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On Common Fixed Point Theorems in Complete Rectangular S-Metric Spaces

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Abstract: *In this paper, we establish the some common fixed point theorems in rectangular S-metric spaces, an advanced generalization of S-metric spaces. We develop new common fixed point theorems that integrate and extend various well-known results in fixed point theory. Our findings are further supported by illustrative examples.*

Keywords: *Rectangular metric spaces; S-metric spaces; Rectangular S-metric spaces*

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I. Introduction and Preliminaries

Frechet [4] originally introduced the concept of a metric space. Over the course of time, mathematicians have delved into diverse extensions and generalizations of metric spaces. Initially, Gahler [5] and Dhage [3] introduced the ideas of 2-metric spaces and D-metric spaces, respectively. Subsequently, Mustafa and Sims [6] extended the theory by introducing G-metric spaces. In more recent developments, Shaban Sedghi [9, 10] introduced the novel concepts of $D^*$and S-metric spaces, providing some of their fundamental properties. Following this, Sedghi [10] focused on advancing the theory of S -metric spaces, deriving new results that have been presented in several papers. In this paper, we find some new results of rectangular S -metric spaces and prove common fixed point theorems on same spaces.

Definition 1.1. [10] Let X be a non empty set and $\mathrm{S}: \mathrm{X}^3 \rightarrow \mathrm{R}^+$, a function that satisfies the following properties:

(i) $\mathrm{S}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = 0$ if and only if $\mathrm{x} = \mathrm{y} = \mathrm{z}$,
(ii) $\mathrm{S}(\mathrm{x}, \mathrm{y}, \mathrm{z}) \leq \mathrm{S}(\mathrm{x}, \mathrm{x}, \mathrm{a}) + \mathrm{S}(\mathrm{y}, \mathrm{y}, \mathrm{a}) + \mathrm{S}(\mathrm{z}, \mathrm{z}, \mathrm{a})$ for all $\mathrm{a}, \mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{X}$ (rectangle inequality).

Then (X, S) is called S -metric space.

Definition 1.2. [1] Let X be a non empty set and $\mathrm{S}: \mathrm{X}^3 \rightarrow \mathrm{R}^+$, a function satisfying the following properties:

(i) $S(x,y,z) = 0$ if and only if $x = y = z$,
(ii) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$ for all $x,y,z \in X$ and all distinct points $a \in X - \{x,y,z\}$. Then (X,S) is called a rectangular S -metric space.

Definition 1.3. [10] Let (X,S) be an S -metric space and $A \subset X$.

(i) A sequence $\{x_n\}$ in X converges to x if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is for every $\in > 0$ there exists $n_o \in N$ such that for $n \geq n_o$, $S(x_n, x_n, x) < \in$. This case, we denote by $\lim_{n\to\infty} x_n = x$ and we say that x is the limit of $\{x_n\}$ in X.

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $\in > 0$, there exists $n_o \in N$ such that $S(x_n, x_n, x_m) < \in$ for each $n, m \geq n_0$.

(iii) The S -metric space(X, S) is said to be complete if every Cauchy sequence is convergent.

Proposition 1.4. [2] Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Lemma 1.5. [10] If (X, S) is an S -metric space, then we have $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

Lemma 1.6. [10] Let (X, S) be an S -metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X converging to x and y respectively, that is, $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n, x_n, y_n) \to S(x, x, y)$ as $n \to \infty$.

Lemma 1.7. [10] Let (X, S) be an S -metric space. If the sequence $\{x_n\}$ in X converges to x, then the limit x is unique.

Lemma 1.8. [10] Let (X, S) be an S -metric space. If the sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is a Cauchy sequence.

Example 1.9. Let $X = N \cup \{0\}$ and define S: $X \times X \times X \to R^+ \cup \{0\}$ *by*

$$S(l,m,n) = \begin{cases} 0 & if \quad l = m = n, \\ lm + mn + nl & if \ l \neq m \neq n. \end{cases}$$

Then (X,S) is a rectangular S -metric space.

