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Interface of Special Functions and Statistics

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Abstract:
In the last two to three decades, significant research has been focused on special functions related to statistical distributions. This includes the Generalized Hurwitz-Lerch Zeta Beta type-2 distributions and Gamma distribution, generalized hypergeometric distribution, Parabolic cylinder distribution, generalized Hurwitz Zeta Beta prime distribution, Mathieu distribution, Plank distribution, Mittag-Leffler distribution, Hurwitz Lerch Zeta distribution. Furthermore, the special cases explored include the Zipf-Mandelbrot distribution, Lotka distribution,Good distribution, Logarithmic Series distribution, right truncated form of the HLZ distribution and Estoup distribution. This study explores the properties and statistical measures like moment generating function, distribution function, survival function, Hazard rate function, mean residual life function, characteristic function, probability generating function, likelihood equations resulting in method of moments.

Keywords: *Special functions, Statistical distributions, Generalized Hurwitz-Lerch Zeta, Gamma Distribution, Hypergeometric Distributions, Parabolic Cylinder Distribution, Moment Generating Function, Survival Function, Hazard Rate Function, Method of Moments.*

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I. Introduction

Special functions are mathematical functions that arise in the solution of ordinary and partial differential equations, integral equations, in various physical and engineering applications. Over time, they have developed into a well-established area of mathematics**.**

Special functions have been an important part of mathematics for a long time. These functions, which include well-known ones like the gamma, beta, and hypergeometric functions, were originally developed to solve complex problems in physics, engineering, and other scientific fields. Over the years, their role has been expanded to statistics, where they are now used to analyze data, find model uncertainties, understand random processes more effectively and help in modeling distributions.

The history of special functions goes back to the 18th and 19th centuries, when mathematicians like Euler, Gauss, and Legendre introduced and studied them. As advanced mathematical tools, theydid the development of these functions. It became essential in solving differential equations, calculating probabilities, and creating models in various areas of science.

In modern statistics, the need for special functions arises because data sets and models can be complex. These functions help simplify complicated calculations and make it easier to find exact solutions. For example, special functions are useful in probability theory, where they are applied to describe different distributions.

The importance of studying special functions in statistics lies in their ability to provide precise answers to problems that cannot be solved by regular mathematical methods. By understanding and applying these functions, statisticians can improve their analysis, make more accurate predictions, and contribute to advances in various scientific disciplines.Special functions like Gamma and Zeta play a key role in statistical analysis.

This work aims to explore the history, development and current applications of special functions in statistics and the growing need for these functions in solving practical problems. This review looks at research, on how these functions are used in statistics, particularly focusing on applications and theoretical developments.

The objective is to identify, analyze, and summarize the key applications and theoretical developments of special functions which are associated with statistical analysis.

The review was conducted by searching peer-reviewed journals, articles and focuses on Mathai's work on special functions. Key terms such as 'special functions,' 'Gamma distribution,' 'Beta distribution,' and 'Hurwitz-Lerch zeta function' were used to identify relevant literature. Articles were selected based on their contribution to the theoretical development and practical applications of these functions in statistics.

II. Reviewof Literature

The article by Mathai, (1966) considers one-parameter families of distribution which can be transformed into exponential type by a one-to-one transformation. It established theorems which illustrated almost all the classical one-parameter families of distributions.

The paper by Mathai and Saxena (1968) explores statistical distributions that exhibit properties similar to the Gamma distribution such as Bessel function distribution, Parabolic Cylinder function distribution and the generalized Hypergeometric distribution.

Mathai and Saxena (1969) works into the use of special functions in describing and characterizing various probability distributions. The authors focus the study on the ratio of the probability law of two variables. It is proved that there exist an infinite number of independent pairs of stochastic variables whose ratios have the various distribution such as F-distribution, student t-distribution and Cauchy law by investigating the probability laws of two independent Stochastic variables.

Mathai and Moschopoulos (1991)discusses a new form of multivariate gamma distribution and explore into the concept and properties of the distribution.

Mathai (1993) provides a clear overview of various special functions, with a focus on their generalized forms. These include the gamma, Beta and Hypergeometric functions, which are widely used in mathematics, Physics and Engineering.

The book focuses on how these functions are applied in areas like probability and statistics. Mathai explores both classical and modern special functions, with the importance on the development of generalized forms. The book is divided into chapters, each focusing on a different family of special functions. Mathai introduces integral transforms, such as the Mellin and Laplace transforms, which are vital tools in manipulating these generalized functions.

Mathai demonstrates how generalized distributions can be formed using these functions, offering deeper insights into statistical modeling, especially in fields that deal with complex, real-world data distributions.It serves both an introduction to classical concepts and a guide to more advanced topics, making it an essential tool for deepening one's understanding of this field.

The paper by Ben Nakhi and Kall (2002) introduced a generalized beta function and its properties and associated probability density functions such as moment generating function, mean residue function and hazard rate function and also shown its figures of shape and scale parameters.

Gupta*et al.* (2008), study a class of Hurwitz-Lerch Zeta (HLZ) distributions by exploring the structural properties, statistical inference and their applications in reliability such as moments, recurrence relations between the moments, mode, p.g.f., distribution function, Reliability function, failure rate, reversed hazard rate and the likelihood equations result in method of moment equations and an example is obtained in which HLZ distribution fits the data perfectly.

