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Sum Perfect Cube Labeling Of Various Graphs

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Abstract:
This paper investigates sum perfect cube graphs, defined as a graph $G = (V, E)$ with p vertices and q edges is said to admit sum perfect cube labeling if there exists a bijection $f: V(G) \rightarrow \{0,1,2,3, \dots, p-1\}$ such that for each edge $e = uv$ the induced map $f^*: E(G) \rightarrow \mathbb{N}$ is defined by, $f^*(uv) = f(u)^3 + 3f(u)f(v)^2 + 3f(u)^2f(v) + f(v)^3$. All edge labels are distinct. The graph which attains such labeling is termed as a sum perfect cube graph. The study focused on identifying graphs where all edges permit such labeling termed sum perfect cube graphs. This paper explores the study of various sum perfect cube graphs.

Keywords: Sum perfect cube graph.

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I. Introduction

All the graphs in this paper are finite and undirected. The symbols $V(G)$ & $E(G)$ denotes the vertex set and edge set of a graph G. An excellence reference on this subject is the survey by J. A. Gallian [2]. The definitions which are useful for the present investigation are below. We refer Gross and Yellen [3], for all kinds of definitions and notations.

Definition 1: A graph $G = (V, E)$ with p vertices and q edges is said to admit sum perfect cube labeling if there exists a bijection $f: V(G) \rightarrow \{0,1,2,3, \dots, p-1\}$ such that for each edge $e = uv$ the induced map $f^*: E(G) \rightarrow \mathbb{N}$ is defined by, $f^*(uv) = f(u)^3 + 3f(u)f(v)^2 + 3f(u)^2f(v) + f(v)^3$. All edge labels are distinct. The graph which attains such labeling is termed as a sum perfect cube graph. [4]

II. Some Existing Results

S. G. Sonchhatra, D. D. Pandya and T. M. Chhaya [4] proved that

- ❖Path P_n is a sum perfect cube graph.
- ❖Star is a sum perfect cube graph.
- ❖Cycle with a chord is a sum perfect cube graph.
- ❖K_n, $n < 4$ is a sum perfect cube graph.
- ❖Every tree is a sum perfect cube graph.

Definition 1.1: The **middle graph** $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it.[5]

Definition 1.2: The ladder graph L_n is defined by $L_n = P_n \times P_2$.[3]

Definition 1.3: The **total graph** $T(G)$ of G has the vertex set $V(G) \cup E(G)$ in which two vertices are adjacent whenever they are either adjacent or incident in G. [5]

Definition 1.4: Square of a graph G is denoted by $\boldsymbol{G^2}$ has the same vertex as of G and two vertices are adjacent in G^2 if they are at a distance of 1 or 2 apart in G. [3]

Definition 1.5: A **triangular snake** T_n is obtained from a path $v_1, v_2, \dots, v_n$ by joining v_i &v_{i+1} to a new vertex u_i, $1 \leq i \leq n-1$. That is every edge of path is replaced by a triangle.[8]

Definition 1.6: A **quadrilateral snake** $\boldsymbol{Q_n}$ is obtained from a path $u_1, u_2, \dots, u_n$ by joining u_i and u_{i+1} to a new vertex v_i and w_i respectively and then joining v_i and w_i.That is every edge of a path is replaced by a cycle C_4.[8]

Definition 1.7: Let P_n be a path with consecutive vertices $u_1, u_2, \dots, u_n$. An **irregular triangular snake** graph is obtained from the path P_n and new vertices $v_1, v_2, \dots, v_{n-2}$ and edges u_iv_i, v_iu_{i+2}, $1 \leq i \leq n-2$. It is denoted as $I(T_n)$. [8]

Definition 1.8: $\boldsymbol{H \odot K_2}$ is a graph obtained from H by attaching triangle to each vertex of the H. [5]

Definition 1.9: Let $L_n = P_n \times P_2$ be the ladder graph with vertex set u_k & v_k, $k = 1,2, \dots, n$. The **alternate triangular belt** is obtained from the ladder by adding the edges $u_{2k+1}v_{2k+2}$, $\forall k = 0,1,2, \dots, n-1$ and $v_{2k}u_{2k+1}$, $\forall k = 1,2, \dots, n-1$.This graph is denoted by $ATB(n)$. [6]

