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On The Closure Properties Of Single-Valued Neutrosophic Primal Topological Spaces

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Abstract:
In this paper, we investigate the concept of closure within the framework of single-valued neutrosophic primal topological spaces. We introduce and define the single-valued neutrosophic primal closure operator and examine its fundamental properties, such as idempotency, extensivity, and preservation under primal continuous mappings. Furthermore, we explore the relationships between the primal closure and other classical closure operators in neutrosophic and fuzzy topologies. Several characterizations are provided to distinguish primal closed sets from general closed sets. Our results contribute to a deeper understanding of the structural behavior of neutrosophic primal topologies and offer a foundation for further applications in uncertainty modeling and decision-making systems.
Keywords: *Single-valued neutrosophic primal; closure and interior operators (including per-primal, α-primal, β-primal forms); single-valued neutrosophic primal continuity; α-continuity; compactness.*

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I. Introduction

The theory of neutrosophic sets, initiated by Smarandache in 1999 [1], provides a flexible mathematical framework for handling indeterminate, inconsistent, and incomplete information. As a generalization of classical, fuzzy, and intuitionistic fuzzy set theories, neutrosophic sets have found extensive applications in logic, decision-making, artificial intelligence, and topology. A particular branch of this theory, known as single-valued neutrosophic sets (SVNS), restricts the membership, indeterminacy, and non-membership degrees to values within the standard unit interval [0,1], thereby enabling practical implementation in real-world scenarios.

Topological structures defined under SVNS—referred to as single-valued neutrosophic topological spaces—have been the subject of growing interest. Among various extensions, the concept of single-valued neutrosophic primal topology offers a nuanced approach to defining open, closed, interior, and closure operators that account for primal properties unique to the neutrosophic environment.

In classical topology, the closure operator plays a fundamental role in understanding the structure of topological spaces. However, in neutrosophic topology—especially within the primal framework—the behavior and properties of closure are less straightforward due to the inherent uncertainty and dual nature of membership and non-membership values. This motivates a deeper investigation into how closure functions in such contexts and how it interacts with other neutrosophic operators.

Recently, neutrosophic theory has gained traction across various branches of mathematics. Salama et al. [2,3] extended Chang's [4] framework for fuzzy topological spaces (Ω,T) to neutrosophic settings. Hur et al. [5,6] further developed the structure of neutrosophic and neutrosophic crisp sets, while Smarandache [7] explored their applications to non-standard intervals.

Recognizing the limitations of Chang's approach, Sostak [8] redefined fuzzy topology as a mapping from fuzzy subsets of Ω to the interval [0,1], establishing what is now referred to as smooth fuzzy topology. Researchers such as Fang et al. [9] and Zahran et al. [10] contributed further by analyzing decomposition techniques in fuzzy continuity, ideal continuity, and α-continuity.

The concept of single-valued neutrosophic sets was introduced by Wang [11], and fuzzy neutrosophic topological spaces in the Sostak sense were developed by Gayyar [12]. Kim [13] defined a foundation for ordinary single-valued neutrosophic topology, while Alsharari et al. [13] introduced stratified filters and quasi-uniformities for single-valued neutrosophic soft sets. Saber et al. [14] presented the Ideals on fuzzy topological spaces to study topological structures in this context. Alsharari, et al. [15] presented the stratified single-valued soft topogenous structures.

Ongoing work [16–26] has continued to expand the theory of single-valued neutrosophic topological spaces (Ω,T), including the definition of open local functions and primal structures [27].

In this work, we define and analyze the single-valued neutrosophic primal closure operator, exploring its properties, comparing it with traditional closure concepts, and identifying its implications for the broader structure of neutrosophic topological spaces. The results obtained here form a basis for further studies on

II. Preliminaries

This section provides a comprehensive introduction to the fundamental concepts and methods underlying Single-Valued Neutrosophic Set Theory (SVN-set) and Single-Valued Neutrosophic Primal Theory (SVN-primal).

Let $\xi^{✠}$ denote the set of all SVN-sets defined on a universe ✠. A structure of the form (✠,$T^{\sigma\tau\pi}$, $P^{\sigma\tau\pi}$) is referred to as a Single-Valued Neutrosophic Primal Topological Space (SVNPTS). In this framework:

- ξ represents the closed interval [0, 1],
- ξ_0 denotes the open interval (0, 1],
- For any α ∈ ξ and ϑ ∈ ✠, define the constant function α(ϑ) = α.