Garg*et al.* (2009), introduced two new statistical distributions and then derived the expressions for the moments, distribution function, the survivor function, the Hazard rate function and the mean residue life function

Saxena*et al.* (2011), introduced two new statistical distributions named as generalized Hurwitz-Lerch Zeta Beta prime distribution and Gamma distribution and investigate their statistical functions such as moments, probability generating function, characteristics function, distribution and survivor function, the hazard rate function and mean residue life functions.

Srivastava*et al.*(2015) provides a comprehensive analysis of various generalized Mathieu-type series and their applications in probability theory. The authors investigate the characteristics of these series and their connections to Hurwitz-Lerch zeta functions.

Jayakumarand Pillai (1993), Jayakumar (2003),Jayakumarand Suresh (2003) provides an in-depth examination of Mittag-Leffler distributions, emphasizing their properties and applications in various fields. The author explores the mathematical foundations of these distributions, particularly their relationships with other known statistical distributions. It significantly contributes to the understanding of Mittag-Leffler distributions, both theoretically and practically.

Haubold*et al.* (2011) had a paper on a brief survey of the Mittag-Leffler functions and their applications and properties

A random variable X has the Mittag-Leffler distribution if its distribution function has the form

$$F_{\propto}(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{k\propto}}{\Gamma(1+k\alpha)}, \; x > 0, \quad 0 < \alpha \leq 1$$

where, α is the scale parameter. If $\alpha = 1$, we get the exponential distribution.

Rafik*et al.* (2018) introduces a new special function termed T(x), which behaves similarly to the error function. The authors provide a closed form for the cumulative distribution function (CDF) of T(x), which allows for better modeling in probability theory.

Liew *et al.* (2022) introduce a new statistical distribution called the Poisson-stopped Hurwitz-Lerch zeta distribution. This distribution combines ideas from the Poisson distribution and the Hurwitz-Lerch zeta function to create a model that can handle more complicated data.

The authors explain the key features of this new distribution, such as how to calculate probabilities and other important statistical measures like moments and generating functions. They also show how this distribution can be useful in real-life situations, such as risk management and queueing systems, where the level of uncertainty can change over time.

The paper presents a new distribution that could be useful for complex data, but more research is needed to test its practical use and compare it with other models.

Statistical properties

1. Ben Nakhi and Kall (2002)introduced the pdf of the generalized beta function and statistical measure.

The pdf of a random variable X associated with generalized beta function is defined by –

f(x) = $\frac{\nu^{-a} x^{u-1}(1+x)^{-\mu-u}\ 2\overset{\omega}{R}1(a,b;c;\frac{-x}{\nu})}{\overset{\omega}{B}\binom{a,b;c;\nu}{u,\mu}}$ x 1, [x>0] (1)

where, a, b, c are complex parameters, $\omega > 0$, Re p, Re u> 0, $|arg\nu|$ <π and c≠0, -1, -2,…,

$\overset{\omega}{B}\binom{a,b;c;\nu}{u,\mu} \triangleq \nu^{-a}\int_0^\infty t^{u-1}\,(1+t)^{-\mu-u} 2\overset{\omega}{R}1(a,b;c;\frac{-t}{\nu})$dt, is a generalized form of beta function and

$2\overset{\omega}{R}1\left(a,b;c;\frac{-x}{\nu}\right) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\omega k)}{\Gamma(c+\omega k)}\frac{x^k}{k!}$, |x|<1, isthe ω −Gauss hypergeometric function.

Distribution function

$$F(x) \triangleq \mathrm{P}(\mathrm{X} \le \mathrm{x}) = \int_0^x f(t)dt = \frac{\overset{\omega}{B}{}_0^x\binom{a,b;c;\nu}{u,\mu}}{\overset{\omega}{B}\binom{a,b;c;\nu}{u,\mu}}$$

Moments

$$E[X^k] = \frac{\nu^{-a}\int_0^\infty t^{k+u-1}(1+t)^{-\mu-u}2\overset{\omega}{R}1(a,b;c;\frac{-t}{\nu})dt}{\overset{\omega}{B}\binom{a,b;c;\nu}{u,\mu}} = \frac{\overset{\omega}{B}\binom{a,b;c;\nu}{u+k,\mu-k}}{\overset{\omega}{B}\binom{a,b;c;\nu}{u,\mu}}$$

Moment generating function of the random variable X is obtained by using Taylor expansion,

$$M(t) = \mathrm{E}[e^{tx}] = \sum_{k=0}^{\infty}\frac{t^k}{k!}E[X^k] = \sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{\overset{\omega}{B}\binom{a,b;c;\nu}{u+k,\mu-k}}{\overset{\omega}{B}\binom{a,b;c;\nu}{u,\mu}}$$

Survival function

$$S(x) = \mathrm{P}(\mathrm{X} \ge \mathrm{x}) = 1 - F(x) = \int_x^\infty f(t)dt = \frac{\overset{\omega}{B}{}_x^\infty\binom{a,b;c;\nu}{u,\mu}}{\overset{\omega}{B}\binom{a,b;c;\nu}{u,\mu}}$$

The Hazard Rate function

$$h(x) = \frac{f(x)}{S(x)} = \frac{\nu^{-a}x^{u-1}(1+x)^{-\mu-u}2\overset{\omega}{R}1(a,b;c;\frac{-x}{\nu})}{\overset{\omega}{B}{}_x^\infty\binom{a,b;c;\nu}{u,\mu}} \text{x } 1,\ [\mathrm{x} > 0]$$