Definition 1.10: A **pentagonal snake graph** $\boldsymbol{PS_n}$ is obtained from the path P_n by replacing every edge of a path by a cycle C_5. [3]

III. Main Results

Theorem 1 The middle graph of a path P_n is a sum perfect cube graph.
Proof: Let $G = M(P_n)$.
$V(G) = \{u_k, v_j : 1 \le k \le n, 1 \le j \le n-1\}$ and
$E(G) = \{(v_k v_{k+1}): 1 \le k \le n-2\} \cup \{(u_k v_k): 1 \le k \le n-1\} \cup \{(u_{k+1} v_k): 1 \le k \le n-1\}$.
So, $|V(G)| = 2n-1$ & $|E(G)| = 3n-4$.
Define $f: V(G) \to \{0,1,2,3,\dots,2n-2\}$ as follows.

$$f(u_k) = 2k-2 \qquad 1 \le k \le n$$
$$f(v_k) = 2k-1 \qquad 1 \le k \le n-1$$

We define edge function $f^*: E(G) \to \mathbb{N}$ as follows.

$$f^*(u_k v_k) = (4k-3)^3 \qquad 1 \le k \le n-1$$
$$f^*(u_{k+1} v_k) = (4k-1)^3 \qquad 1 \le k \le n-1$$
$$f^*(v_k v_{k+1}) = (4k)^3 \qquad 1 \le k \le n-2$$

All edges are distinct. f^* is injective function. Hence middle graph of a path P_n is a sum perfect cube graph.
Illustration: A Sum perfect cube labeling of $M(P_4)$ is shown in Figure-1.

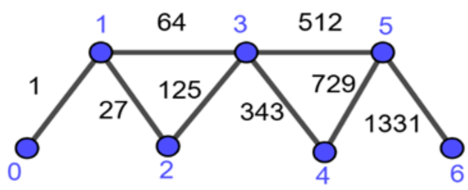

Figure-1 $\boldsymbol{M(P_4)}$

Theorem 2 The ladder graph L_n is a sum perfect cube graph.
Proof: Let $G = L_n$.
$V(G) = \{u_k, v_k : 1 \le k \le n\}$ and
$E(G) = \{(u_k u_{k+1}): 1 \le k \le n-1\} \cup \{(v_k v_{k+1}): 1 \le k \le n-1\} \cup \{(u_k v_k): 1 \le k \le n\}$.
So, $|V(G)| = 2n$ & $|E(G)| = 3n-2$.
Define $f: V(G) \to \{0,1,2,3,\dots,2n-1\}$ as follows.

$$f(u_k) = 2k-1 \qquad 1 \le k \le n$$
$$f(v_k) = 2k-2 \qquad 1 \le k \le n$$

We define edge function $f^*: E(G) \to \mathbb{N}$ as follows.

$$f^*(u_k u_{k+1}) = (4k)^3 \qquad 1 \le k \le n-1$$
$$f^*(u_k v_k) = (4k-3)^3 \qquad 1 \le k \le n$$
$$f^*(v_k v_{k+1}) = (4k-2)^3 \qquad 1 \le k \le n-1$$

All edges are distinct. f^* is injective function. Hence Ladder graph is a sum perfect cube graph.
Illustration: A sum perfect cube labeling of L_3 is shown in Figure-2.

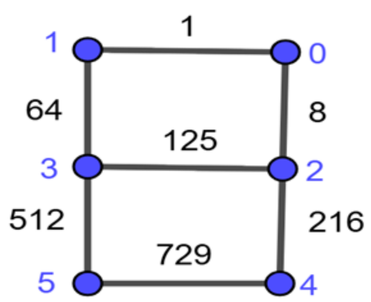

Figure-2 $\boldsymbol{L_3}$

Theorem 3 The total graph of a path $P_n, T(P_n)$ is a sum perfect cube graph.
Proof: Let $G = T(P_n)$.
$V(G) = \{u_k, v_j : 1 \le k \le n-1, 1 \le j \le n\}$ and

$E(G) = \{(u_k u_{k+1}),: 1 \le k \le n-2\} \cup \{(v_j v_{j+1}): 1 \le j \le n-1\} \cup \{(u_k v_j): 1 \le k, j \le n-1\} \cup \{(u_k v_{j+1}): 1 \le k, j \le n-1\}$.
So, $|V(G)| = 2n-1$ & $|E(G)| = 4n-5$.
Define $f: V(G) \to \{0,1,2,3,\ldots,2n-2\}$ as follows.

