IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661,p-ISSN: 2278-8727, Volume 20, Issue 5, Ser. 1 (Sept. – Oct. 2024), PP 11-17
www.iosrjournals.org

Arithmetic Sequential Graceful Labeling For Arrow Graphs

P. Sumathi[1], G. Geetha Ramani[2]
[1]Department Of Mathematics,
C. Kandaswami Naidu College For Men, Chennai, Tamil Nadu, India.
[2]Department Of Mathematics,
New Prince Shri Bhavani College Of Engineering And Technology,
Chennai, Tamil Nadu, India.

Abstract:
Let G be a simple, finite, connected, undirected, non-trivial graph with p vertices and q edges. $V(G)$ be the vertex set and $E(G)$ be the edge set of G. Let $f: V(G) \to \{a, a+d, a+2d, a+3d, \dots, a+2qd\}$ where $a \geq 0$ and $d \geq 1$ is an injective function. If for each edge $uv \in E(G)$, $f^*: E(G) \to \{d, 2d, 3d, 4d, \dots, qd\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is a bijective function then the function f is called arithmetic sequential graceful labeling. The graph with arithmetic sequential graceful labeling is called arithmetic sequential graceful graph. In this paper, we prove that one side arrow graphs $AR_\eta^2, AR_\eta^3, AR_\eta^5$ and double-sided arrow graphs $D(AR_\eta^2), D(AR_\eta^3)$ are arithmetic sequential graceful graph.
Keywords: *Graceful labeling, Arithmetic sequential graceful labeling, Grid graph, one side arrow graph, double-sided arrow graph.*

Date of Submission: 10-09-2024 Date of Acceptance: 20-09-2024

I. Introduction

A fascinating area of research in graph theory is labeling. Giving values to edges or vertices is the process of labeling. It was Alexander Rosa [2] who first proposed the idea of graceful labeling. Later, a few labeling techniques were presented. See Gallian's dynamic survey [3] for further details. V J Kaneria , Meera Meghpara , H M Makadia Pasaribu[4] proved that grid graph is graceful graph . V. J. Kaneria, H. M. Makadia and M. M. Jariya [5] proved that arrow and double arrow graph is graceful graph. Here are the some of the definitions which are helpful in this article.

II. Definitions

Definition 2.1:

A function f is called graceful labeling of graph $G = (V.E)$ if $f: V \to \{0,1, \dots, q\}$ is injective and the induced function $f^*: E \to \{1,2, \dots, q\}$ defined as $f^*(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E$. A graph G is called graceful graph if it admits a graceful labeling.

Definition 2.2:

An arrow graph AR_η^t with width t and length η is formed by connecting a vertex v to the superior vertices of $P_m \times P_n$ by m new edges from one end.

Definition 2.3:

A double arrow graph $D(AR_\eta^t)$ with width t and length η is formed by connecting two vertices v & w to the superior vertices of $P_m \times P_n$ by $(m+m)$ new edges from both the ends.

Definition 2.4:

Let G be a simple, finite, connected, undirected, non-trivial graph with p vertices and q edges. $V(G)$ be the vertex set and $E(G)$ be the edge set of G. Let $f: V(G) \to \{a, a+d, a+2d, a+3d, \dots, a+2qd\}$ where a ≥ 0 and $d \geq 1$ is an injective function. If for each edge $uv \in E(G)$, $f^*: E(G) \to \{d, 2d, 3d, 4d, \dots, qd\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is a bijective function then the function f is called arithmetic sequential graceful labeling. The graph with arithmetic sequential graceful labeling is called arithmetic sequential graceful graph.

III. Main Results

Theorem 3.1:
Arrow graph AR_η^2 is arithmetic sequential graceful graph, when $\eta \geq 1$.