The Mean Residual life function

$$K(x) = \mathrm{E}[\mathrm{X} - \mathrm{x}|\mathrm{X} \ge \mathrm{x}] = \frac{\int_x^\infty (t-x)f(t)dt}{S(x)} = \frac{\int_x^\infty tf(t)dt}{S(x)} - x$$

Since,

$$\int_x^\infty tf(t)dt = \frac{\overset{\omega}{B}{}_x^\infty\binom{a,b;c;\nu}{u+1,\mu-1}}{\overset{\omega}{B}\binom{a,b;c;\nu}{u,\mu}}$$

therefore,

$$K(x) = \frac{\overset{\omega}{B_x^\infty}\binom{a,b;c;\nu}{u+1,\mu-1}}{\overset{\omega}{B_x^\infty}\binom{a,b;c;\nu}{u,\mu}} - x$$

For a=0, we get the mean residual life function K(x) of the beta distribution of second kind $\frac{B_x^\infty(u+1,\mu-1)}{B_x^\infty(u,\mu)} - x$.

2. Hurwitz-Lerch Zeta distribution (Refer Gupta *et al.* (2008))

The probability mass function is given as

$P(X=k) = \frac{\theta^k}{T(\theta,s,a)(a+k)^{s+1}}$, (k ∈ ℕ; s≥0; 0≤a≤1; 0< θ ≤1) (2)

where, $T(\theta,s,a) = \sum_{k=1}^{\infty}\frac{\theta^k}{(k+a)^{s+1}} = \theta\,\Phi(\theta, s+1, a+1)$ and $\Phi(z,s,a) = \sum_{k=0}^{\infty}\frac{z^k}{(k+a)^s}$, is the Lerch transcendent defined for a ∉ {-1,-2,…} and s ∈ ℂ when |z|<1 or Re s>1 when |z|=1. Here θ is the scale parameter, s is the shape parameter and a is the location parameter.

The special cases of the HLZ distribution are as follows:-

a) Riemann Zeta distribution: If θ=1and a=0 in (2), we have

$$P(X=k) = \frac{1}{\Phi(1,s+1,1)k^{s+1}} = \frac{1}{\zeta(s+1)k^{s+1}}, (k \in N;\ s>0)$$

Here, s is the shape parameter and $\zeta(s+1)$is the Riemann Zeta (RZ) function.

b) Zipf-Mandelbrot distribution: If θ=1 in equation (2), we have

$$P(X=k) = \frac{1}{\Phi(1,s+1,a+1)(a+k)^{s+1}} = \frac{1}{\zeta(s+1,a+1)(a+k)^{s+1}},\ (k \in N;\ s>0;\ a>0)$$

Here, s is the shape parameter and a is the location parameter.

c) Lotka distribution: If a=0, θ=1 and s=1 in equation (2), we have

$$P(X=k) = \frac{1}{[\Phi(1,2,1)]k^2}, (k \in N;\ \Phi(1,2,1) = \frac{\pi^2}{6})$$

Here, all the parameters are constant.

d) Good distribution: iIf a=0 in equation (2), we have

$$P(X=k) = \frac{\theta^k}{[\theta\Phi(\theta,s+1,1)]k^{s+1}},\ (k \in \mathbb{N};\ s>0;\ 0<\theta<1)$$

where, θ is the scale parameter and s is the shape parameter.

e) Logarithmic series distribution: if s=0 and a=0 in equation (2), we have

$$P(X=k) = \frac{\theta^k}{[\theta\Phi(\theta,1,1)]k},\ (k \in \mathbb{N};\ 0<\theta<1)$$ (Refer Liewand Ong(2012))

Here, θ is the scale parameter.

f) Right truncated form of the HLZ distribution:

$P(X=k) = \frac{\theta^k}{T^*(\theta,s,a)(a+k)^{s+1}}$, (k= $1,\dots,n$; s>0; 0< θ ≤1); 0≤a≤1

Here, s is the shape parameter, a is the location parameter and

$$T^*(\theta,s,a) = \theta\Phi(\theta,1,1) - \theta^{n+1}\Phi(\theta,s+1,a+n+1)$$

g) Estoup distribution: If a=0, θ=1 and s=0 in eq. (2), we get

$$P(X=k) = \frac{1}{[\Phi_n(1,1,1)]k}, (k= 1,\dots,n)$$

where, $\Phi_n(1,1,1) = \sum_{k=1}^{n}\frac{1}{k} = H_n$(harmonic number).