$$f(u_1) = 0$$
$$f(v_1) = 1$$
$$f(v_2) = 2$$
$$f(u_{2k}) = 4k \qquad 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor$$
$$f(u_{2k+1}) = 4k+2 \qquad 1 \le k \le \left\lfloor \frac{n-2}{2} \right\rfloor$$
$$f(v_{2k+1}) = 4k-1 \qquad 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor$$
$$f(v_{2k+2}) = 4k+1 \qquad 1 \le k \le \left\lfloor \frac{n-2}{2} \right\rfloor$$

We define edge function $f^*: E(G) \to \mathbb{N}$ as follows.

$$f^*(u_1 u_2) = 64$$
$$f^*(u_1 v_1) = 1$$
$$f^*(v_k v_{k+1}) = (2k+1)^3, k = 1,2$$
$$f^*(u_k v_2) = (4k-2)^3, k = 1,2$$
$$f^*(u_{k+1} u_{k+2}) = (4k+6)^3 \qquad 1 \le k \le n-3$$
$$f^*(u_{k+1} v_{k+2}) = (4k+3)^3 \qquad 1 \le k \le n-2$$
$$f^*(u_{k+2} v_{k+2}) = (4k+5)^3 \qquad 1 \le k \le n-3$$
$$f^*(v_{k+2} v_{k+3}) = (4k+4)^3 \qquad 1 \le k \le n-3$$

All edges are distinct. f^* is injective function. Hence total graph of a path P_n, T(P_n) graph is a sum perfect cube graph.
Illustration: A sum perfect cube labeling of T(P_4) is shown in Figure-3.

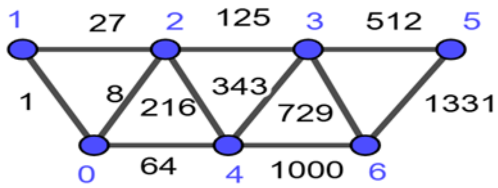

Figure-3 T(P_4)

Theorem 4 The P_n^2 graph is a sum perfect cube graph.
Proof: Let $G = P_n^2$.
$V(G) = \{u_k : 1 \le k \le n\}$ and
$E(G) = \{(u_k u_{k+1}): 1 \le k \le n-1\} \cup \{(u_k u_{k+2}): 1 \le k \le n-2\}$.
So, $|V(G)| = n$ & $|E(G)| = 2n-3$.
Define $f: V(G) \to \{0,1,2,3,\ldots,n-1\}$ as follows.

$$f(u_k) = k-1 \qquad 1 \le k \le n$$

We define edge function $f^*: E(G) \to \mathbb{N}$ as follows.

$$f^*(u_k u_{k+1}) = (2k-1)^3 \qquad 1 \le k \le n-1$$
$$f^*(u_{2k-1} u_{2k+1}) = (4k-2)^3 \qquad 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor$$
$$f^*(u_{2k} u_{2k+2}) = (4k)^3 \qquad 1 \le k \le \left\lfloor \frac{n-2}{2} \right\rfloor$$

All edges are distinct. f^* is injective function.
Hence the graph P_n^2 is a sum perfect cube graph.
Illustration: A sum perfect cube labeling of P_4^2 is shown in Figure-6.

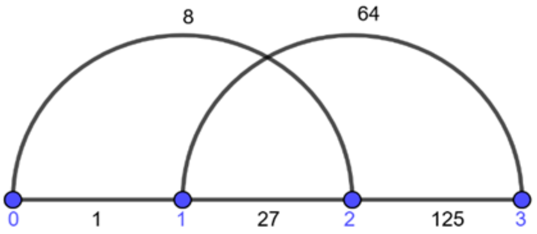

Figure-4 $P_4{}^2$

Theorem 5 The triangular snake T_n is a sum perfect cube graph.
Proof: Let $G = T_n$
$V(G) = \{u_k, v_j : 1 \le k \le n-1, 1 \le j \le n\}$ and
$E(G) = \{(u_k\, v_{k+1}): 1 \le k \le n-1\} \cup \{(u_k\, v_k): 1 \le k \le n-1\} \cup \{v_k v_{k+1}: 1 \le k \le n-1\}$.
So, $|V(G)| = 2n-1$ & $|E(G)| = 3n-3$
Define $f: V(G) \to \{0,1,2,3, \dots, 2n-2\}$ as follows.