Proof:

Let $G = AR_\eta^2$ be an arrow graph obtained by connected a vertex v with superior vertices of $P_2 \times P_\eta$ by two new edges. Let $v_{i,j}(1 \leq i \leq 2, 1 \leq j \leq \eta)$ be the vertices of $P_2 \times P_\eta$.
Join v with $v_{i,1}(1 \leq i \leq 2)$by two new edges to get G.
$V(G) = \{v_{i,j}: 1 \leq i \leq 2 ; 1 \leq j \leq \eta\} \cup \{v\}$
$E(G) = \{v_{i,j}\, v_{i,j+1}: 1 \leq i \leq 2 ; 1 \leq j \leq \eta - 1\} \cup \{v\, v_{i,1}: 1 \leq i \leq 2\}$
Here $|V| = 2\eta + 1$, $|E| = 3\eta$.
We define a function $f: V(G) \to \{a, a+d, a+2d, a+3d, \dots, a+2qd\}$.
The vertex labeling are as follows,
$f(v) = a, f(v_{2,1}) = a + d$
$f(v_{1,j}) = a + \left[3n - \frac{3(j-1)}{2}\right] d, when\ j \equiv 1(mod 2)\ for\ all\ j = 1,2,\dots,\eta$
$f(v_{1,j}) = a + \left[\frac{3j}{2} - 1\right] d, when\ j \equiv 0(mod 2)\ for\ all\ j = 1,2,\dots,\eta$
$f(v_{2,j}) = a + [f(v_{1,j-1}) - 2(-1)^j]d, \quad for\ all\ j = 1,2,\dots,\eta$
From the function $f^*: E(G) \to \{d, 2d, 3d, 4d, \dots, qd\}$ we get the edge labels of the arrow graph AR_η^2 as follows

Table:1 Edge labels of the graph AR_η^2

$f^*(u\,v)$	Edge labels	Value
$f^*(v\, v_{1,j})$	$\left\lvert\left[3n - \frac{3(j-1)}{2}\right] d\right\rvert$	$j = 1$
$f^*(v_{1,j} v_{1,k})$	$\left\lvert\left[3n - \frac{3(j-1)}{2} - \frac{3k}{2} + 1\right] d\right\rvert$	$j = 1, k = 2$
$f^*(v_{2,j} v_{2,k})$	$\lvert[1 - f(v_{1,k-1}) + 2(-1)^k]d\rvert$	$j = 1, k = 2$
$f^*(v_{1,j} v_{2,k})$	$\left\lvert\left[\frac{3j}{2} - 1 - f(v_{1,k-1}) + 2(-1)^k\right] d\right\rvert$	$j \equiv 0(mod 2), j = 1,2,\dots,\eta - 1$ $k \equiv 0(mod 2), k = 1,2,\dots,\eta - 1$
$f^*(v_{1,j} v_{2,k})$	$\left\lvert\left[3n - \frac{3(j-1)}{2} - f(v_{1,k-1}) + 2(-1)^k\right] d\right\rvert$	$j \equiv 1(mod 2), j = 1,2,\dots,\eta - 1$ $k \equiv 1(mod 2), k = 1,2,\dots,\eta - 1$

It is clear that the function f is injective and also table 1 shows that

$f^*: E \to \{d, 2d, 3d, 4d, \dots, qd\}$ is bijective. Hence f is arithmetic sequential graceful labeling and the graph AR_η^2 is arithmetic sequential graceful graph.

Example 3.1.1: Arrow graph of AR_5^2 and its graceful labeling shown in figure-1.

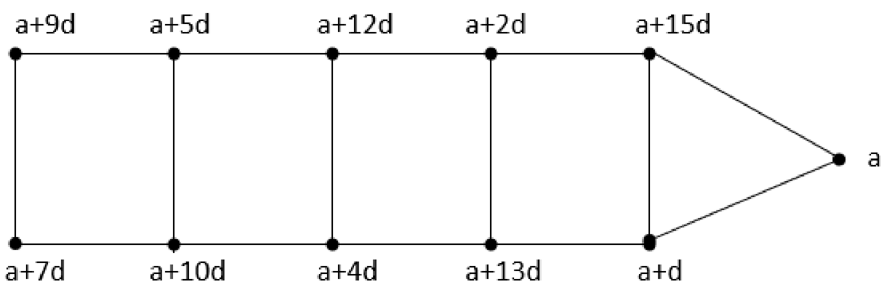

Figure -1: Arrow graph of AR_5^2 and its graceful labeling.