Definition 1 ([7])
Let ✠ be a non-empty set. An n-set on ✠ is defined as:

$$D = \{\langle \kappa, \sigma\_D(\vartheta), \tau\_D(\vartheta), \pi\_D(\vartheta)\rangle \mid \vartheta \in ✠\},$$

where:

- $\pi\_D(\vartheta)$: Degree of membership,
- $\tau\_D(\vartheta)$: Degree of indeterminacy,
- $\sigma\_D(\vartheta)$: Degree of non-membership, with all values in the extended interval (0^-, 1^+).

Definition 2 ([11])
A set D is a Single-Valued Neutrosophic Set (SVN-set) on ✠ if:

$$D = \{\langle \kappa, \sigma\_D(\vartheta), \tau\_D(\vartheta), \pi\_D(\vartheta)\rangle \mid \vartheta \in ✠\},$$

where

- σ_D: ✠ $\to \xi$ (falsity membership),
- τ_D: ✠ $\to \xi$ (indeterminacy membership),
- π_D: ✠ $\to \xi$ (truth membership).

Special cases:

- Null SVN-set (denoted 0): $\forall\vartheta \in$ ✠, $\sigma\_D(\vartheta) = 0, \tau\_D(\vartheta) = 1, \pi\_D(\vartheta) = 1.$
- Absolute SVN-set (denoted Π): $\forall\vartheta \in$ ✠, $\sigma\_D(\vartheta) = 1, \tau\_D(\vartheta) = 0, \pi\_D(\vartheta) = 0.$

Definition 3 ([11])
The complement of an SVN-set D is denoted D^c and defined as:

- $\sigma\_D^c(\vartheta) = \pi\_D(\vartheta),$
- $\tau\_D^c(\vartheta) = \tau\_D(\vartheta),$
- $\pi\_D^c(\vartheta) = \sigma\_D(\vartheta).$

Definition 4 ([28])
Given $D, Z \in \xi^{✠}$

- $D \subseteq Z \Leftrightarrow \forall\vartheta \in$ ✠, $\sigma\_D(\vartheta) \le \sigma\_Z(\vartheta), \tau\_D(\vartheta) \ge \tau\_Z(\vartheta), \pi\_D(\vartheta) \ge \pi\_Z(\vartheta).$
- $D = Z \Leftrightarrow D \subseteq Z \; and \; Z \subseteq D.$

Definition 5 ([29])
Let $D, Z \in \xi^{✠}$. Then:

- $D \wedge Z = \langle min(\sigma\_D, \sigma\_Z), max(\tau\_D, \tau\_Z), max(\pi\_D, \pi\_Z)\rangle$
- $D \vee Z = \langle max(\sigma\_D, \sigma\_Z), min(\tau\_D, \tau\_Z), min(\pi\_D, \pi\_Z)\rangle$

Definition 6 ([17])
An SVN Topological Space (SVNTS) is a tuple (✠,T^σ, T^τ, T^π), where each T is a mapping $\xi \to \xi^{✠}$ satisfying:

(T1) $T^\sigma(0) = T^\sigma(1) = 1,\; T^\tau(0) = T^\tau(1) = 0,\; T^\pi(0) = T^\pi(1) = 0.$

(T2) For any D, Z ∈ $\xi^{✠}$:

$$T^\sigma(D \wedge Z) \ge T^\sigma(D) \wedge T^\sigma(Z),$$
$$T^\tau(D \wedge Z) \le T^\tau(D) \vee T^\tau(Z),$$

$$T^{\pi}(D \wedge Z) \leq T^{\pi}(D) \vee T^{\pi}(Z).$$

- (T3) For any family $\{D_j\} \subseteq \xi^{✠}$:

$$T^{\sigma}(\vee D_j) \geq \vee T^{\sigma}(D_j), T^{\tau}(\vee D_j) \leq \vee T^{\tau}(D_j), \quad T^{\pi}(\vee D_j) \leq \vee T^{\pi}(D_j).$$

Theorem 1 ([27])

Let $(✠, T^{\sigma\tau\pi},)$ be an SVNTS. Define the operator $CT\sigma\tau\pi : \xi^{✠} \times \xi_0 \rightarrow \xi^{✠}$ by:

$$CT\sigma\tau\pi(D,r) = \vee\{Z \in \xi^{✠} \,|\, \Pi \subseteq D, \quad T^{\sigma}(Z^{c}) \geq r, \quad T^{\pi}(Z^{c}) \leq 1-r, \quad T^{\pi}(Z^{c}) \leq 1-r\}.$$

Definition 7 ([27])

Similarly, define the interior operator $IT\sigma\tau\pi : \xi^{✠} \times \xi_0 \rightarrow \xi^{✠}$ by:

$$IT\sigma\tau\pi(D,r) = \wedge\{Z \in \xi^{✠} \,|\, D \supseteq Z, \quad T^{\sigma}(Z) \geq r, \quad T^{\pi}(Z) \leq 1-r, \quad T^{\pi}(Z) \leq 1-r\}.$$