Moments

By Gupta (1974), HLZ distribution is a special case of the modified power series distribution,

$$P(X=x) = \frac{A(x)(g(\theta))^x}{f(\theta)} \quad (x \in \mathbb{B})$$

where $\mathbb{B}$ is a subset of the set $\mathbb{N}_0$ of non-negative integer, A(x) > 0, f(θ) and g(θ) are positive, finite and differentiable function of θ, then we have

$$g(\theta) = \theta, \qquad f(\theta) = T(\theta,s,a), \qquad A(x) = \frac{1}{(x+a)^{s+1}}$$

thus,

$$E(X) = \frac{g(\theta)f'(\theta)}{f(\theta)g'(\theta)} = \frac{\theta\sum_{k=1}^{\infty}\frac{k\,\theta^{k-1}}{(a+k)^{s+1}}}{\sum_{k=1}^{\infty}\frac{\theta^{k}}{(a+k)^{s+1}}} = \frac{\sum_{k=1}^{\infty}\frac{k\,\theta^{k}}{(a+k)^{s+1}}}{\sum_{k=1}^{\infty}\frac{\theta^{k}}{(a+k)^{s+1}}} = \frac{1}{T(\theta,s,a)}\sum_{k=1}^{\infty}\frac{k\,\theta^{k}}{(a+k)^{s+1}} = \mu \qquad (2.1)$$

Mode

For finding the mode,

$\frac{P(X=k)}{P(X=k-1)} = \theta\left(1-\frac{1}{a+k}\right)^{s+1}$, $0<\theta \le 1$, (k ∈ $\mathbb{N}\backslash\{1\}$) (2.2)

This means that, as k → ∞, the above ratio → θ. It has only one mode at the point x = 1.

Probability generating function

$\psi(t)$= E(e^X) $= \sum_{k=0}^{\infty} t^k P(X=k) = \frac{T(\theta t,s,a)}{T(\theta,s,a)}$, ($0 <\theta t< 1$) (2.3)

Distribution function

P(X ≤ k) = 1 - $\theta^k \frac{T(\theta,s,a+k)}{T(\theta,s,a)}$, (k ∈ ℕ) (2.4)

Survival function

$$s(k) = 1 - F(x) = \theta^{k-1}\frac{T(\theta,s,a+k-1)}{T(\theta,s,a)}, \quad (k \in \mathbb{N}) \qquad (2.5)$$

Failure rate (Hazard rate) function

$$r(k) = \frac{P(X=k)}{P(X\ge k)} = \frac{1}{(a+k)^{s+1}\,T(\theta,s,a+k)}, \quad (k \in \mathbb{N}) \qquad (2.6)$$

Reversed hazard Rate

$$r^*(k) = \frac{P(X=k)}{P(X\le k)} = \frac{\theta^k}{[T(\theta,s,a)-\theta^k T(\theta,s,a+k)](a+k)^{s+1}}, (k \in \mathbb{N}) \quad (2.7)$$

Mean Time Between Failure

$$\text{MTBF} = \sum_{k=0}^{\infty}\theta^{k-1}\frac{T(\theta,s,a+k-1)}{T(\theta,s,a)} = \frac{1}{T(\theta,s,a)}\sum_{k=0}^{\infty}\theta^{k-1}\,T(\theta,s,a+k-1)$$

Mean Time To Failure

$$\text{MTTF} = \sum_{k=0}^{\infty}k\,\theta^{k-1}\frac{T(\theta,s,a+k-1)}{T(\theta,s,a)} = \frac{1}{T(\theta,s,a)}\sum_{k=0}^{\infty}k\,\theta^{k-1}\,T(\theta,s,a+k-1)$$

HLZ distribution life distribution has

(i) log-concave (log-convex) probability mass function (pmf) if

$$\frac{p(t+2)p(t)}{[p(t+1)]^2} \le (\ge) 1, (t \ge 0) \quad \Rightarrow \quad \left[\frac{(a+t+1)^2}{(a+t+2)(a+t)}\right]^{s+1} \le (\ge) 1 \qquad (2.8)$$

(ii) IFR/DFR (increasing failure rate/Decreasing failure rate) if the failure rate is non-decreasing/non-increasing of ‘k’ in eq. (2.6).

(iii) IFRA/DFRA (increasing failure rate average/decreasing failure rate average) if

$$\{S(k)\}^{1/k} = \left\{\frac{\theta^{k-1}T(\theta,s,a+k-1)}{T(\theta,s,a)}\right\}^{1/k}, \qquad (2.9)$$

is a decreasing (increasing) sequence for k $\epsilon\mathbb{N}$.

(iv) NBU/NWU (new better than used/new worse than used) if

$\frac{S(t+x)}{S(t)} \le (\ge)\, S(x) \;\Rightarrow \frac{\theta^x T(\theta,s,a+t+x)}{T(\theta,s,a+t-1)} \le (\ge)\, S(x)$ (2.10)

(v) NBUE/NWUE (new better than used in expectation/new worse than used in expectation)

$$\sum_{j=0}^{\infty} S(t+j) \le (\ge) \sum_{j=0}^{\infty} S(j)$$

$$\Rightarrow \sum_{j=0}^{\infty}\frac{\theta^{t+j-1}T(\theta,s,a+t+j-1)}{T(\theta,s,a)} \le (\ge) \sum_{j=0}^{\infty}\frac{\theta^{j-1}T(\theta,s,a+j-1)}{T(\theta,s,a)}$$

$$\Rightarrow \sum_{j=0}^{\infty}\theta^{t+j-1}T(\theta,s,a+t+j-1) \le (\ge) \sum_{j=0}^{\infty}\theta^{j-1}T(\theta,s,a+j-1) \qquad (2.11)$$