Theorem 3.2:
Arrow graph AR_η^3 is arithmetic sequential graceful graph, when $\eta \geq 2$.

Proof:

Let $G = AR_\eta^3$ be an arrow graph obtained by connected a vertex v with superior vertices of $P_3 \times P_\eta$ by 3 new edges. Let $v_{i,j} (1 \le i \le 3, 1 \le j \le \eta)$ be the vertices of $P_3 \times P_\eta$.
Join v with $v_{i,1} (1 \le i \le 3)$ by 3 new edges to get G.
$V(G) = \{v_{i,j} : 1 \le i \le 3 ; 1 \le j \le \eta\} \cup \{v\}$
$E(G) = \{v_{i,j}\, v_{i,j+1} : 1 \le i \le 3 ; 1 \le j \le \eta - 1\} \cup \{v\, v_{i,1} : 1 \le i \le 3\}$
Here $|V| = 3\eta + 1$, $|E| = 5\eta$.
We define a function $f: V(G) \rightarrow \{a, a+d, a+2d, a+3d, \dots, a+2qd\}$.
The vertex labeling are as follows,
$f(v) = a, f(v_{2,1}) = a + 2d$
$f(v_{1,j}) = a + \left[5\eta - \frac{5(j-1)}{2}\right] d, when\ j \equiv 1 (mod 2)\ for\ all\ j = 1,2, \dots, \eta - 1$
$f(v_{1,j}) = a + \left[\frac{5j}{2} - 1\right] d, when\ j \equiv 0 (mod 2)\ for\ all\ j = 1,2, \dots, \eta - 1$
$f(v_{2,j}) = a + [f(v_{1,j-1}) - 3(-1)^j] d\ \ for\ all\ j = 1,2, \dots, \eta - 1$
$f(v_{3,j}) = a + [f(v_{1,j}) + (-1)^j] d\ \ for\ all\ j = 1,2, \dots, \eta - 1$
$f(v_{1,\eta}) = a + [f(v_{1,\eta-1}) - 5(-1)^\eta] d$
$f(v_{2,\eta}) = a + [f(v_{2,\eta-1}) + 4(-1)^\eta] d$
$f(v_{3,\eta}) = a + [f(v_{3,\eta-1}) - 6(-1)^\eta] d$
From the function $f^*: E(G) \rightarrow \{d, 2d, 3d, 4d, \dots, qd\}$ we get the edge labels of the graph AR_η^3 as follows

Table:2 Edge labels of the graph AR_η^3

$f^*(u\,v)$	Edge labels	Value
$f^*(v\, v_{i_1,j})$	$\left\|\left[5\eta - \frac{5(j-1)}{2}\right] d\right\|$	$i_1 = 1, j = 1$
$f^*(v\, v_{2,j})$	$\|[f(v_{1,j-1}) - 3(-1)^j] d\|$	$j = 1$
$f^*(v\, v_{3,j})$	$\|[f(v_{1,j}) + (-1)^j] d\ \|$	$j = 1$
$f^*(v_{i_1,j} v_{i_2,k})$	$\left\|\left[\frac{5j}{2} - 1 - f(v_{i_1,k-1}) + 3(-1)^k)\right] d\right\|$	$i_1 = 1, i_2 = 2, j \equiv 0(mod 2)$ $for\ all\ j = 1,2, \dots, \eta - 1,$ $k \equiv 0(mod 2) for\ all\ k = 1,2, \dots, \eta - 1$
$f^*(v_{i_1,j} v_{i_2,k})$	$\left\|\left[5n - \frac{5(j-1)}{2} - f(v_{i_1,k-1}) + 3(-1)^k)\right] d\right\|$	$i_1 = 1, i_2 = 2, j \equiv 1(mod 2)$ $for\ all\ j = 1,2, \dots, \eta - 1,$ $k \equiv 1(mod 2) for\ all$ $k = 1,2, \dots, \eta - 1$
$f^*(v_{i_2,j} v_{i_3,k})$	$\|[f(v_{i_1,j-1}) - 3(-1)^j) - f(v_{i_1,k}) - (-1)^k)] d\|$	$i_1 = 1, i_2 = 2, i_3 = 3, j \equiv$ $0(mod 2), for\ all\ j = 1,2, \dots, \eta - 1,$ $k \equiv 0(mod 2) for\ all$ $k = 1,2, \dots, \eta - 1$
$f^*(v_{i_2,j} v_{i_3,k})$	$\|[f(v_{i_1,j-1}) - 3(-1)^j) - f(v_{i_1,k}) - (-1)^k)] d\|$	$i_1 = 1, i_2 = 2, i_3 = 3, j \equiv$ $1(mod 2), for\ all\ j = 1,2, \dots, \eta - 1,$ $k \equiv 1(mod 2) for\ all$ $k = 1,2, \dots, \eta - 1$