$$f(u_k) = 2k-1 \qquad 1 \le k \le n-1$$

$$f(v_k) = 2k-2 \qquad 1 \le k \le n.$$

We define edge function $f^*: E(G) \to \mathbb{N}$ as follows.

$$f^*(v_k v_{k+1}) = (4k-2)^3 \qquad 1 \le k \le n-1$$
$$f^*(u_k v_{k+1}) = (4k-1)^3 \qquad 1 \le k \le n-1$$
$$f^*(u_k v_k) = (4k-3)^3 \qquad 1 \le k \le n-1$$

All edges are distinct. f^* is injective function. Hence triangular snake T_n is a sum perfect cube graph.
Illustration: A sum perfect cube labeling of T_3 is shown in Figure-7.

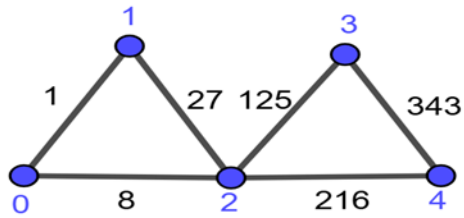

Figure-5 T_3

Theorem 6 The quadrilateral snake Q_n is a sum perfect cube graph.
Proof: Let $G = Q_n$
$V(G) = \{u_k, v_j, w_j : 1 \le k \le n, 1 \le j \le n-1\}$ and
$E(G) = \{(u_k u_{k+1}): 1 \le k \le n-1\} \cup \{(v_k w_k): 1 \le k \le n-1\} \cup \{(u_k v_k): 1 \le k \le n-1\} \cup \{(u_{k+1} w_k): 1 \le k \le n-1\}$.
So, $|V(G)| = 3n-2$ & $|E(G)| = 4n-4$.
Define $f: V(G) \to \{0,1,2,3, \dots, 3n-3\}$ as follows.

$$f(u_1) = 0$$
$$f(u_{2k+1}) = 6k-1 \qquad 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor$$
$$f(u_{2k}) = 6k-4 \qquad 1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$$
$$f(v_k) = 3k-2 \qquad 1 \le k \le n-1$$
$$f(w_k) = 3k \qquad 1 \le k \le n-1$$

We define edge function $f^*: E(G) \to \mathbb{N}$ as follows.

$$f^*(u_1 u_2) = 8$$
$$f^*(u_1 v_1) = 1$$
$$f^*(u_{k+1} u_{k+2}) = (6k+1)^3 \qquad 1 \le k \le n-2$$
$$f^*(u_{k+1} v_{k+1}) = (6k)^3 \qquad 1 \le k \le n-2$$
$$f^*(u_{k+1} w_{k+1}) = (6k-1)^3 \qquad 1 \le k \le n-1$$
$$f^*(v_k w_k) = (6k-2)^3 \qquad 1 \le k \le n-1$$

All edges are distinct. f^* is injective function.
Hence quadrilateral snake Q_n is a sum perfect cube graph.

Illustration: A sum perfect cube labeling of Q_3 is shown in Figure-6.

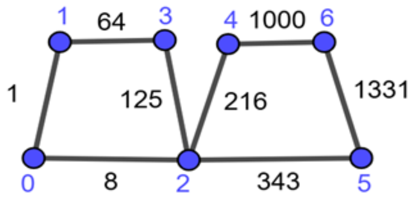

Figure-6 $\boldsymbol{Q_3}$

Theorem 7 The irregular triangular snake IT_n is a sum perfect cube graph.
Proof: Let $G = IT_n$.
$V(G) = \{u_k, v_j : 1 \le k \le n, 1 \le j \le n-2\}$ and
$E(G) = \{(u_k u_{k+1}): 1 \le k \le n-1\} \cup \{(u_k v_k): 1 \le k \le n-2\} \cup \{(u_{k+2} v_k): 1 \le k \le n-2\}$.
So, $|V(G)| = 2n-2$ & $|E(G)| = 3n-5$..
Define $f: V(G) \to \{0,1,2,3,\dots,2n-3\}$ as follows.
Case 1 n is even.