(vi) DMRL/IMRL (decreasing mean residual life/ increasing mean residual life) if

$$\mu(t+1) \leq (\geq)\, \mu(t), \ (t \geq 0)$$

Since, $MRL = \mu(t) = \sum_{k=0}^{\infty} \frac{(k-t)}{T(\theta,s,a+k)(a+k)^{s+1}}$, we get

$$\sum_{k=0}^{\infty} \frac{(k-t-1)}{T(\theta,s,a)} \leq (\geq) \sum_{k=0}^{\infty} \frac{(k-t)}{T(\theta,s,a)} \tag{2.12}$$

(vii) HNBUE/HNWUE (harmonic new better than used in expectation/harmonic new worse than used in expectation) if

$$\sum_{j=0}^{\infty} S(j) \leq (\geq)\, \mu \left(1-\frac{1}{\mu}\right)^{k}, \qquad (\mu = E(x) < \infty) \tag{2.13}$$

(viii) DRHR/IRHR (decreasing reversed hazard rate/increasing reversed hazard rate) if

$$\frac{P(X=k)}{P(X\leq k)} = \frac{\theta^{k}}{(a+k)^{s+1}[T(\theta,s,a)-\theta^{k}T(\theta,s,a+k)} \tag{2.14}$$

is decreasing/increasing in k.

Statistical inference

The likelihood function is given by ,

$$L(\theta, s, a) = \frac{\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n}[(a+x_i)^{s+1}T(\theta,s,a)]},$$

The log-likelihood function is given by

$$\ln L = \sum_{i-1}^{n} x_i \, log\theta - \sum_{i=1}^{n}(s+1)\log(a+x_i) - nlogT(\theta,s,a)$$

Likelihood equations are

$$\frac{\partial lnL}{\partial\theta} = \frac{\sum_{i-1}^{n} x_i}{\theta} - n\frac{\frac{\partial T(\theta,s,a)}{\partial\theta}}{T(\theta,s,a)} \tag{2.15}$$

$$\frac{\partial lnL}{\partial s} = -\sum_{i-1}^{n}\log(a+x_i) - n\frac{\frac{\partial T(\theta,s,a)}{\partial s}}{T(\theta,s,a)} \tag{2.16}$$

$$\frac{\partial lnL}{\partial a} = -\sum_{i=1}^{n}\frac{(s+1)}{(a+x_i)} - n\frac{\frac{\partial T(\theta,s,a)}{\partial a}}{T(\theta,s,a)} \tag{2.17}$$

Since,

$$\frac{\frac{\partial T(\theta,s,a)}{\partial\theta}}{T(\theta,s,a)} = \frac{1}{T(\theta,s,a)}\frac{\partial}{\partial\theta}\sum_{k=1}^{\infty}\frac{\theta^{k}}{(a+k)^{s+1}} = \frac{1}{T(\theta,s,a)}\sum_{k=1}^{\infty}\frac{k\theta^{k-1}}{(a+k)^{s+1}} = \frac{1}{\theta T(\theta,s,a)}\sum_{k=1}^{\infty}\frac{k\theta^{k}}{(a+k)^{s+1}} = \frac{1}{\theta}\mu, \text{ from eq. (2.1)} \tag{2.18}$$

Also,

$$\frac{\frac{\partial T(\theta,s,a)}{\partial s}}{T(\theta,s,a)} = \frac{1}{T(\theta,s,a)}\frac{\partial}{\partial s}\sum_{k=1}^{\infty}\frac{\theta^{k}}{(a+k)^{s+1}} = \frac{1}{T(\theta,s,a)}\sum_{k=1}^{\infty}\frac{\theta^{k}}{(a+k)^{s+1}}\ln(a+k)(-1) = -E[\ln(a+X)] \tag{2.19}$$

and,

$$\frac{\frac{\partial T(\theta,s,a)}{\partial a}}{T(\theta,s,a)} = \frac{1}{T(\theta,s,a)}\frac{\partial}{\partial a}\sum_{k=1}^{\infty}\frac{\theta^{k}}{(a+k)^{s+1}} = \frac{1}{T(\theta,s,a)}\sum_{k=1}^{\infty}(s+1)\frac{\theta^{k}}{(a+k)(a+k)^{s+1}}(-1) = -(s+1)\,E\left(\frac{1}{a+X}\right) \tag{2.20}$$

By solving eq. (2.15) and using (2.18) we get, $\bar{x}$ = E(x) = μ, which implies the method of moments equation is a solution of the likelihood equation.

On solving eq. (2.16) and using (2.19) we get,

$$\sum_{k=1}^{\infty}\frac{\ln(a+x_i)}{n} = E[\ln(a+X)]$$

By solving eq. (2.17) and using (2.20) we get,

$$\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{(a+x_i)} = E\left(\frac{1}{a+X}\right)$$

The following result establishes a relation between log-concavity(log-convexity) of the pmf and IFR(DFR) distributions.

Theorem 1. Let $\eta(t) = 1 - \frac{P(X=t+1)}{P(X=t)}$ and $\Delta\,\eta(t) = \eta(t+1) - \eta(t) = \left(\frac{p(t+1)}{p(t)} - \frac{p(t+2)}{p(t+1)}\right)$

Then the following statement hold true:

(i) If $\Delta\,\eta(t) > 0$ (log-concavity), then r(t) is non-decreasing (IFR).
(ii) If $\Delta\,\eta(t) < 0$ (log-convexity), then r(t) is non-increasing (IFR).
(iii) If $\Delta\,\eta(t) = 0$ for all t, then the hazard rate is a constant.