It is clear that the function f is injective and also table 2 shows that

$f^*: E \rightarrow \{d, 2d, 3d, 4d, \dots, qd\}$ is bijective. Hence f is arithmetic sequential graceful labeling and the graph AR_η^3 is arithmetic sequential graceful graph.

Example 3.2.1: Arrow graph of AR_6^3 and its graceful labeling shown in figure-2.

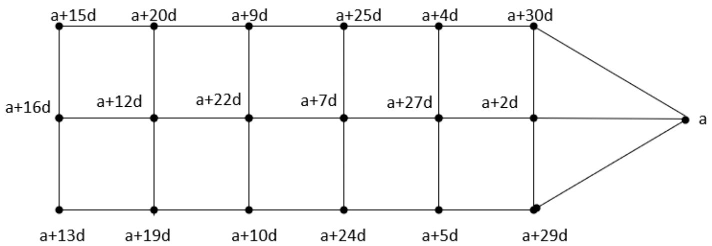

Figure -2: Arrow graph of AR_6^3 and its graceful labeling.

Theorem 3.3:
Arrow graph AR_{η}^{5} is arithmetic sequential graceful graph, when $\eta \geq 2$.

Proof:
Let $G = AR_{\eta}^{5}$ be an arrow graph obtained by connected a vertex v with $v_{i,1}(1 \leq i \leq 5)$ by 5 new edges. Let $v_{i,j}(1 \leq i \leq 5, 1 \leq j \leq \eta)$ be the vertices of $P_5 \times P_{\eta}$.
Join v with $v_{i,1}(1 \leq i \leq 5)$ by 5 new edges to get G.
$V(G) = \{v_{i,j}: 1 \leq i \leq 5 ; 1 \leq j \leq \eta\} \cup \{v\}$
$E(G) = \{v_{i,j}\, v_{i,j+1}: 1 \leq i \leq 5 ; 1 \leq j \leq \eta - 1\} \cup \{v\, v_{i,1}: 1 \leq i \leq 5\}$
Here $|V| = 5\eta + 1$, $|E| = 9\eta$.
We define a function $f: V(G) \rightarrow \{a, a + d, a + 2d, a + 3d, \ldots, a + 2qd\}$.
The vertex labeling are as follows,
$f(v) = a, f(v_{2,1}) = a + 3d, f(v_{4,1}) = a + 4d$
$f(v_{1,j}) = a + \left[9n - \frac{9(j-1)}{2}\right] d, when\ j \equiv 1(mod2)\ for\ all\ j = 1,2, \ldots, \eta - 1$
$f(v_{1,j}) = a + \left[\frac{9j}{2} - 2\right] d, when\ j \equiv 0(mod2)\ for\ all\ j = 1,2, \ldots, \eta - 1$
$f(v_{i,j}) = a + \left[f(v_{1,j}) + (-1)^j \frac{(i-1)}{2}\right] d\ \ for\ all\ \ i = 3,5\ \&\ j = 1,2, \ldots, \eta - 1$
$f(v_{i,j}) = a + [f(v_{i-1,j-1}) - 5(-1)^j] d\ \ for\ all\ \ i = 2,4\ \&\ j = 1,2, \ldots, \eta - 1$
$f(v_{1,\eta}) = a + [f(v_{1,n-1}) - 6(-1)^{\eta}] d$
$f(v_{2,\eta}) = a + [f(v_{2,n-1}) + 7(-1)^{\eta}] d$
$f(v_{3,\eta}) = a + [f(v_{3,\eta-1}) - 8(-1)^{\eta}] d$
$f(v_{4,\eta}) = a + [f(v_{4,\eta-1}) + 10(-1)^{\eta}] d$
$f(v_{5,\eta}) = a + [f(v_{5,\eta-1}) - 11(-1)^{\eta}] d$
From the function $f^*: E(G) \rightarrow \{d, 2d, 3d, 4d, \ldots, qd\}$ we get the edge labels of the graph AR_{η}^{5} as follows