Definition 8 ([27])

A triple of mappings $P^{\sigma}, P^{\tau}, P^{\pi} : \xi^{✠} \rightarrow \xi$ is called a SVN-primal if:

(P1)) Boundary conditions:

$$P^{\sigma}(1) = 0, \quad P^{\sigma}(0) = 1, \quad T^{\tau}(1) = 0, \quad T^{\tau}(0) = 1, \quad T^{\pi}(1) = 0, \quad T^{\pi}(0) = 1.$$

(P2) For all $D, Z \in \xi^{✠}$:

$$P^{\sigma}(D \wedge Z) \leq P^{\sigma}(D) \vee P^{\sigma}(Z), \quad P^{\tau}(D \wedge Z) \geq P^{\tau}(D) \wedge P^{\tau}(Z), \quad T^{\pi}(D \wedge Z) \geq T^{\pi}(D) \wedge T^{\pi}(Z).$$

(P3) If $D \leq Z \Rightarrow P^{\sigma}(Z) \leq P^{\sigma}(D), P^{\tau}(Z) \geq P^{\tau}(D), T^{\pi}(Z) \geq T^{\pi}(D)$.

If $P \leq P^{\star}$ (pointwise), we say that P is finer than $P^{\star}$.

The structure $(✠, T^{\sigma\tau\pi}, P^{\sigma\tau\pi})$ is called a SVNPTS.

Definition 9 ([27])

Let $(✠, T^{\sigma\tau\pi}, P^{\sigma\tau\pi})$, be a single-valued neutrosophic topological space (svnpts). For each point $\vartheta \in ✠$ and for each $D \in \xi^{✠}$, the single-valued neutrosophic primal open local function $D_r^{\star}(T^{\sigma\tau\pi}, P^{\sigma\tau\pi})$ is defined as the union of all svn-points $x_{\{t,s,e\}}$ such that: If $Z \in Q_{T^{\sigma\tau\pi}}(x_{\{t,s,e\}}, r)$, and $P^{\sigma}(R) \geq r$, $P^{\pi}(R) \leq 1-r$,, $P^{\pi}(R) \leq 1-r$, then there exists at least one $\vartheta \in ✠$ such that the following conditions hold:

$$\sigma\_D(\vartheta) + \sigma\_Z(\vartheta) - 1 > \sigma\_R(\vartheta),$$
$$\pi\_D(\vartheta) + \pi\_Z(\vartheta) - 1 \leq \tau\_R(\vartheta),$$
$$\pi\_D(\vartheta) + \pi\_Z(\vartheta) - 1 \leq \pi\_R(\vartheta).$$

In this article, we denote this local function as $D_r^{\star}$ without loss of generality.

Remark 1 ([27])

Let $(✠, T^{\sigma\tau\pi}, P^{\sigma\tau\pi})$, be a single-valued neutrosophic topological space (svnpts) and let $D \in \xi^{✠}$. Define the following operators:

$$C^{\star}\sigma\tau\pi(D,r) = D \vee D_r^{\star}$$
$$I^{\star}\sigma\tau\pi(D,r) = D \wedge [(D^{c})]_r^{\star}$$

The operator $C^{\star}\sigma\tau\pi$ is referred to as an *svn-primal closure operator*. The structure $(T^{\sigma}(P^{\sigma}), T^{\tau}(P^{\tau}), T^{\pi}(P^{\pi}))$ forms the *svnt* generated by $C^{\star}\sigma\tau\pi$ $\mathcal{C}^{\star}_{T^{\sigma\tau\pi}}$, where:
$T^{\star}(I)(D) = \vee\{r: C^{\star}\sigma\tau\pi(D^{c}, r) = D^{c}\}$.

Theorem 2 ([27])

Let $(✠, T^{\sigma\tau\pi})$ be an SVNTS (soft valued neutrosophic triple system), and let P_1 and P_2 be two SVN-primals on ✠. Then, for each $D, Z \in \xi^{✠}$ and $r \in \xi_0$, the following properties hold:

1. If $D \leq Z$, then $D_r^{\star} \leq Z_r^{\star}$.
2. If $P_1^{\sigma} \leq P_2^{\sigma}, P_1^{\tau} \geq P_2^{\tau}$ and $P_1^{\pi} \geq P_2^{\pi}$, then: $D_r^{\star}(T^{\sigma\tau\pi}, P_1^{\sigma\tau\pi}) \geq D_r^{\star}(T^{\sigma\tau\pi}, P_2^{\sigma\tau\pi})$
3. $D_r^{\star} = C^{\star}\sigma\tau\pi(D,r) \leq CT\sigma\tau\pi(D,r)$.
4. $(D_r^{\star})_r^{\star} \leq D_r^{\star}$.
5. If $P^{\sigma}(D) \geq r$, $P^{\pi}(D) \leq 1-r$, $P^{\pi}(D) \leq 1-r$, and P^π ≤ 1 − r, then $(D \vee Z)_r^{\star} = D_r^{\star} \vee Z_r^{\star} = D_r^{\star}$
6. $D_r^{\star} \vee Z_r^{\star} = (D \vee Z)_r^{\star}$
7. $D_r^{\star} \wedge Z_r^{\star} \geq (D \vee Z)_r^{\star}$.

III. Primal Semi-Closure Operator

Definition 10 Let $(✠, T^{\sigma\tau\pi}, P^{\sigma\tau\pi})$ be a *svnpts* and $r \in \xi_0$. Then $D \in \xi^{✠}$ is called:

(1)*r-svnpo* iff $D \leq IT\sigma\tau\pi(D_r^{\star}, r)$.

(2)*r-svnspo* iff $D \leq C^{\star}\sigma\tau\pi(IT\sigma\tau\pi(D,r), r)$.

(3)*r-svnppo* iff $D \leq IT\sigma\tau\pi(C^{\star}\sigma\tau\pi(D,r),r)$.
The complement of *r-svnpo* (resp, *r-svnspo*, *r-svnppo-set*) are called *r-svnpc* (resp, *r-svnspc*, *r-svnppc*).
Example 1 Presume that ✠ $= \{u1, u2, u3\}$; define the *svn-sets* $Z1, Z2, Z3, R \in \xi$✠ as follows

$$Z1 = \langle(0.5,0.5,0.5), (0.5,0.5,0.5), (0.5,0.5,0.5)\rangle,$$
$$Z2 = \langle(0.4,0.4,0.4), (0.5,0.5,0.5), (0.5,0.5,0.5)\rangle,$$
$$Z3 = \langle(0.4,0.4,0.4), (0.4,0.4,0.4), (0.4,0.4,0.4)\rangle,$$
$$R = \langle(0.1,0.1,0.1), (0,0,0), (0,0,0)\rangle.$$

Define the mapping, P^{σ}, P^{τ}, P^{π} : $\xi^{✠} \rightarrow \xi\ \xi$ and T^{σ}, T^{τ}, T^{π} : $\xi^{✠} \rightarrow \xi$ as follows:

$$T^{e}(\mathrm{D}) = \begin{cases} 1, & \text{if } \mathrm{Z} = \overline{0}, \\ 1, & \text{if } \mathrm{Z} = \overline{1}, \\ \frac{1}{2}, & \text{if } Z = \{Z1, Z2\}, \\ 0, & \text{otherwise} , \end{cases} \qquad P^{\varrho}(\Pi) = \begin{cases} 1, & \text{if } \mathrm{Z} = \overline{0}, \\ \frac{1}{4}, & \text{if } \mathrm{Z} = \Upsilon, \\ \frac{1}{2}, & \text{if } \overline{0} < \mathrm{Z} < \mathrm{R}, \\ 0, & \text{otherwise} , \end{cases}$$

$$T^{\varsigma}(\Pi) = \begin{cases} 0, & \text{if } \mathrm{Z} = \overline{0}, \\ 0, & \text{if } \mathrm{Z} = \overline{1}, \\ \frac{1}{2}, & \text{if } Z = \{Z2, Z3\}, \\ 1, & \text{otherwise} , \end{cases} \qquad P^{\varsigma}(\Pi) = \begin{cases} 0, & \text{if } \mathrm{Z} = \overline{0}, \\ \frac{3}{4}, & \text{if } \mathrm{Z} = \Upsilon, \\ \frac{1}{2}, & \text{if } \overline{0} < \mathrm{Z} < \mathrm{R}, \\ 1, & \text{otherwise}, \end{cases}$$

$$\mathfrak{T}^{\varphi}(\Pi) = \begin{cases} 0, & \text{if } \mathrm{Z} = \overline{0}, \\ 0, & \text{if } \mathrm{Z} = \overline{1}, \\ \frac{1}{4}, & \text{if } Z = \{Z2, Z3\}, \\ 1, & \text{otherwise} , \end{cases} \qquad P^{\varphi}(\Pi) = \begin{cases} 0, & \text{if } \mathrm{Z} = \overline{0}, \\ \frac{2}{3}, & \text{if } \mathrm{Z} = \Upsilon, \\ \frac{1}{2}, & \text{if } \overline{0} < \mathrm{Z} < \mathrm{R}, \\ 1, & \text{otherwise}, \end{cases}$$

Suppose that $\Theta = \langle(0.3,0.3,0.3), (0.3,0.3,0.3), (0.3,0.3,0.3)\rangle, then$, Θ is $\frac{1}{2}$ - *svnppo* but it is not $\frac{1}{2}$-*svnspo*.