II. Main results

Theorem 2.1. Let (X,S) be a complete rectangular S -metric space and let f and g be a self mapping on X. Assume that f and g satisfies the following conditions:

$$S(fx, fx, fy) \leq h[S(fx, fx, gy) + S(fy, fy, gx)],$$

where $0 \leq h < 0.2$, also,

(i) $f(X) \subseteq g(X)$,

(ii) If $g(X)$ is complete.

Then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof: Let x_0 be any point in X. Since $fx_0 \subseteq f(X)$ and $f(X) \subseteq g(X)$, there exists a point x_1 in X such that $fx_0 = gx_1$. As $x_1 \in X$, it follows that $fx_1 \subseteq f(X)$. Thus, we can select x_2 in X such that $fx_1 = gx_2$. Repeating this process iteratively, we construct a sequence $\{x_n\}$ in X, where $x_{n+1} \in X$, satisfies $fx_n = gx_{n+1}$ for all n.

Now, consider

$$S(gx_{n+1}, gx_{n+1}, gx_n) = S(fx_n, fx_n, fx_{n-1})$$

$$\leq h[S(fx_n, fx_n, gx_{n-1}) + S(fx_{n-1}, fx_{n-1}, gx_n)]$$

$$= h[S(gx_{n+1}, gx_{n+1}, gx_{n-1}) + S(gx_n, gx_n, gx_n)]$$

$$\leq hS(gx_{n+1}, gx_{n+1}, gx_{n-1})$$

$$\leq h[S(gx_{n+1}, gx_{n+1}, gx_n) + S(gx_{n+1}, gx_{n+1}, gx_n)$$

$$+ S(gx_{n-1}, gx_{n-1}, gx_n)]$$

$$= h[2S(gx_{n+1}, gx_{n+1}, gx_n) + S(gx_{n-1}, gx_{n-1}, gx_n)]$$

$$\Rightarrow (1-2h)S(gx_{n+1}, gx_{n+1}, gx_n) \leq hS(gx_{n-1}, gx_{n-1}, gx_n)$$

$$\Rightarrow \quad S(gx_{n+1}, gx_{n+1}, gx_n) \leq \frac{h}{(1-2h)} S(gx_{n-1}, gx_{n-1}, gx_n)$$

$$= kS(gx_n, gx_n, gx_{n-1})$$

$$S(gx_{n+1}, gx_{n+1}, gx_n) \leq k^n S(gx_1, gx_1, gx_0).$$

For $m, n \in N$ and some $N \in N$, with $n > m$, we have

$$S(gx_n, gx_n, gx_m) \leq S(gx_n, gx_n, gx_{n-1}) + S(gx_n, gx_n, gx_{n-1}) + S(gx_m, gx_m, gx_{n-1})$$

$$\leq k^{n-1}S(gx_1, gx_1, gx_0) + k^{n-1}S(gx_1, gx_1, gx_0) + S(gx_m, gx_m, gx_{n-1})$$

$$= 2k^{n-1}S(gx_1, gx_1, gx_0) + S(gx_m, gx_m, gx_{n-1})$$

$$\leq 2k^{n-1}z + [S(gx_m, gx_m, gx_{n-2}) + S(gx_m, gx_m, gx_{n-2})$$

$$+ S(gx_{n-1}, gx_{n-1}, gx_{n-2})]$$

$$\leq 2k^{n-1}z + [2S(gx_m, gx_m, gx_{n-2}) + k^{n-2}S(gx_1, gx_1, gx_0)]$$

$$= 2k^{n-1}z + k^{n-2}z + 2S(gx_m, gx_m, gx_{n-2})$$

$$\leq 2k^{n-1}z + k^{n-2}z + 2[2S(gx_m, gx_m, gx_{n-3})$$

$$+ S(gx_{n-2}, gx_{n-2}, gx_{n-3})]$$

$$\leq 2k^{n-1}z + k^{n-2}z + 4S(gx_m, gx_m, gx_{n-3}) + 2k^{n-3}z$$

$$\leq 2k^{n-1}z + k^{n-2}z + 2k^{n-3}z + 4[2S(gx_m, gx_m, gx_{n-4})$$

$$+S(gx_{n-3}, gx_{n-3}, gx_{n-4})]$$

$$\leq 2k^{n-1}z + k^{n-2}z + 2k^{n-3}z + 8S(gx_m, gx_m, gx_{n-4}) + 4k^{n-4}z$$

$$= 2k^{n-1}z + k^{n-2}z + 2k^{n-3}z + 4k^{n-4}z + \cdots$$

$$= 2k^{n-1}z + k^{n-2}z(1 + \frac{2}{k} + \frac{4}{k^2} + \frac{8}{k^3} + \cdots)$$