From this theorem, the following implications hold true

IFR(DFR) ⇒ IFRA(DFRA) ⇒ $NBU(NWU)$ ⇒ $NBUE(NWUE)$ ⇒ HNBUE(HNWUE)

and

IFR(DFR) ⇒DMRL(IMRL) ⇒ $NBUE(NWUE)$ ⇒ HNBUE(HNWUE).

Theorem 2. In this case, we find that

$$\Delta\,\eta(k) = \theta\left[\left(\frac{a+k}{a+k+1}\right)^{s+1} - \left(\frac{a+k+1}{a+k+2}\right)^{s+1}\right]$$

If b = a+k+1, we get

$$\Delta\,\eta(k) = \theta\left[\left(\frac{b-1}{b}\right)^{s+1} - \left(\frac{b}{b+1}\right)^{s+1}\right] = \theta\left(\frac{(b^2-1)^{s+1}-(b^2)^{s+1}}{[b(b+1)]^{s+1}}\right)$$

This implies that $\Delta\,\eta(k) < 0$. Thus, the HLZ distribution has a log-convex pmf and is infinitely divisible.
So, we can say that the HLZ distribution is DFR, DFRA, NWU, NWUE, HNWUE and IMRL.
Monotonicity of the reversed hazard rate

Lemma. Suppose that X is a non-negative discrete random variable. If the pmf, that is, f(k) of X is decreasing in k, then X has DRHR.

Theorem 3. The HLZ distribution has DRHR.
Proof.Since P(X=k) is decreasing in k (from eq. 2.16), using the above lemma, the result of this theorem follows easily.

Theorem 4. Let F be a discrete life distribution and let the corresponding probability mass function be denoted by the sequence $\{f_k\}_{k\in\mathbb{N}_0}$. If F has decreasing failure rate, then it has DRHR.

Theorem 5. Each of the following closure properties holds true.
(a) If a sequence of the HLZ distribution converges to a limiting distribution, then the limiting distribution has DRHR.
(b) If the components of a parallel system have independent life times with the HLZ distributions, then the system life time has DRHR.
(c) If the components of a k-out-of-n system have independent life times with identical HLZ distributions, then the system life time has DRHR.
(d) The convolution of two HLZ distributions produces a DRHR distribution.

3. Generalized Hurwitz-Lerch Zeta Beta type-2 Distribution (Garg*et al.* (2009))
The p.d.f. of a random variable is defined as

$$f(x) = \begin{cases} \frac{\Gamma\gamma\; x^{\beta-1}(1+x)^{-\gamma}\,\Phi_\alpha^*\left(\frac{xz}{1+x},s,a\right)}{\Gamma\gamma\;\Gamma(\gamma-\beta)\Phi_{\alpha,\beta,\gamma}(z,s,a)}, & x>0 \\ 0, & otherwise \end{cases} \qquad (3)$$

where, $\Phi_\alpha^*(z,s,a) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\frac{z^n}{(n+a)^s}$, (a ≠ 0, -1, -2,… s ∈ C when |z|<1 and Re(s-α) > 0, when |z| = 1) is generalized Hurwitz-Lerch Zeta function.
and$\Phi_{\alpha,\beta,\gamma}(z,s,a) = \sum_{n=0}^{\infty}\frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n n!}\frac{z^n}{(n+a)^s}$, is the extension of general HLZ function and $(\gamma, a \neq 0,-1,-2,\dots s\in C, when\ |z|<1\ and\ Re(\gamma+s-\alpha-\beta)>0, when\ |z|=1)$.Here, β and γ are shape parameters and z is the scale parameter. On taking $\alpha = 0$ in f(x), we get beta distribution.

Distribution function,
It is given by,

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_0^x f(t)dt = \frac{B_{\alpha,\beta;\gamma}^{0,x}(z,s,a)}{\Phi_{\alpha,\beta;\gamma}(z,s,a)},$$

where $B_{\alpha,\beta;\gamma}^{0,x}(z,s,a) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^x t^{\beta-1}(1+t)^{-\gamma}\Phi_\alpha^*(\frac{tz}{1+t},s,a)dt$, which is known as the incomplete generalized beta type-2 function.

Moments

$$E(x^k) = \int_0^\infty x^k\ f(x)\ dx = \frac{(-1)^k(\beta)_k}{(1-\gamma+\beta)_k}\frac{\Phi_{\alpha,\beta+k;\gamma}(z,s,a)}{\Phi_{\alpha,\beta,\gamma}(z,s,a)}$$

Further, they obtain the Mellin Transform, Laplace Transform and Fourier Transform (characteristic function) of f(x) respectively as follows

$$E(x^{(t-1)}) = M[f(x);t] = \int_0^\infty x^{(t-1)} f(x)\,dx = \frac{(-1)^{(t-1)}(\beta)_{t-1}}{(1-\gamma+\beta)_{t-1}} \frac{\Phi_{\alpha,\beta+t-1;\gamma}(z,s,a)}{\Phi_{\alpha,\beta,\gamma}(z,s,a)},$$

$$E(x^{-tx}) = L[f(x);t] = \int_0^\infty x^{-tx} f(x)dx = \frac{1}{\Phi_{\alpha,\beta,\gamma}(z,s,a)} \sum_{k=0}^{\infty} \frac{t^k(\beta)_k}{(1-\gamma+\beta)_k\, k!} \Phi_{\alpha,\beta+k;\gamma}(z,s,a)$$