$$f(u_1) = 0$$
$$f(u_{2k}) = 4k-2 \qquad 1 \le k \le \frac{n-2}{2}$$
$$f(u_{2k+1}) = 4k-1 \qquad 1 \le k \le \frac{n-2}{2}$$
$$f(u_n) = 2n-3$$
$$f(v_{2k-1}) = 4k-3 \qquad 1 \le k \le \frac{n-2}{2}$$
$$f(v_{2k}) = 4k \qquad 1 \le k \le \frac{n-2}{2}$$

Case 2 n is odd.

$$f(u_1) = 0$$
$$f(u_{2k}) = 4k-2 \qquad 1 \le k \le \frac{n-1}{2}$$
$$f(u_{2k+1}) = 4k-1 \qquad 1 \le k \le \frac{n-1}{2}$$
$$f(v_{2k-1}) = 4k-3 \qquad 1 \le k \le \frac{n-1}{2}$$
$$f(v_{2k}) = 4k \qquad 1 \le k \le \frac{n-3}{2}$$

We define edge function $f^*: E(G) \to \mathbb{N}$ as follows.
When $\boldsymbol{n}$ is even.

$$f^*(u_1 u_2) = 8$$
$$f^*(u_1 v_1) = 1$$
$$f^*(u_n v_{n-2}) = (4n-7)^3$$
$$f^*(u_{k+1} u_{k+2}) = (4k+1)^3 \qquad 1 \le k \le n-3$$
$$f^*(u_{n-1} u_n) = (4n-8)^3$$
$$f^*(u_{2k+1} v_{2k-1}) = (8k-4)^3 \qquad 1 \le k \le \frac{n-2}{2}$$
$$f^*(u_{2k+2} v_{2k}) = (8k+2)^3 \qquad 1 \le k \le \frac{n-4}{2}$$
$$f^*(u_{2k} v_{2k}) = (8k-2)^3 \qquad 1 \le k \le \frac{n-2}{2}$$
$$f^*(u_{2k+1} v_{2k+1}) = (8k)^3 \qquad 1 \le k \le \frac{n-4}{2}$$

When $\boldsymbol{n}$ is odd.

$$f^*(u_1 u_2) = 8$$
$$f^*(u_1 v_1) = 1$$
$$f^*(u_{k+1} u_{k+2}) = (4k+1)^3 \qquad 1 \le k \le n-2$$
$$f^*(u_{2k+1} v_{2k-1}) = (8k-4)^3 \qquad 1 \le k \le \frac{n-1}{2}$$

$$f^*(u_{2k+2}v_{2k}) = (8k+2)^3 \qquad 1 \le k \le \frac{n-3}{2}$$
$$f^*(u_{2k}v_{2k}) = (8k-2)^3 \qquad 1 \le k \le \frac{n-3}{2}$$
$$f^*(u_{2k+1}v_{2k+1}) = (8k)^3 \qquad 1 \le k \le \frac{n-3}{2}$$

All edges are distinct. f^* is injective function.
Hence irregular triangular snake IT_n is a sum perfect cube graph.
Illustration: A sum perfect cube labeling of IT_4 is shown in Figure-7.

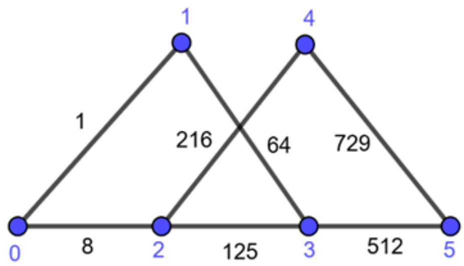

Figure-7 IT_4

Theorem 8 $H \odot K_2$ graph of a path P_n is a sum perfect cube graph.
Proof: Let $G = H \odot K_2$ graph of a path P_n.