$$= 2k^{n-1}z + k^{n-2}z(1 + (\frac{2}{k}) + \left(\frac{2}{k}\right)^2 + \left(\frac{2}{k}\right)^3 + \cdots)$$

$$= 2k^{n-1}z + k^{n-2}z(1 - \frac{2}{k})^{-1},$$

where $z = S(gx_1, gx_1, gx_0)$, as $n \to \infty$ and since $k < 1$, we have,

$$\lim_{n\to\infty} S(gx_n, gx_n, gx_m) = 0.$$

Thus, $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exist q in $g(X)$ such that $gx_n \to$ q, as $n \to \infty$. Consequently, we can find p in X such that $g(p) = q$. Thus,

$$S(fx_n, fx_n, fp) \leq h[S(fx_n, fx_n, gp,) + S(fp, fp, gx_n)$$

$$= h[S(gx_{n+1}, gx_{n+1}, gp,) + S(fp, fp, gx_n)$$

$$= h[S(q, q, q) + S(fp, fp, q)]$$

Letting $n \to \infty$,

$$S(q, q, fp) \leq hS(q, q, fp).$$

Since $S(q, q, q) = 0$ and $h < 1$, this is true if $S(q, q, fp) = 0 \Rightarrow f(p) = q$. Therefore, $f(p) = g(p) = q$. Hence, q is the point of coincidence of f and g. Now, we show that f and g have a unique point of coincidence. For this, assume that there exists another point $w \in X$ such that $f(w) = g(w) = w$.

Now

$$S(q, q, w) = S(fp, fp, fw)$$

$$\leq h[S(fp, fp, gw) + S(fw, fw, gp)]$$

$$= h[S(q, q, w) + S(w, w, q)]$$

$$\leq h[S(q, q, w) + S(q, q, w)$$

$$= 2hS(q,q,w).$$

Since $h \in [0, 0.2)$. It is true if $S(q,q,w) = 0$. So $q = w$. Hence, it is prove that, f and g have a unique point of coincidence. Since, f and g are weakly compatible, so by Preposition 1.4, f and g have a unique common fixed point in X.

Theorem 2.2. Let (X, S) be a complete rectangular S-metric space. Suppose that mapping $f: X \to X$ satisfies the following condition:

$$S(fx,fx,fy) \leq \alpha_1 S(x,x,y) + \alpha_2 S(x,x,fx) + \alpha_3 S(y,y,fy) + \alpha_4 S(x,x,fy) + \alpha_5 S(y,y,fx),$$

for all $x, y \in X$, where $\alpha i \geq 0$ for each $i \in \{1,2,3,4,5\}$ and $\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 + \alpha_5 < 1$. Then f has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point of X, and define x_n by $x_{2n+1} = \mathrm{f}x_{2n}$, $x_{2n+2} = \mathrm{f}x_{2n+1}$, $n = 0, 1, 2, \cdots$

Now,

$$S(x_{2n+1},x_{2n+1},x_{2n+2}) = S(\mathrm{f}x_{2n},\mathrm{f}x_{2n},fx_{2n+1})$$