$$E(x^{wtx}) = F[f(x);t] = \int_0^\infty x^{wtx} f(x)dx = \frac{1}{(1-\gamma+\beta)_k} \sum_{k=0}^{\infty} \frac{(-\omega t)^k(\beta)_k}{(1-\gamma+\beta)_k} \Phi_{\alpha,\beta+k;\gamma}(z,s,a),$$

where, $\omega = \sqrt{(-1)}$

Survival function

It is expressed as the following,

$$S(x) = 1 - F(x) = \int_x^\infty f(t)dt = \frac{B^{x,\infty}_{\alpha,\beta;\gamma}(z,s,a)}{\Phi_{\alpha,\beta;\gamma}(z,s,a)},$$

where, $B^{x,\infty}_{\alpha,\beta;\gamma}(z,s,a) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_x^\infty t^{\beta-1}(1+t)^{-\gamma} \Phi^*_\alpha(\frac{tz}{1+t},s,a)dt$, which is known as the complementary incomplete generalized beta type-2 function.

Hazard rate function

$$h(x) = \frac{f(x)}{S(x)} = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \frac{x^{\beta-1}(1+x)^{-\gamma} \Phi^*_\alpha(\frac{xz}{1+x},s,a)}{B^{x,\infty}_{\alpha,\beta;\gamma}(z,s,a)}.$$

Mean residual life function

For a random variable *x*, we have

$$K(x) = E[X - x|X \geq x] = \frac{1}{S(x)} \int_x^\infty (t-x)f(t)dt = \frac{\beta\, B^{x,\infty}_{\alpha,\beta+1;\gamma}(z,s,a)}{(\gamma-\beta-1)B^{x,\infty}_{\alpha,\beta;\gamma}(z,s,a)} - x.$$

4. The Generalized Hurwitz-Lerch Zeta Gamma Distribution (Refer Garg*et al.* (2009)):

$$f(x) = \begin{cases} \frac{b^s x^{s-1} e^{-ax}\ 2F1(\alpha,\beta;\gamma;ze^{-ax})}{\Gamma s\ \Phi_{\alpha,\beta,\gamma}\left(z,s,\frac{a}{b}\right)}, & x > 0 \\ 0, & otherwise \end{cases} \tag{4}$$

where, $\Phi_{\alpha,\beta,\gamma}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} \frac{z^n}{(n+a)^s}$ is the HLZ function and $2F1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!}$, is Gauss' Hypergeometric function, $(\gamma, a \neq 0,-1,-2,\ldots s \in C, when\ |z| < 1\ and\ Re(\gamma+s-\alpha-\beta) > 0, when\ |z| = 1)$.Here, a and b are the scale parameters and s is the shape parameter.

If b = a and$\alpha = 0$ in f(x), it reduces to gamma distribution.

If $\beta = \gamma$, we get unified Plank distribution.

If $\alpha = 1$, we get generalized Plank distribution, defined by Nadarajah and Kotz (2006).

Distribution function

$$F(x) = \frac{\Gamma^{0,x}_{\alpha,\beta;\gamma}(z,s,a,b)}{\Phi_{\alpha,\beta,\gamma}(z,s,\frac{a}{b})},$$

where, $\Gamma^{0,x}_{\alpha,\beta;\gamma}(z,s,a,b) = \frac{b^s}{\Gamma(s)} \int_0^x t^{s-1} e^{-at}\ 2F1\,(\alpha,\beta;\gamma;ze^{-bt})dt$ is the incomplete generalized gamma function

Moments

$$E(x^k) = \int_0^\infty x^k f(x)\,dx = \frac{(s)_k}{b^k} \frac{\Phi_{\alpha,\beta;\gamma}(z,s+k,\frac{a}{b})}{\Phi_{\alpha,\beta,\gamma}(z,s,\frac{a}{b})}$$

The Mellin Transform, Laplace Transform and Fourier Transform (characteristic function) are as follows

$$E(x^{(t-1)}) = M[f(x);t] = \int_0^\infty x^{(t-1)} f(x)\,dx = \frac{(s)_{t-1}}{b^{t-1}} \frac{\Phi_{\alpha,\beta;\gamma}(z,s+t-1,\frac{a}{b})}{\Phi_{\alpha,\beta,\gamma}(z,s,\frac{a}{b})},$$

$$E(x^{-tx}) = L[f(x);t] = \int_0^\infty x^{-tx} f(x)dx = \frac{\Phi_{\alpha,\beta;\gamma}(z,s,\frac{a+t}{b})}{\Phi_{\alpha,\beta,\gamma}(z,s,\frac{a}{b})}$$

$$E(x^{wtx}) = F[f(x);t] = \int_0^\infty x^{wtx} f(x)dx = \frac{\Phi_{\alpha,\beta;\gamma}(z,s,\frac{a-\omega t}{b})}{\Phi_{\alpha,\beta,\gamma}(z,s,\frac{a}{b})},$$

where, $\omega = \sqrt{(-1)}$

Survival function

$$S(x) = 1 - \mathrm{F(x)} = \frac{\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b)}{\Phi_{\alpha,\beta,\gamma}(z,s,\frac{a}{b})}$$

where, $\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b) = \frac{b^s}{\Gamma(s)}\int_x^{\infty} t^{s-1}e^{-at}\ \ 2F1\ (\alpha,\beta;\gamma;ze^{-bt})dt$, is called the complementary incomplete generalized gamma function.