$$V(G) = \{u_k, v_k, r_k, s_k, t_k, w_k : 1 \le k \le n\}$$

$E(G) = \{u_k u_{k+1} : 1 \le k \le n-1\} \cup \{v_k v_{k+1} : 1 \le k \le n-1\} \cup \{u_k r_k : 1 \le k \le n\} \cup \{u_k t_k : 1 \le k \le n\} \cup \{v_k s_k : 1 \le k \le n\} \cup \{v_k w_k : 1 \le k \le n\} \cup \{r_k t_k : 1 \le k \le n\} \cup \{s_k w_k : 1 \le k \le n\} \cup \{u_{\frac{n+1}{2}} v_{\frac{n+1}{2}} : n\ odd\}$ OR
$E(G) = \{u_k u_{k+1} : 1 \le k \le n-1\} \cup \{v_k v_{k+1} : 1 \le k \le n-1\} \cup \{u_k r_k : 1 \le k \le n\} \cup \{u_k t_k : 1 \le k \le n\} \cup \{v_k s_k : 1 \le k \le n\} \cup \{v_k w_k : 1 \le k \le n\} \cup \{r_k t_k : 1 \le k \le n\} \cup \{s_k w_k : 1 \le k \le n\} \cup \{u_{\frac{n}{2}+1} v_{\frac{n}{2}} : n\ even\}$.
So, $|V(G)| = 6n$ & $|E(G)| = 8n-1$.
Define $f : V(G) \to \{0,1,2,3,\ldots,6n-1\}$ as follows.
Case 1 n is even.

$$f(u_{2k-1}) = 6k-4 \qquad 1 \le k \le \frac{n}{2}$$
$$f(u_{2k}) = 6k-2 \qquad 1 \le k \le \frac{n}{2}$$
$$f(v_{2k-1}) = 3n-4+6k \qquad 1 \le k \le \frac{n}{2}$$
$$f(v_{2k}) = 3n-2+6k \qquad 1 \le k \le \frac{n}{2}$$
$$f(r_k) = 3k-3 \qquad 1 \le k \le n$$
$$f(t_{2k-1}) = 6k-5 \qquad 1 \le k \le \frac{n}{2}$$
$$f(t_{2k}) = 6k-1 \qquad 1 \le k \le \frac{n}{2}$$
$$f(s_k) = 3n+3k-3 \qquad 1 \le k \le n$$
$$f(w_{2k-1}) = 3n-5+6k \qquad 1 \le k \le \frac{n}{2}$$
$$f(w_{2k}) = 3n-1+6k \qquad 1 \le k \le \frac{n}{2}$$

Case 2 n is odd.

$$f(u_{2k-1}) = 6k-4 \qquad 1 \le k \le \frac{n+1}{2}$$
$$f(u_{2k}) = 6k-2 \qquad 1 \le k \le \frac{n-1}{2}$$
$$f(v_{2k-1}) = 3n-5+6k \qquad 1 \le k \le \frac{n+1}{2}$$
$$f(v_{2k}) = 3n-1+6k \qquad 1 \le k \le \frac{n-1}{2}$$

$$f(r_k) = 3k - 3 \qquad 1 \le k \le n$$
$$f(t_{2k-1}) = 6k - 5 \qquad 1 \le k \le \frac{n+1}{2}$$
$$f(t_{2k}) = 6k - 1 \qquad 1 \le k \le \frac{n-1}{2}$$
$$f(s_k) = 3n + 3k - 3 \qquad 1 \le k \le n$$
$$f(w_{2k-1}) = 3n - 4 + 6k \quad 1 \le k \le \frac{n+1}{2}$$
$$f(w_{2k}) = 3n - 2 + 6k \qquad 1 \le k \le \frac{n-1}{2}$$

We define edge function $f^*: E(G) \rightarrow \mathbb{N}$ as follows.

When n is even.

$$f^*(u_k u_{k+1}) = (6k)^3 \qquad 1 \le k \le n-1$$
$$f^*(u_{2k-1} r_{2k-1}) = (12k - 10)^3 \qquad 1 \le k \le \frac{n}{2}$$
$$f^*(u_{2k} r_{2k}) = (12k - 5)^3 \qquad 1 \le k \le \frac{n}{2}$$
$$f^*(r_{2k-1} t_{2k-1}) = (12k - 11)^3 \qquad 1 \le k \le \frac{n}{2}$$
$$f^*(r_{2k} t_{2k}) = (12k - 4)^3 \qquad 1 \le k \le \frac{n}{2}$$
$$f^*(u_k t_k) = (6k - 3)^3 \qquad 1 \le k \le n$$
$$f^*\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) = (6n)^3$$
$$f^*(v_k v_{k+1}) = (6n + 6k)^3 \qquad 1 \le k \le n-1$$
$$f^*(v_{2k-1} s_{2k-1}) = (6n - 10 + 12k)^3 \qquad 1 \le k \le \frac{n}{2}$$
$$f^*(v_{2k} s_{2k}) = (6n - 5 + 12k)^3 \qquad 1 \le k \le \frac{n}{2}$$
$$f^*(w_{2k-1} s_{2k-1}) = (6n - 11 + 12k)^3 \qquad 1 \le k \le \frac{n}{2}$$
$$f^*(w_{2k} s_{2k}) = (6n - 4 + 12k)^3 \qquad 1 \le k \le \frac{n}{2}$$
$$f^*(v_k w_k) = (6n - 3 + 6k)^3 \qquad 1 \le k \le n$$