Definition 11 A mapping PSC : $\xi^{✠} \times \xi_0 \rightarrow \xi^{✠}$ is called a single-valued neutrosophic primal semi-closure operator (for short, PSC) on ✠ if, for all D,Z∈ $\xi^{✠}$ and $r, t \in \xi_0$, the following axioms hold:
1. $PSC(\overline{0}, r) = \overline{0}$;
2. $D \leq PSC(D, r)$;
3. $PSC(D, r) \vee PSC(Z, r) = PSC(D \vee Z, r)$;
4. $PSC(D, t) \leq PC(D, r)\ if\ t \leq r$;
5. $PSC(PSC(D, r), r) = PSC(D, r)$.

The pair (✠, PSC) is referred to as a single-valued neutrosophic primal semiclosure space (*svnpscs*).

If $_1$ and PSC2 are single-valued neutrosophic semi-closure operators on ✠, then PSC1 is said to be finer than $PSC2$, denoted $PSC2 \leq PC1, if\ PSC1(D, r) \leq PSC2(D, r)$ for every D∈ $\xi^{✠}$ and $r \in \xi_0$.

Theorem 3 *Let* (✠, $T^{\sigma\tau\pi}$) *be an svnts. Then, for any* D∈ $\xi^{✠}$ *and* r ∈ ξ_0, *we define an operator* PSC : $\xi^{✠} \times \xi_0 \rightarrow \xi^{✠}$ *as follows:*

$$PSC\sigma\tau\pi(D, r) = \wedge\{Z \in \xi^{✠} : D \leq Z, \quad Z\ is\ r - svnpc\}.$$

Then, (✠, $PSC\sigma\tau\pi$) *is an svnpscs.*
Proof Suppose that (✠, $PSC\sigma\tau\pi$) is an *SVNTS*. Then, (1), (2) and (4) follows directly from the definition of $PSC\sigma\tau\pi$.
(3) Since $D, Z \leq D \cup Z$ we obtain $PSC\sigma\tau\pi(Z, r) \leq PSC\sigma\tau\pi(D \cup Z, r)$ and $PSC\sigma\tau\pi(D, r) \leq PSC\sigma\tau\pi(D \cup Z, r)$, therefore,

$$PSC\sigma\tau\pi(D, r) \cup PSC\sigma\tau\pi(Z, r) \leq PSC\sigma\tau\pi(D \cup Z, r).$$

Let (✠, $T^{\sigma\tau\pi}$) be an *svnts*. From (2), we have

$D \leq PSC\sigma(D, r)$, $\quad [PSC\sigma(D, r)]^{c} \leq C\sigma(int\sigma([PSC\sigma(D, r)]^{c}, r), r)$

$$[PSC\sigma(D, r)]^{c} \geq C\sigma(int\sigma([PSC\sigma(D, r)]^{c}, r), r),$$
$$[PSC\sigma(D, r)]^{C} \geq C\sigma(int\sigma([PSC\sigma(D, r)^{c}, r), r)$$

$D \geq PSC\tau(D, r)$, $\quad [PSC\tau(D, r)]^{C} \geq C\tau(int\tau([PSC\tau(D, r)]^{c}, r), r)$

$$[PSC\tau(D, r)]^{c} \leq C\tau(int\tau([PSC\tau(D, r)]^{c}, r), r),$$
$$[PSC\tau(D, r)]^{c} \leq C\tau(int\tau([PSC\tau(D, r)]^{c}, r), r),$$

$D \geq PSC\tau(D, r)$, $\quad [PSC\pi(D, r)]^{C} \geq C\pi(int\pi([PSC\pi(D, r)]^{c}, r), r)$

$$[PSC\pi(D,r)]^c \le C\pi(int\pi([PSC\pi(D,r)]^c,r),r),$$
$$[PSC\pi(D,r)]^c \le C\pi(int\tau([PSC\pi(D,r)]^c,r),r),$$

And

$Z \le PSC\sigma(Z,r)$, $\quad [PSC\sigma(Z,r)]^c \le C\sigma(int\sigma([PSC\sigma(Z,r)]^c,r),r)$

$$[PSC\sigma(Z,r)]^c \ge C\sigma(int\sigma([PSC\sigma(Z,r)]^c,r),r),$$
$$[PSC\sigma(Z,r)]^C \ge C\sigma(int\sigma([PSC\sigma(Z,r)^c,r),r)$$