Hazard rate function

$$\mathrm{h(x)} = \frac{f(x)}{S(x)} = \frac{b^s e^{-ax}\, x^{s-1} 2F1\ (\alpha,\beta;\gamma;ze^{-bx})}{\Gamma(s)\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b)}$$

Mean Residual Life Function

$$K(x) = \mathrm{E[X - x|X \geq x]} = \frac{1}{S(x)}\int_x^{\infty}(t-x)f(t)dt = \frac{s}{b}\frac{\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s+1,a,b)}{\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,\frac{a}{b})} - x\ .$$

5. Generalized Hurwitz-Lerch Zeta prime distribution: (Refer Saxena *et al.* (2011)):

$$f(x) = \begin{cases} \frac{\Gamma\gamma\ x^{\lambda-1}\Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\left(\frac{zx^{\rho}}{(1+x)^{\kappa}},s,a\right)}{\Gamma(\lambda)\,\Gamma(\nu-\lambda)(1+x)^{\nu}\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)}, & x > 0 \\ 0, & x \leq 0 \end{cases} \qquad (5)$$

Here, μ, λ are shape parameters and z is the scale parameter which satisfy$\Re\{\nu\}$>$\Re\{\lambda\}$>0,s ∈ ℂ, $\kappa > \rho$>0, σ>0.
For $\sigma = \rho = \kappa = 1$ we get HLZ Beta prime distribution discussed by Garg*et al.*(2009).
If $\sigma = \rho = \kappa = 1\ and\ \mu = 0$, we get Beta Prime distribution (or Beta distribution of second kind)

Distribution function,

$$F(x) = \frac{\varphi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)}$$

Survival function

$$S(x) = \frac{\bar{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)}$$

Probability generating function

$$G_x(t) = \mathrm{E}(e^{-tX}) = \frac{1}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)}\sum_{r=0}^{\infty}\frac{(\lambda)_r}{(1+\lambda-\nu)_r}\frac{t^r}{r!}\Phi_{\lambda+r,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)$$

Characteristic function

$$\phi_x(t) = \mathrm{E}(e^{itX}) = \frac{1}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)}\sum_{r=0}^{\infty}\frac{(\lambda)_r}{(1+\lambda-\nu)_r}\frac{(-it)^r}{r!}\Phi_{\lambda+r,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)$$

Hazard Rate function

$$h(x) = \frac{f(x)}{S(x)} = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\frac{x^{\lambda-1}}{(1+x)^{\nu}}\frac{\Phi_{\lambda,\mu,\nu}^{(\sigma,\kappa-\rho)}(z,s,a)}{\bar{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)},$$

where, $\bar{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)$ is the upper and lower incomplete (complementary) φ – functions.

6. Generalized Hurwitz-Lerch Zeta Gamma distribution: (ReferSaxena *et al.* (2011)):

$$f(x) = \begin{cases} \frac{b^s x^{s-1}e^{-ax}\ \ 2\Psi1^*\left[\left(\begin{smallmatrix}(\lambda,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{smallmatrix}\middle| ze^{-bx}\right)\right]}{\Gamma s\ \ \Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}\left(z,s,\frac{a}{b}\right)}, & x > 0 \\ 0, & x \leq 0 \end{cases} \qquad (6)$$

Here, a, b are scale parameters and s is the shape parameter. R{a}>R{s}>0, when |z| ≤ 1 (z≠10) and R(s) > 1 when z = 1.For $\sigma = \rho = \kappa = 1$, we get HLZ Gamma distribution discussed by *Garg et al.*.If $\sigma = \rho = \kappa = 1$, b

= a, λ=0, equation (6) reduces to the gamma distribution. If $\sigma = \rho = \kappa = 1$, μ=ν, λ=1, equation (6) reduces to generalized Plank distribution.

Generating function

$$G_x(t) = \mathrm{E}(e^{-tY}) = \frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,(a+t)/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,\frac{a}{b})}$$

Characteristic function

$$\phi_x(t) = \mathrm{E}(e^{itY}) = \frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,(a-it)/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,\frac{a}{b})}$$

III. Conclusion:

Many authors have defined and studied different forms of special functions, but only a few researchers have introduced the special functions into a statistical probability distribution. The common and well-known special function distributions are beta, gamma, Hypergeometric and so on.

This literature review highlights the significance of special functions in the field of statistics. These functions, including the Gamma, Hurwitz-Lerch Zeta, Beta, and others, play a key role in understanding various statistical distributions. By examining works from Mathai, Saxena, and other researchers, we see how special functions help in modeling complex real-world data and solving statistical problems.Special functions have been widely applied in probability theory and reliability studies,offering tools for calculating important statistical properties like moments, survivor functions, and hazard rates. The development of new distributionsdemonstrates how these functions continue to evolve to meet modern statistical challenges.

Studying special functions provides deep insights and practical tools that improve our ability to analyze and interpret data. This review emphasizes the importance of continuing research in this area to develop new methods for solving emerging problems in statistical analysis.

There is a significant scope for further research in this area such as estimating parameters, testing hypotheses, and creating confidence intervals. Further studies can also focus on using Bayesian methods and exploring reliability and survival analysis.

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