When n is odd.

$$f^*(u_k u_{k+1}) = (6k)^3 \qquad 1 \le k \le n-1$$
$$f^*(u_{2k-1} r_{2k-1}) = (12k - 10)^3 \qquad 1 \le k \le \frac{n+1}{2}$$
$$f^*(u_{2k} r_{2k}) = (12k - 5)^3 \qquad 1 \le k \le \frac{n-1}{2}$$
$$f^*(r_{2k-1} t_{2k-1}) = (12k - 11)^3 \qquad 1 \le k \le \frac{n+1}{2}$$
$$f^*(r_{2k} t_{2k}) = (12k - 4)^3 \qquad 1 \le k \le \frac{n-1}{2}$$
$$f^*(u_k t_k) = (6k - 3)^3 \qquad 1 \le k \le n$$
$$f^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = (6n)^3$$
$$f^*(v_k v_{k+1}) = (6n + 6k)^3 \qquad 1 \le k \le n-1$$
$$f^*(v_{2k-1} s_{2k-1}) = (6n - 11 + 12k)^3 \qquad 1 \le k \le \frac{n+1}{2}$$
$$f^*(v_{2k} s_{2k}) = (6n - 4 + 12k)^3 \qquad 1 \le k \le \frac{n-1}{2}$$
$$f^*(w_{2k-1} s_{2k-1}) = (6n - 10 + 12k)^3 \qquad 1 \le k \le \frac{n+1}{2}$$
$$f^*(w_{2k} s_{2k}) = (6n - 5 + 12k)^3 \qquad 1 \le k \le \frac{n-1}{2}$$
$$f^*(v_k w_k) = (6n - 3 + 6k)^3 \qquad 1 \le k \le n$$

All edges are distinct. f^* is injective function. Hence $H \odot K_2$ graph of path P_n is a sum perfect cube graph.

Illustration: A sum perfect cube labeling of $H \odot K_2$ graph of path P_3 is shown in Figure-8.

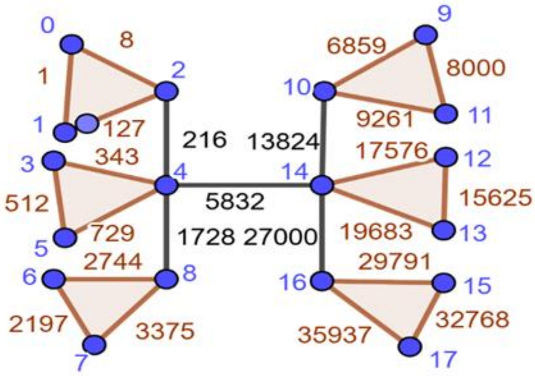

Figure-8 $H \odot K_2$ graph of a path P_3

Theorem 9 An alternate triangular belt $ATB(n)$ is a sum perfect cube graph.
Proof: Let $G = ATB(n)$.
$V(G) = \{u_k: 1 \le k \le n\} \cup \{v_k: 1 \le k \le n\}$ and
$E(G) = \{(u_k u_{k+1}): 1 \le k \le n-1\} \cup \{(v_k v_{k+1}): 1 \le k \le n-1\} \cup \{(u_k v_k): 1 \le k \le n\} \cup \left\{(u_{2k-1} v_{2k}): 1 \le k \le \frac{n}{2}\right\} \cup \left\{(u_{2k+1} v_{2k}): 1 \le k \le \frac{n-2}{2}\right\}$, n is even.
$E(G) = \{(u_k u_{k+1}): 1 \le k \le n-1\} \cup \{(v_k v_{k+1}): 1 \le k \le n-1\} \cup \{(u_k v_k): 1 \le k \le n\} \cup \left\{(u_{2k-1} v_{2k}): 1 \le k \le \frac{n-1}{2}\right\} \cup \left\{(u_{2k+1} v_{2k}): 1 \le k \le \frac{n-1}{2}\right\}$, n is odd.
So, $|V(G)| = 2n$ & $|E(G)| = 4n - 3$.
Define $f: V(G) \cup E(G) \to \{0,1,2,3,\dots,2n-1\}$ as follows.