$Z \ge PSC\tau(Z,r)$, $\quad [PSC\tau(Z,r)]^C \ge C\tau(int\tau([PSC\tau(Z,r)]^c,r),r)$

$$[PSC\tau(Z,r)]^c \le C\tau(int\tau([PSC\tau(Z,r)]^c,r),r),$$
$$[PSC\tau(Z,r)]^c \le C\tau(int\tau([PSC\tau(Z,r)]^c,r),r),$$

$Z \ge PSC\tau(Z,r)$, $\quad [PSC\pi(Z,r)]^C \ge C\pi(int\pi([PSC\pi(Z,r)]^c,r),r)$

$$[PSC\pi(Z,r)]^c \le C\pi(int\pi([PSC\pi(Z,r)]^c,r),r),$$
$$[PSC\pi(Z,r)]^c \le C\pi(int\tau([PSC\pi(Z,r)]^c,r),r)$$

It implies that $D \cup Z \le PSC\sigma\tau\pi(D,r) \cup PSC\sigma\tau\pi(Z,r)$ and

$[PSC\sigma(D,r) \cup [PSC\sigma(Z,r)]^c = [PSC\sigma(D,r)]^c \cap [PSC\sigma(Z,r)]^C$
$\le C\sigma(in\,\sigma([PSC\sigma(D,r)]^c,r),r) \cap C\sigma(int\sigma([PSC\sigma(Z,r)]^c,r),r)$
$= C\sigma(int\sigma([PSC\sigma(D,r)]^c \cap [PSC\sigma(Z,r)]^c,r),r)$
$= C\sigma(int\sigma([PSC\sigma(D,r) \cup PSC\sigma(Z,r)]^c,r),r),$
$[PSC\tau(D,r) \cup [PSC\tau(Z,r)]^c = [PSC\tau(D,r)]^c \cap [PSC\tau(Z,r)]^c$
$\ge C\tau(int\tau([PSC\tau(D,r)]^c,r),r) \cap C\tau(int\tau([PSC\tau(Z,r)]^c,r),r)$
$= C\tau(int\tau([PSC\tau(D,r)]^c \cap [PSC\tau(Z,r)]^c,r),r)$
$= C\tau(int\tau([PSC\tau(D,r) \cup PSC\tau(Z,r)]^c,r),r),$
$[PSC\pi(D,r) \cup [PSC\pi(Z,r)]^c = [PSC\pi(D,r)]^c \cap [PSC\pi(Z,r)]^c$
$\ge C\tau(int\pi([PSC\pi(D,r)]^c,r),r) \cap C\pi(int\pi([PSC\pi(Z,r)]^c,r),r)$
$= C\pi(int\pi([PSC\pi(D,r)]^c \cap [PSC\pi(Z,r)]^c,r),r)$
$= C\pi(int\pi([PSC\pi(D,r) \cup PSC\pi(Z,r)]^c,r),r),$

Hence, $PSC\sigma\tau\pi(D,r) \vee PSC\sigma\tau\pi(Z,r) \ge PSC\sigma\tau\pi(D \vee Z,r)$,thus

$$PSC\sigma\tau\pi(D,r) \vee PSC\sigma\tau\pi(Z,r) = PSC\sigma\tau\pi(D \vee Z,r)$$

(5) Suppose that there exists $r \in \zeta_0$, D∈ $\zeta^{✠}$ and κ ∈✠ such that

$$T^{\sigma}_{PSC\sigma T\sigma}(PSC\sigma(D,r),r)(\upsilon) > T^{\sigma}_{PSC\sigma T\sigma}((D,r),r)(\upsilon).$$
$$T^{\tau}_{PSC\sigma T\sigma}(PSC\tau(D,r),r)(\upsilon) \le T^{\tau}_{PSC\sigma T\sigma}((D,r),r)(\upsilon).$$
$$T^{\pi}_{PSC\sigma T\sigma}(PSC\pi(D,r),r)(\upsilon) \le T^{\pi}_{PSC\sigma T\sigma}((D,r),r)(\upsilon).$$