$$\leq \alpha_1 S(x_{2n},x_{2n},x_{2n+1}) + \alpha_2 S(x_{2n},x_{2n},fx_{2n})$$

$$+ \alpha_3 S(x_{2n+1},x_{2n+1},fx_{2n+1}) +$$

$$\alpha_4 S(x_{2n},x_{2n},fx_{2n+1}) + \alpha_5 S(x_{2n+1},x_{2n+1},fx_{2n})$$

$$= \alpha_1 S(x_{2n},x_{2n},x_{2n+1}) + \alpha_2 S(x_{2n},x_{2n},x_{2n+1})$$

$$+\alpha_3 S(x_{2n+1},x_{2n+1},x_{2n+2}) + \alpha_4 S(x_{2n},x_{2n},x_{2n+2})$$

$$+\alpha_5 S(x_{2n+1},x_{2n+1},x_{2n+1})$$

$$= \alpha_1 S(x_{2n},x_{2n},x_{2n+1}) + \alpha_2 S(x_{2n},x_{2n},x_{2n+1})$$

$$+\alpha_3 S(x_{2n+1},x_{2n+1},x_{2n+2}) + \alpha_4 [S(x_{2n},x_{2n},x_{2n+1})$$

$$+S(x_{2n},x_{2n},x_{2n+1}) + S(x_{2n+2},x_{2n+2},x_{2n+1})]$$

$$= \alpha_1 S(x_{2n},x_{2n},x_{2n+1}) + \alpha_2 S(x_{2n},x_{2n},x_{2n+1})$$

$$+\alpha_3 S(x_{2n+1},x_{2n+1},x_{2n+2})$$

$$+ \alpha_4 [2S(x_{2n},x_{2n},x_{2n+1}) + S(x_{2n+1},x_{2n+1},x_{2n+2})]$$

$$= (\alpha_1 + \alpha_2 + 2\alpha_4) S(x_{2n},x_{2n},x_{2n+1})$$

$$+(\alpha_3 + \alpha_4) S(x_{2n+1},x_{2n+1},x_{2n+2})$$

$$\Rightarrow (1 - (\alpha_3 + \alpha_4))\, S(x_{2n+1},x_{2n+1},x_{2n+2}) \leq (\alpha_1 + \alpha_2 + 2\alpha_4) S(x_{2n},x_{2n},x_{2n+1})$$

$$\Rightarrow\ S(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq \left(\frac{\alpha_1 + \alpha_2 + 2\alpha_4}{(1-(\alpha_3+\alpha_4))}\right) S(x_{2n}, x_{2n}, x_{2n+1})$$

$$= \delta S(x_{2n}, x_{2n}, x_{2n+1})$$

$$\leq \delta^{2n+1} S(x_0, x_0, x_1).$$

Where $\delta = \frac{\alpha_1 + \alpha_2 + 2\alpha_4}{(1-(\alpha_3+\alpha_4))} < 1$. Then, For $m, n \in N$ and some $N \in N$, with $n > m$, we have

$$S(x_n, x_n, x_m) \leq S(x_n, x_n, x_{n+1}) + S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1})$$

$$\leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1})$$

$$\leq 2\delta^n S(x_0, x_0, x_1) + [S(x_m, x_m, x_{n+2})$$

$$+S(x_m, x_m, x_{n+2}) + S(x_{n+1}, x_{n+1,} x_{n+2})]$$

$$= 2\delta^n z + \left[2S(x_m, x_m, x_{n+2}) + S(x_{n+1}, x_{n+1,} x_{n+2})\right]$$

$$\leq 2\delta^n z + [2S(x_m, x_m, x_{n+2}) + \delta^{n+1} z]$$

$$\leq 2\delta^n z + \delta^{n+1} z + 2\,[S(x_m, x_m, x_{n+3}) + S(x_m, x_m, x_{n+3})$$

$$+ S(x_{n+2}, x_{n+2,} x_{n+3})]$$

$$\leq 2\delta^n z + \delta^{n+1} z + 2[2S(x_m, x_m, x_{n+3}) + \delta^{n+2} z]$$

$$\leq 2\delta^n z + \delta^{n+1} z + 2\delta^{n+2} z + 4[S(x_m, x_m, x_{n+4})$$

$$+S(x_m, x_m, x_{n+4}) + S(x_{n+3}, x_{n+3,} x_{n+4})]$$

$$\leq 2\delta^n z + \delta^{n+1} z + 2\delta^{n+2} z + 4[2S(x_m, x_m, x_{n+4}) + \delta^{n+3} z]$$

$$\leq 2\delta^n z + \delta^{n+1} z + 2\delta^{n+2} z + 4\delta^{n+3} z + 8\delta^{n+4} z + \cdots$$

