0\}$ is a closed subset of $B(\mathcal{H})$ under scalar multiplication.*

Proof. Suppose $T \in B(\mathcal{H})$ is normal and $\alpha \in \mathbb{C}$. Then $(\alpha T)^*(\alpha T) = \overline{\alpha}\alpha T^*T = \alpha\overline{\alpha}TT^* = (\alpha T)(\alpha T)^*$ which shows that αT is normal.
Next, suppose $\{T_k\}$ is a sequence of normal operators converging to $T \in B(\mathcal{H})$. Then

$$\|T^*T - TT^*\| \leq \|T^*T - T_k^*T_k\| + \|T_k^*T_k - TT^*\| \longrightarrow 0$$

as $k \longrightarrow \infty$. Hence $T^*T = TT^*$ and therefore T is normal.

Theorem 3.3 *If $T \in B(\mathcal{H})$ is normal then T^n is normal for any $n \in \mathcal{N}$.*

Proof. Since T is normal, $\mathcal{N} = \{T : [T^*, T] = 0\}$. By mathematical induction or simple calculation $(T^*T)^n = T^{*n}T^n = T^nT^{*n}$.

Theorem 3.4 *Let $T \in B(\mathcal{H})$. If $T \in \theta \cap \mathcal{B}$, the $T \in \mathcal{Q}$.*

Proof. See ([3] and [5]).

Theorem 3.5 *If $T \in B(\mathcal{H})$ is normal and S is unitarily equivalent to T then S is normal.*

Proof. Normality of T implies that $[T, T^*] = 0$. Suppose $S = U^*TU$, for some unitary operator $U \in B(\mathcal{H})$. Then

$$[S, S^*] = [U^*TU, U^*T^*U] = U^*[T^*, T]U = 0.$$

Hence S is normal. This proves the claim.

4 Quasiaffine Sets of some Operators

An operator $W : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ is a unilateral weighted shift if there exists an orthonormal basis $\{e_n : n = 0, 1, 2, ...\}$ and a sequence of scalars $\{\alpha_n\}$ such that $We_n = \alpha_n e_{n+1}$, for all $n = 0, 1, 2, ...$ If $\alpha_n = 1$ for all $n = 0, 1, 2, ..$, then W is called the unilateral shift or forward shift operator and is usually denoted by S. Clearly, $S(e_0, e_1, e_2, ...) = (0, e_0, e_1, e_2, e_3, ...)$.
It is known (see [6], Proposition 2.1) that a weighted shift is hyponormal if and only if its weight sequence $\{\alpha_n\}$ is increasing (that is, $\alpha_{n+1} \geq \alpha_n$). Clearly, the unilateral shift is a hyponormal operator on $\mathcal{H} = \ell^2(\mathbb{N})$.

Note that the quasiaffine transform of an operator T may not have exactly the same properties as T. We may have T being a quasiaffine transform of S without T inheriting many of the properties of S.

Example. Let $H = \ell^2(\mathbb{N})$. Define $W : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ by

$$We_0 = e_1, We_1 = \sqrt{2}e_2, We_n = e_{n+1},$$

for all $n = 2, 3, 4, ...$ Then there exists $X \in B(\mathcal{H})$ such that

$$Xe_0 = e_0, We_1 = e_1, We_n = \frac{1}{\sqrt{2}}e_n,$$

for all $n = 2, 3, 4, ...$ With respect to the orthonormal basis $e_n : n = 0, 1, 2, ...$ of $\mathcal{H}$, X has an infinite matrix representation given by $X = diag(1, 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, ...)$. It is clear that $Ker(X) = \{0\}$ and $\overline{Ran(X)} = \mathcal{H}$ and hence a quasiaffinity and $XW = SX$, where S is the unilateral shift on $\mathcal{H}$. But the weight sequence for W is $\{1, \sqrt{2}, 1, 1, 1, ...\}$ which is not increasing. So W is not hyponormal.

Theorem 4.1 *Let $T \in B(\mathcal{H})$ be hyponomal and let $A \in B(\mathcal{H})$ be a quasiaffine transform of T. Then $Ker(A - \lambda I) = Ker(A - \lambda I)^2$ for every $\in \mathbb{C}$.*

Proof. See ([6], Proposition 2.3).

Let $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ denote the open unit disc and $\overline{\mathbb{D}}\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ its topological closure.

Proposition 4.2 *If $T \in B(\mathcal{H})$ is a quasiaffine transform of a hyponormal operator L, then $\sigma(L) \subseteq \sigma(T)$.*

Proof. See [4].