By the definition of $PSC\sigma\tau\pi$, there exists an D∈ $\zeta^{✠}$ with D≥Z and Z that is *svnpscs* such that

$$T^{\sigma}_{PSC\sigma T\sigma}(PSC\sigma(D,r),r)(\upsilon) > T^{\sigma}(Z)(\upsilon) > T^{\sigma}_{PSC\sigma T\sigma}((D,r),r)(\upsilon).$$
$$T^{\tau}_{PSC\sigma T\sigma}(PSC\tau(D,r),r)(\upsilon) \le T^{\tau}(Z)(\upsilon) \le T^{\tau}_{PSC\sigma T\sigma}((D,r),r)(\upsilon).$$
$$T^{\pi}_{PSC\sigma T\sigma}(PSC\pi(D,r),r)(\upsilon) \le T^{\pi}(Z)(\upsilon) \le T^{\pi}_{PSC\sigma T\sigma}((D,r),r)(\upsilon).$$

Since $PSC\sigma\tau\pi(D,R) \le Z$ and Z that is *svnpscs*, by the definition of $PSC\sigma\tau\pi(PSC\sigma\tau\pi)$, we have

$$T^{\sigma}_{PSC\sigma T\sigma}(PSC\sigma(D,r),r)(\upsilon) \le T^{\sigma}(Z)(\upsilon), \quad T^{\tau}_{PSC\sigma T\sigma}(PSC\tau(D,r),r)(\upsilon) > T^{\tau}(Z)(\upsilon),$$
$$T^{\pi}_{PSC\sigma T\sigma}(PSC\pi(D,r),r)(\upsilon) > T^{\pi}(Z)(\upsilon).$$

It is a contradiction. Thus, $PSC(PSC(D,r),r) = PSC(D,r)$. Hence, PSC is a single valued neutrosophic primal closure operator in ✠.

Theorem 4 *Let* $(F, PSC\sigma\tau\pi)$ *be an SVNPSCS and* D∈ $\zeta^{✠}$ *and* $r \in \zeta_0$. *Define the mapping* $T^{\tau\tau\pi}_{PSC\sigma T\sigma}: \zeta^{✠} \to \zeta$ *by*

$$T^{\sigma}_{PSC\sigma}(D,r) = \vee\{r \in \xi_0:\ PSC\sigma(D^c,r) = D^c\}.$$
$$T^{\tau}_{PSC\sigma}(D,r) = \wedge\{r \in \xi_0:\ PSC\tau(D^c,r) = D^c\}.$$
$$T^{\tau}_{PSC\sigma}(D,r) = \wedge\{r \in \xi_0:\ PSC\tau(D^c,r) = D^c\}.$$

Then,

1) $T^{\sigma}_{PSC\sigma}$ *is an SVNTS on* F*;*
2) $PSC\sigma\tau\pi$ *is finer than* SC.

Proof (T1) Let $(F, PSC\sigma\tau\pi)$ be an *SVNSCS*. Since $PSC(\overline{0},r) = \overline{0}$ and $PSC(\overline{1},r) = \overline{1}$; for every $r \in \zeta_0$,

(T2) Let $(F, PSC\sigma\tau\pi)$ be an *SVNSCS*. Suppose that there exists D1, D2∈ $\zeta^{✠}$ such that

$$T^{\sigma}_{PSC\sigma T\sigma}(D1 \wedge D2) < T^{\sigma}_{PSC\sigma T\sigma}(D1) \wedge T^{\sigma}_{PSC\sigma T\sigma}(D2),$$
$$T^{\tau}_{PSC\sigma T\sigma}(D1 \wedge D2) > T^{\tau}_{PSC\sigma T\sigma}(D1) \vee T^{\sigma}_{PSC\sigma T\sigma}(D2),$$
$$T^{\pi}_{PSC\sigma T\sigma}(D1 \wedge D2) > T^{\pi}_{PSC\sigma T\sigma}(D1) \vee T^{\sigma}_{PSC\sigma T\sigma}(D2),$$

there exists $r \in \zeta_0$ such that

$$T^{\sigma}_{PSC\sigma T\sigma}(D1 \wedge D2) > r > T^{\sigma}_{PSC\sigma T\sigma}(D1) \wedge T^{\sigma}_{PSC\sigma T\sigma}(D2),$$
$$T^{\tau}_{PSC\sigma T\sigma}(D1 \wedge D2) \leq 1 - r \leq T^{\tau}_{PSC\sigma T\sigma}(D1) \vee T^{\sigma}_{PSC\sigma T\sigma}(D2),$$
$$T^{\pi}_{PSC\sigma T\sigma}(D1 \wedge D2) \leq 1 - r \leq T^{\pi}_{PSC\sigma T\sigma}(D1) \vee T^{\sigma}_{PSC\sigma T\sigma}(D2),$$

For each $i \in \{1,2\}$, there exists $r \in \zeta_0$ with $PSC(Di, ri) = Di$ such that

$$r < ri \leq T^{\sigma}_{PSC\sigma T\sigma}(Di),\ \ T^{\tau}_{PSC\sigma T\sigma}(Di) \leq 1 - ri < -r,\quad T^{\pi}_{PSC\sigma T\sigma}(Di) \leq 1 - ri < -r.$$