Table:3 Edge labels of AR_{η}^{5}

$f^*(u\, v)$	**Edge labels**	**Value**
$f^*(v\, v_{1,j})$	$\left\|\left[\frac{9(j-1)}{2} - 9\eta\right] d\right\|$	$j = 1$
$f^*(v\, v_{i,j})$	$\left\|\left[\frac{(-1)^j(i-1)}{2} - f(v_{1,j})\right] d\right\|$	$i = 3,5\ \&\ j = 1$
$f^*(v_{1,j}\, v_{i,j})$	$\left\|\left[24 - \frac{9(j-1)}{2}\right] d\right\|$	$i = 2, j = 1$
$f^*(v_{2,j}\, v_{i,j})$	$\left\|\left[3 - f(v_{1,j}) - \frac{(-1)^j(i-1)}{2}\right] d\right\|$	$i = 3, j = 1$
$f^*(v_{i,j}\, v_{4,j})$	$\left\|\left[f(v_{1,j}) + \frac{(-1)^j(i-1)}{2} - 4\right] d\right\|$	$i = 3, j = 1$
$f^*(\, v_{4,j} v_{i,j})$	$\left\|\left[4 - f(v_{1,j}) - \frac{(-1)^j(i-1)}{2}\right] d\right\|$	$i = 5, j = 1$

It is clear that the function f is injective and also table 3 shows that
$f^*: E \rightarrow \{d, 2d, 3d, 4d, \ldots, qd\}$ is bijective. Hence f is arithmetic sequential graceful labeling and the graph is arithmetic sequential graceful graph.
Example 3.3.1: Arrow graph of AR_{3}^{5} and its graceful labeling shown in figure-3.

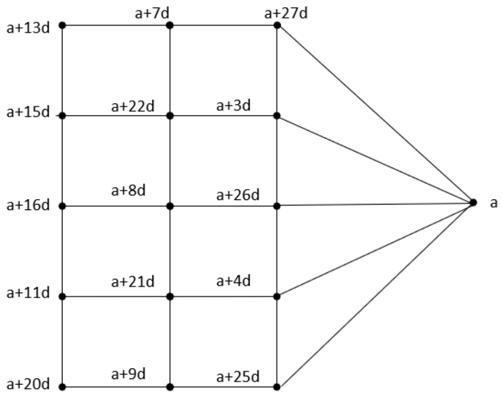

Figure-3: Arrow graph of AR_3^5 and its graceful labeling.

Theorem 3.4:
Double side arrow graph $D(AR_\eta^2)$ is arithmetic sequential graceful graph, when $\eta \geq 2$.