Recall that $T \in B(\mathcal{H})$ is *bounded below* if there exists a constant $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$, for all $x \in \mathcal{H}$. If $T \in B(\mathcal{H})$ is bounded below and has dense range, then it is invertible.

Theorem 4.3 *(Bounded Inverse Theorem): Let $T \in B(\mathcal{H}, \mathcal{K})$. Then $Ker(T) = \{0\}$ if and only if T is injective if and only if $T^{-1} : Ran(T) \longrightarrow \mathcal{H}$ exists.*

Corollary 4.4 *Let $T \in B(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent, (a). T is bounded below.*
(b). $T^{-1} : Ran(T) \longrightarrow \mathcal{H}$ exists and is bounded.
(c). $\overline{Ran(T)} = Ran(T)$.

Remark. Note that if $T \in B(\mathcal{H}, \mathcal{K})$ is bounded below, then $Ker(T) = \{0\}$ and so T^{-1} : $Ran(T) \longrightarrow \mathcal{H}$ exists. It remains to show that T^{-1} is bounded. Let $y \in Ran(T) \subseteq \mathcal{K}$. Then there exists $x \in \mathcal{H}$ such that $Tx = y$. Thus

$$\|T^{-1}y\| = \|T^{-1}Tx\| = \|x\| \leq \frac{1}{\alpha}\|Tx\| = \frac{1}{\alpha}\|y\|, \tag{4.1}$$

for all $y \in \mathcal{K}$.

Proposition 4.5 *If $T \in B(\mathcal{H})$ is invertible and $S \in B(\mathcal{K})$ is hyponormal and $X \in B(\mathcal{H}, \mathcal{K})$ has dense range and $XT = SX$, then S is invertible.*

Proof. Clearly, $Ran(X) \subseteq Ran(S)$ and so $\mathcal{K} = \overline{Ran(X)} \subseteq \overline{Ran(S)}$, which implies that $\overline{Ran(S)} = \mathcal{K}$. Hence $Ran(S)$ is dense in $\mathcal{K}$. It remains to show that S is bounded below on $Ran(X)$. Let $y \in Ran(T)$. Then there exists $x \in \mathcal{H}$ such that $Tx = y$, that is x=T$^{-1}y.Thenusing(4.1), wededucethat\|S(Xx)\| = \|XTx\| \geq \frac{1}{\|T\|}\|Xx\|.Thisprovestheclaim.$

Remark. From Proposition 4.5, it follows that if an invertible operator T is densely intertwined by a hyponormal operator S, then S is invertible. Since $XT = SX$, then either T and S are both invertible or both non-invertible. A consequence of Proposition 4.5, is that quasisimilar hyponormal operators S and T have equal spectra, since for any $\lambda \in \mathbb{C}$, the operators $S - \lambda I$ and $T - \lambda I$ are both invertible or both non-invertible, and hence $\sigma(S) = \sigma(T)$.

Theorem 4.6 *Let $T \in B(\mathcal{H})$ be a contraction which is a quasiaffine transform of the unilateral shift $S \in B(\mathcal{H})$. Then $\sigma(T) = \overline{\mathbb{D}}$.*

Proof. There exists a quasiaffinity X such that $XT = SX$. Clearly every $\lambda \in \mathbb{D}$ is an eigenvalue of T^* (that is, $\lambda \in \sigma_p(T^*)$) with $dimKer(T^* - \lambda I) \geq dimKer(S^*)$. Therefore $\sigma(T) = \overline{\mathbb{D}}$.

Theorem 4.7 *Let $T \in B(\mathcal{H})$ be a contraction such that $XT = SX$ where X is a quasiaffinity and S is a unilateral shift. Let $\mathcal{M} \subseteq \mathcal{H}$ be a T-invariant subspace of $\mathcal{H}$(that is, $\mathcal{M} \in Lat(T)$). Then the map*

$$\varphi : Lat(T) \longmapsto Lat(S)$$

defined by $\varphi : M \longmapsto \overline{XM}$ is an isomorphism.

Theorem 4.8 *Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are quasisimilar, then $\mathcal{Q}_s(A) = \mathcal{Q}_s(A)$.*

Remark. Theorem 4.8 says that two quasisimilar operators have equal quasisimilarity orbits.

V. Discussion

The notions of intertwining sets, quasiaffine sets or orbits, commutators, commutants and double com- mutants of operators are very useful in solving the classical Carathéodory interpolation problems (see [8]) . Intertwining operators also find applications in solving ordinary and partial differential equations (see [1]) and also in the construction of exactly solvable or quantum mechanical systems described by Hamil- tonians (see [2]) and quantification of how well two observables described by operators can be measuredsimultaneously in the Heisenberg Uncertainty Principle in Quantum mechanics.

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