$$= 2\delta^n z + \delta^{n+1} z(1 + 2\delta + 4\delta^2 + 8\delta^3 + \cdots)$$

$$= 2\delta^n z + \delta^{n+1} z(1 + (2\delta) + (2\delta)^2 + (2\delta)^3 + \cdots$$

$$S(x_n, x_n, x_m) \leq 2\delta^n z + \delta^{n+1} z\left(\frac{1}{1-2\delta}\right),$$

where $z = S(x_0, x_0, x_1)$. Since $\delta < 1$, as $n \to \infty$, we have,

$$\lim_{n\to\infty} S(gx_n, gx_n, gx_m) = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exist q in X such that $x_n \to$ q, as $n \to \infty$. Now,

$$S(x_n, x_n, fq) \leq S(x_n, x_n, x_{n+1}) + S(x_n, x_n, x_{n+1}) + S(fq, fq, x_{n+1})$$

$$= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, fq)$$

$$\leq 2\delta^n z + [S(x_{n+1}, x_{n+1}, x_{n+2}) +$$

$$S(x_{n+1}, x_{n+1}, x_{n+2}) + S(fq, fq, x_{n+2})]$$

$$= 2\delta^n z + [2S(x_{n+1}, x_{n+1}, x_{n+2}) + S(fq, fq, x_{n+2})]$$

$$\leq 2\delta^n z + [2\delta^{n+1} z + S(x_{n+2}, x_{n+2}, fq)]$$

$$\leq 2\delta^n z + 2\delta^{n+1} z + [\, S(x_{n+2}, x_{n+2}, x_{n+3})$$

$$+ S(x_{n+2}, x_{n+2}, x_{n+3}) + S(fq, fq, x_{n+3})]$$

$$= 2\delta^n z + 2\delta^{n+1} z + [\, 2S(x_{n+2}, x_{n+2}, x_{n+3}) + S(x_{n+3}, x_{n+3}, fq)]$$

$$\leq 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + S(x_{n+3}, x_{n+3}, fq)$$

$$\leq 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + [\, S(x_{n+3}, x_{n+3}, x_{n+4})$$

$$+ S(x_{n+3}, x_{n+3}, x_{n+4}) + S(fq, fq, x_{n+4})]$$

$$= 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + [2S(x_{n+3}, x_{n+3}, x_{n+4}) + S(x_{n+4}, x_{n+4}, fq)]$$

$$\leq 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + 2\delta^{n+3} z + S(x_{n+4}, x_{n+4}, fq)$$

$$\leq 2\delta^n z + 2\delta^{n+1} z + 2\delta^{n+2} z + 2\delta^{n+3} z + \cdots$$

$$\leq 2\delta^n z(1 + \delta + \delta^2 + \delta^3 + \cdots)$$

$$= 2\delta^n z\left(\frac{1}{1-\delta}\right),$$

where $z = S(x_0, x_0, x_1)$. Since $\delta < 1$, as $n \to \infty$, we have,

$$\lim_{n\to\infty} S(q, q, fq) = 0 \Rightarrow \text{fq} = \text{q}.$$

Hence, f has a fixed point in X. To prove uniqueness, suppose that if w is another fixed point of f , then

$$S(q, q, w) = S(fp, fp, fw)$$

$$\leq \alpha_1 S(q, q, w) + \alpha_2 S(q, q, fq) + \alpha_3 S(w, w, fw)$$

$$+ \alpha_4 S(q, q, fw) + \alpha_5 S(w, w, fq)$$

$$= \alpha_1 S(q,q,w) + \alpha_2 S(q,q,q) + \alpha_3 S(w,w,w) +$$

$$\alpha_4 S(q,q,w) + \alpha_5 S(w,w,q)$$

$$= \alpha_1 S(q,q,w) + \alpha_4 S(q,q,w) + \alpha_5 S(q,q,w)$$

$$S(q,q,w) \leq (\alpha_1 + \alpha_4 + \alpha_5) S(q,q,w)$$

Since, $\alpha i > 0$. This gives $S(q,q,w) = 0$. Hence, $q = w$. It is prove f has a unique fixed point in X.

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