Proof:
Let $G = D(AR_\eta^2)$ be double sided arrow graph obtained by connected two vertices v & v' with $P_2 \times P_\eta$ by 4 new edges in both sides. Let $v_{i,j}(1 \leq i < 2, 1 \leq j \leq \eta)$ be the vertices of $P_2 \times \eta$.
Join v with $v_{i,1}(1 \leq i \leq 2)$ and v' with $v_{i,\eta}(1 \leq i \leq 2)$ by 4 new edges to get G.
$V(G) = \{v_{i,j}: 1 \leq i \leq 2 ; 1 \leq j \leq \eta\} \cup \{v\} \cup \{v'\}$
$E(G) = \{v_{i,j}\, v_{i,j+1}: 1 \leq i \leq 2 ; 1 \leq j \leq \eta - 1\} \cup \{v\, v_{i,1}: 1 \leq i \leq 2\} \cup \{v'\, v_{i,\eta}: 1 \leq i \leq 2\}$
Here $|V| = 2\eta + 2$, $|E| = 3\eta + 2$.
We define a function $f: V(G) \to \{a, a+d, a+2d, a+3d, \dots, a+2qd\}$.
The vertex labeling are as follows,
$f(v) = a, f(v_{2,1}) = a + d$
$f(v_{1,j}) = a + \left[3n + 2 - \left(\frac{3(j-1)}{2}\right)\right] d, when\ j \equiv 1(mod2)\ for\ all\ j = 1,2, \dots, \eta - 1$
$f(v_{1,j}) = a + \left[\frac{3j}{2} - 1\right] d, when\ j \equiv 0(mod2)\ for\ all\ j = 1,2, \dots, \eta - 1$
$f(v_{2,j}) = a + [f(v_{1,j-1}) - 2(-1)^j] d\ \ for\ all\ \ j = 2,3, \dots, \eta - 1$
$f(v_{1,\eta}) = a + [f(v_{1,\eta-1}) - 4(-1)^\eta] d$
$f(v_{2,\eta}) = a + [f(v_{2,\eta-1}) + 6(-1)^\eta] d$
$f(v') = a + [f(v_{1,\eta}) - 2(-1)^\eta] d$
From the function $f^*: E(G) \to \{d, 2d, 3d, 4d, \dots, qd\}$ we get the edge labels of the graph $D(AR_\eta^2)$ as follows

Table:4 Edge labels of $D(AR_\eta^2)$

$f^*(u\, v)$	**Edge labels**	**Value**
$f^*(v\, v_{1,j})$	$\left\lvert\left[3n + 2 - \left(\frac{3(j-1)}{2}\right)\right] d\right\rvert$	$j = 1$
$f^*(v\, v_{2,1})$	$\lvert d \rvert$	
$f^*(v_{i,j}\, v_{k,j})$	$\left\lvert\left[\frac{3j}{2} - 1 - f(v_{1,j-1}) + 2(-1)^j\right] d\right\rvert$	$i = 1, k = 1,$ $j \equiv 0(mod2)\ for\ all$ $j = 1,2, \dots, \eta - 1$
$f^*(v_{i,j}\, v_{k,j})$	$\left\lvert\left[3n + 2 - \left(\frac{3(j-1)}{2}\right) - f(v_{1,j-1}) + 2(-1)^j\right] d\right\rvert$	$i = 1, k = 2,$ $j \equiv 1(mod2)\ for\ all$ $j = 1,2, \dots, \eta - 1$

It is clear that the function f is injective and also table 4 shows that
$f^*: E \to \{d, 2d, 3d, 4d, \dots, qd\}$ is bijective. Hence f is arithmetic sequential graceful labeling and the graph is arithmetic sequential graceful graph.

Example 3.3.1: Double sided arrow graph of $D(AR_4^2)$ and its graceful labeling shown in figure-4.

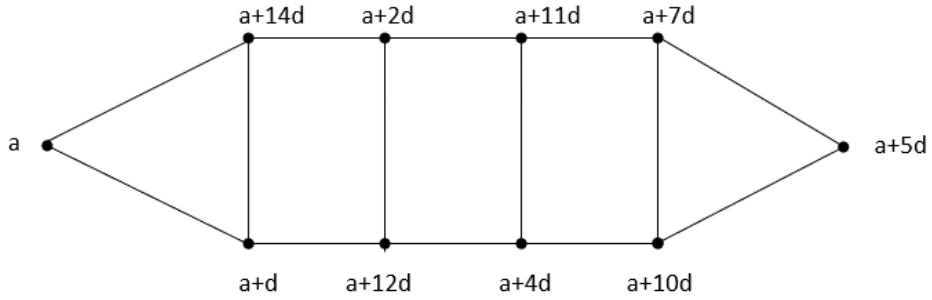

Figure-4: Double sided arrow graph of $D(AR_4^2)$ and its graceful labeling.