In addition, since $PSC\sigma\tau\pi(Di) = Di$ by (2) and (4) of Definition 11 , for any $i \in \{1,2\}$,

$$PSC\sigma\tau\pi(D1 \cup D2, r) = D1 \cup D2.$$

It follows that $T^{\sigma}_{PSC\sigma T\sigma}(D1 \wedge D2) \geq r,\ T^{\tau}_{PSC\sigma T\sigma}(D1 \wedge D2) \leq 1 - r$ and $T^{\pi}_{PSC\sigma T\sigma}(D1 \wedge D2) \leq 1 - r$. It is a contradiction. Thus, for every For any D1,D2 $\in \xi^{Ŧ}$: $T^{\sigma}(D1 \wedge D2) \geq T^{\sigma}(D1) \wedge T^{\sigma}(D2), T^{\tau}(D1 \wedge D2) \leq T^{\tau}(D1) \vee T^{\tau}(D2),\ T^{\pi}(D1 \wedge D2) \leq T^{\pi}(D1) \vee T^{\pi}(D2).$

(T3) Obvious.

Example 1 Presume that $F = \{u1, u2\}$; define the *svn-sets* $Z1, Z2 \in \xi^{Ŧ}$ as follows

$$Z1 = \langle(0.1,0.1), (0.3,0.3), (0.3,0.3)\rangle,$$
$$Z2 = \langle(0.4,0.4,), (0.1,0.1), (0.1,0.1)\rangle.$$

We define the mapping PSC : $\zeta^{Ŧ} \times \zeta_0 \rightarrow \zeta^{Ŧ}$ as follows:

$$\mathrm{PSC(D,r)} = \begin{cases} 1, & \text{if } D = \overline{0}, \\ Z1 \cap Z2, & \text{if } 0 \neq D \leq Z1 \cap Z2,\ 0 < r < \frac{1}{2}, \\ Z1, & \text{if } D \leq Z1,\quad D \nleq Z2,\ 0 < r < \frac{1}{2} \\ Z2, & \text{if } D \leq Z2,\quad D \nleq Z1,\ 0 < r < \frac{1}{2}, \\ Z1 \cup Z2, & \text{if } 0 \neq D \leq Z1 \cap Z2,\quad 0 < r < \frac{1}{2}, \\ 1, & \text{otherwise.} \end{cases}$$

Then, PSC is a single-valued neutrosophic closure operator.

From Theorem 2, we have a single-valued neutrosophic topology $\left(T^{\sigma}_{PSC,}\ T^{\tau}_{PSC} T^{\pi}_{PSC}\right)$ on F as follows:

$$T^{\sigma}_{PSC,}\ (\mathrm{D}) = \begin{cases} 1, & \text{if } D = \overline{0}\ or\ \overline{1} \\ \frac{2}{3}, & \text{if } D = (Z1)^c, \\ \frac{1}{2}, & \text{if } D = (Z2)^c, \\ \frac{1}{2}, & \text{if } D = (Z1)^c \cup (Z1)^c, \\ \frac{1}{2}, & \text{if } 0 \neq D \leq Z1 \cap Z, \\ 0, & \text{otherwise.} \end{cases}$$

$$T^{\tau}_{PSC}\ (\mathrm{D}) = \begin{cases} 0, & \text{if } D = \overline{0}\ or\ \overline{1} \\ \frac{1}{3}, & \text{if } D = (Z1)^c, \\ \frac{1}{2}, & \text{if } D = (Z2)^c, \\ \frac{1}{2}, & \text{if } D = (Z1)^c \cup (Z1)^c, \\ \frac{1}{2}, & \text{if } 0 \neq D \leq Z1 \cap Z, \\ 1, & \text{otherwise.} \end{cases}$$

$$T_{PSC}^{\pi}\ (\mathrm{D}) = \begin{cases} 0, & \text{if } \mathrm{D} = \overline{0} \ or\ \overline{1} \\ \frac{1}{3}, & \text{if } \mathrm{D} = (\mathrm{Z1})^c, \\ \frac{1}{2}, & \text{if } D = (\mathrm{Z2})^c, \\ \frac{1}{2}, & \text{if } D = (\mathrm{Z1})^c \cup (\mathrm{Z1})^c, \\ \frac{1}{2}, & \text{if } 0 \neq \mathrm{D} \leq \mathrm{Z1} \cap \mathrm{Z}, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, the $T_{PSC,}^{\sigma\tau\pi}$ is a single-valued neutrosophic topology on F.

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