$$f(u_k) = 2k - 2 \qquad 1 \le k \le n$$
$$f(v_k) = 2k - 1 \qquad 1 \le k \le n$$

We define edge function $f^*: E(G) \to \mathbb{N}$ as follows.

$$f^*(u_k u_{k+1}) = (4k-2)^3 \qquad 1 \le k \le n-1$$
$$f^*(v_k v_{k+1}) = (4k)^3 \qquad 1 \le k \le n-1$$
$$f^*(u_k v_k) = (4k-3)^3 \qquad 1 \le k \le n$$
$$f^*(u_{2k-1} v_{2k}) = (8k-5)^3 \qquad 1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$$
$$f^*(u_{2k+1} v_{2k}) = (8k-1)^3 \qquad 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor$$

All edges are distinct. f^* is injective function. Hence alternate triangular belt $ATB(n)$ graph is a sum perfect cube graph.
Illustration: A sum perfect cube labeling of $ATB(3)$ is shown in Figure-9.

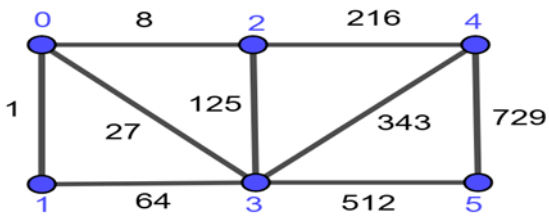

Figure-9 $ATB(3)$

Theorem 10 The Pentagonal snake PS_n graph is a sum perfect cube graph.
Proof: Let $G = PS_n$
$V(G) = \{u_k, v_j, w_j, x_j : 1 \le k \le n, 1 \le j \le n-1\}$ and
$E(G) = \{(u_k u_{k+1}): 1 \le k \le n-1\} \cup \{(v_k x_k): 1 \le k \le n-1\} \cup \{(u_k v_k): 1 \le k \le n-1\} \cup \{(u_{k+1} w_k): 1 \le k \le n-1\} \cup \{(x_k w_k): 1 \le k \le n-1\}$.
So, $|V(G)| = 4n - 3$ & $|E(G)| = 5n - 5$.
Define $f: V(G) \to \{0,1,2,3,\dots,4n-4\}$ as follows.

$$f(u_k) = 4k - 4 \qquad 1 \le k \le n$$

$$f(v_k) = 4k - 3 \qquad 1 \le k \le n-1$$
$$f(w_k) = 4k - 1 \qquad 1 \le k \le n-1$$
$$f(x_k) = 4k - 2 \qquad 1 \le k \le n-1$$

We define edge function $f^*: E(G) \to \mathbb{N}$ as follows.

$$f^*(u_k u_{k+1}) = (8k-4)^3 \qquad 1 \le k \le n-1$$
$$f^*(u_k v_k) = (8k-7)^3 \qquad 1 \le k \le n-1$$
$$f^*(u_{k+1} w_k) = (8k-1)^3 \qquad 1 \le k \le n-1$$
$$f^*(u_k x_k) = (8k-5)^3 \qquad 1 \le k \le n-1$$
$$f^*(x_k w_k) = (8k-3)^3 \qquad 1 \le k \le n-1$$

All edges are distinct. f^* is injective function. Hence pentagonal snake PS_n is a sum perfect cube graph.
Illustration: A sum perfect cube labeling of PS_3 is shown in Figure-10.

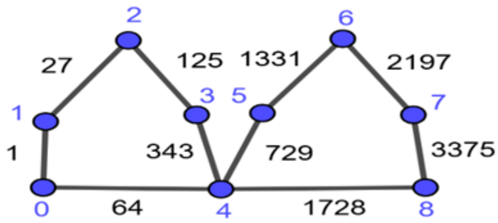

Figure-10 PS_3

IV. Conclusion

We have investigated sum perfect cube labeling of Ladder graph, Total graph of a path P_n, Alternate triangular belt graph, Middle graph of a path P_n, ${P_n}^2$ graph, Triangular snake graph, Quadrilateral snake graph, Irregular triangular snake graph, $H \odot K_2$ graph of a path P_n, pentagonal snake graph. We can discuss more similar results for various graphs.

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