Theorem 3.5:
Double side arrow graph $D(AR_\eta^3)$ is arithmetic sequential graceful graph, when $\eta \geq 2$.

Proof:
Let $G = D(AR_\eta^3)$ be double sided arrow graph obtained by connected two vertices v & v' with $P_3 \times P_\eta$ by 6 new edges in both sides. Let $v_{i,j} (1 \leq i \leq 3, 1 \leq j \leq \eta)$ be the vertices of $P_3 \times P_\eta$.
Join v with $v_{i,1} (1 \leq i \leq 3)$ and v' with $v_{i,\eta} (1 \leq i \leq 3)$ by 6 new edges to get G.
$V(G) = \{v_{i,j}: 1 \leq i \leq 3 ; 1 \leq j \leq \eta\} \cup \{v\} \cup \{v'\}$
$E(G) = \{v_{i,j}\, v_{i,j+1}: 1 \leq i \leq 3 ; 1 \leq j \leq \eta - 1\} \cup \{v\, v_{i,1}: 1 \leq i \leq 3\} \cup \{v'\, v_{i,\eta}: 1 \leq i \leq 3\}$
Here $|V| = 3\eta + 2$, $|E| = 5\eta + 3$.
We define a function $f: V(G) \to \{a, a + d, a + 2d, a + 3d, \ldots, a + 2qd\}$.
The vertex labeling are as follows,
$f(v) = a, f(v_{2,1}) = a + 2d$
$f(v_{1,j}) = a + \left[5\eta + 3 - \left(\frac{5(j-1)}{2}\right)\right] d, when\ j \equiv 1 (mod 2)\ for\ all\ j = 1,2, \ldots, \eta - 1$
$f(v_{1,j}) = a + \left[\frac{5j}{2} - 1\right] d, when\ j \equiv 0 (mod 2)\ for\ all\ j = 1,2, \ldots, \eta - 1$
$f(v_{2,j}) = a + [f(v_{1,j-1}) - (-1)^j] d\ \ for\ all\ \ j = 2,3, \ldots, \eta - 1$
$f(v_{3,j}) = a + [f(v_{1,j}) + (-1)^j] d\ \ for\ all\ \ j = 1,2,3, \ldots, \eta - 1$
$f(v_{1,\eta}) = a + [f(v_{1,\eta-1}) - 8(-1)^\eta] d$
$f(v_{2,\eta}) = a + [f(v_{2,\eta-1}) + 7(-1)^\eta] d$
$f(v_{3,\eta}) = a + [f(v_{3,\eta-1}) - 9(-1)^\eta] d$
$f(v') = a + [f(v_{2,\eta}) - (-1)^\eta] d$
From the function $f^*: E(G) \to \{d, 2d, 3d, 4d, \ldots, qd\}$ we get the edge labels of the graph $D(AR_\eta^3)$ as follows

Table:5 Edge labels of $D(AR_\eta^3)$

$f^*(u\,v)$	Edge labels	Value
$f^*(v\, v_{1,j})$	$\left\|\left[5\eta + 3 - \left(\frac{5(j-1)}{2}\right)\right] d\right\|$	$j = 1$
$f^*(v\, v_{2,j})$	$\|[f(v_{1,j-1}) - (-1)^j] d\|$	$j = 1$
$f^*(v\, v_{3,j})$	$\|[f(v_{1,j}) + (-1)^j] d\|$	$j = 1$
$f^*(v_{i,j}\, v_{i+1,k})$	$\left\|\left[5\eta + 3 - \left(\frac{5(j-1)}{2}\right) - f(v_{1,k-1}) + (-1)^k\right] d\right\|$	$i = 1, k = 1,3, \ldots, (\eta - 1)$ $j \equiv 1 (mod 2)\ for\ all$ $j = 1,2, \ldots, \eta - 1$
$f^*(v_{i,j}\, v_{i+1,k})$	$\left\|\left[\frac{5j}{2} - 1 - f(v_{1,j-1}) + (-1)^j\right] d\right\|$	$i = 1, k = 1,3, \ldots, (\eta - 1),$ $j \equiv 0 (mod 2)\ for\ all$ $j = 1,2, \ldots, \eta - 1$
$f^*(v_{2,j}\, v_{3,j})$	$\|[f(v_{1,j-1}) - 3(-1)^j - f(v_{1,j}) - (-1)^j] d\|$	$j = 1,2, \ldots, \eta - 1$

It is clear that the function f is injective and also table 5 shows that
$f^*: E \to \{d, 2d, 3d, 4d, \ldots, qd\}$ is bijective. Hence f is arithmetic sequential graceful labeling and the graph is arithmetic sequential graceful graph.

Example 3.3.1: Double sided arrow graph of $D(AR_6^3)$ and its graceful labeling shown in figure-5.

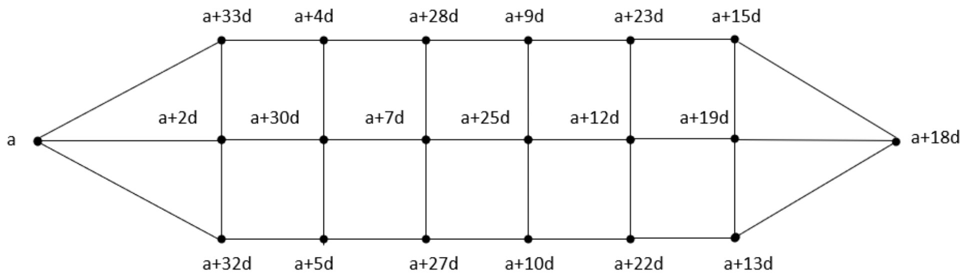

Figure-5: Double sided arrow graph of $D(AR_6^3)$ and its graceful labeling.

IV. Conclusion

We showed here arithmetic sequential graceful labeling of arrow graph. Here we proved five new results. we discussed graceful of $AR_\eta^2, AR_\eta^3, AR_\eta^5$ and double-sided arrow graphs $D(AR_\eta^2), D(AR_\eta^3)$.Labeling pattern is demonstrated by means of illustrations, which provide better understanding of derived results. Analysing arithmetic sequential graceful on other families of graph are our future work.

References

[1] Barnes, J. A., & Harary, F. (1983). Graph Theory In Network Analysis. Social Networks, 5(2), 235-244.
[2] Rosa, A. (1967). On Certain Valuations Of The Vertices Of A Graph, Theory Of Graphs (Internat. Symposium, Rome, July 1966).
[3] Gallian J A, “A Dynamic Survey Of Graph Labeling”, The Electronic Journal Of Combinatorics, (2020), Pp. 77-81.
[4] V J Kaneria1 , Meera Meghpara , H M Makadia Pasaribu, Graceful Labeling For Grid Related Graphs, International Journal Of Mathematics And Soft Computing, Vol.5, No.1 (2015), 111 - 117. DOI:10.26708/IJMSC.2015.1.5.13
[5] V. J. Kaneria, H. M. Makadia And M. M. Jariya, Graceful Labeling Of Arrow Graphs And Double Arrow Graphs, Malaya J. Mat. 3(4)(2015) 382–386.
[6] Sumathi P, Geetha Ramani G (2022) Arithmetic Sequential Graceful Labeling On Star Related Graphs. Indian Journal Of Science And Technology 15(44): 2356-2362. Https://Doi.Org/ 10.17485/IJST/V15i44.1863
[7] Sumathi P, Geetha Ramani G,Arithmetic Sequential Graceful Labeling On Grid Related Graphs, Samdarshi, ISSN: 2581-3986, Vol 16 Issue 4 (September 2023).