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Perfect Codes For Join Zero Divisor Graphs Of A Ring Zn

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Abstract
In this paper, we construct the perfect codes for a star zero divisor graphs of a ring z_n, when $n = 2p$, $p > 2$ prime and join zero divisor graphs of rings z_n and z_m when $n = 2p$ and $m = p2$, $p > 2$ prime . Using these we find codewords and minimum Hamming distance of the code.
Keywords: *Hamming bound, Perfect code, Non- zero zero divisors, Zero –divisor graph, Star graph, join of two graphs,*

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I. Intorduction

Richard Hamming and Marcel Golay introduced the concept of perfect codes in 1940[1], Hamming developing the binary Hamming code in 1950[23]. These codes are fundamental to the theory of error-correcting codes

N. Biggs is credited with initiating the study of perfect codes in graphs in 1973[2], extending the concept from coding theory to distance-transitive graphs, while J. Kratochvíl further developed the theory by introducing t-perfect codes in general graphs in the late 1980s. Michel Mollard, studied On perfect codes in Cartesian products of graphs in 2011[3]. Rongquan Feng, He Huang and Sanming Zhou, studied Perfect codes in circulant graphs in 2017[4].

In abstract algebra, the theory of rings has its origin in the early 19^{th} century when the commutative and non-commutative rings are being explored. A ring is one of the fundamental algebraic structures consisting of a set with two binary operations which are addition and multiplication [5].

In 1988, the zero-divisor graph was first introduced by Beck. Beck introduced idea of presenting zero-divisor graph of a commutative ring R in 1988[5]. Anderson and Livingston [1] introduced and studied the subgraph Γ(R) (of G(R)) whose vertices are the non-zero zero-divisors of R and the authors studied the interplay between the ring-theoretic properties of a commutative ring and the graph theoretic properties of its zero-divisor graph. Let $Z^*(R) = Z(R) \setminus (0)$, be the set of non-zero zero-divisors of R. The zero-divisor graph of R, denoted by Γ (R), is a simple undirected graph with all non-zero zero-divisors as vertices and two distinct vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. Thus Γ (R) is the null graph if and only if R is an integral domain. Lu Danchenga and Wu Tongsuob studied, On bipartite zero-divisor graphs in2009[6]. Nurhidayah Zaid, studied ,the perfect codes of commuting zero divisor graph of some matrices of dimension two in 2021[7] J. Phys.

D. F. Anderson and Philip S. Livingston are credited with introducing the defined term and concept of the "join graph zero divisor graph" or the standardized zero-divisor graph in graph theory, based on the work of Irving Beck. Beck originally introduced a zero-divisor graph in 1990[8], and Anderson and Livingston later modified and standardized its definition in 1999[9] for commutative rings.

II. Preliminaries

Graph theory related definitions:

- Let Z_n be the ring of integers modulo n. The zero-divisor graph of Z_n, denoted by Γ(Zn), is the graph with vertex set $V(\Gamma(Z_n)$ and for distinct vertices a and b are adjacent if and only if $a.b = 0 (\mathrm{mod}\ n)$. Clearly, $\Gamma(Z_n)$ is the null graph if and only if Z_n is an integral domain.
- A star graph is a graph with a single central vertex connected to all other vertices, which are called leaves. If a star graph has n vertices, one vertex has a degree of n-1 (the central vertex), and the other n-1 vertices each have a degree of 1. It is denoted by $K_{1,n}$

- The join of two graphs $G_1$and G_2 with disjoint vertex sets is a new graph formed by taking the union of $G_1$and G_2 and adding an edge between every vertex of $G_1$to every vertex of G_2.

i.e, V(G_1) and V(G_2) are the vertex sets and E(G_1) and E(G_2)are the edge sets, then the join graph G_1+G_2 has:
Vertex set: $V(G_1)\cup V(G_2)$ and edge set: $E(G_1)\cup E(G_2)\cup\{uv| u\in V(G_1), v\in V(G_2)\}$

Coding theory related definitions:

- The **weight w of a code word** is the number of 1s in the word. The "**distance of a code**" refers to its minimum distance, which is the smallest Hamming distance between any two distinct codewords in the code.
- **Hamming bound:** If C is a code of length n and distance d = 2t +1or d = 2t +2, then

|C| $\leq \frac{2^n}{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{t}}$

The Hamming bound is upper bound for the number of words in a code of length n and distance d=2t+1.
Note that t = [(d-1)/2], a code of distance d will correct all error patterns of weight less than or equal to [(d-1)/2]. Such a code correct all error patterns of weight less than or equal to t.

- A code C of length n and odd distance d = 2t +1 is called a **perfect code**, if C attains the Hamming bound.

i.e, |C| $= \frac{2^n}{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{t}}$

- An **Error-Correcting Code** (ECC), is a method of encoding data in a way that allows for the detection and correction of errors that may occur during data transmission or storage
- A **perfect t-correcting code** in coding theory is an error-correcting code that achieves the Hamming bound
- A perfect code which corrects all error patterns of weight less than or equal to t = [(d-1)/2], then we say that t is a **perfect t- error correcting code**.

III. Main Section

From[1], if $0 \leq t \leq n$ and if v is a word of length n, then the no.of words of length n of distance at most t from v is precisely $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{t}$

Let t and n are integers, $0 \leq t \leq n,$ then the symbol $\binom{n}{t} = \frac{n!}{t!(n-t)!}$

An unordered collection of t objects can be chosen from a set n objects.
Thus $\binom{n}{t}$ is the number of words of length n and weight t.
Since there are 2^n words of length n, if t = n
$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{t} = 2^n$

Let t= 0, then $\binom{n}{0}$ = 1= 2^0, so |C| $=\frac{2^n}{\binom{n}{0}}=2^n$. The only code with 2^n codewords of length n in C = K^n, K^n is a perfect code.

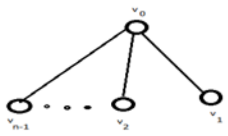
Theorem 3.1: If the length of the code C is n and of odd distance d = n, then C is a perfect code,
Proof: Let a code C of length n and distance d= 2t+1=n
From Definition, |C| $= \frac{2^n}{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{t}}$, C is a perfect code■.

If C be a perfect code of length n and distance d=2t+1, then C will correct all error patterns of weight less than or equal to t and no other error patterns**.**

Let C is a perfect code of length n and distance d=2t+1, each of the 2^n words on K^n lies within distance t of exactly one codeword. If enable to count the no .of codewords of minimum non-zero weight in a perfect code.

For every prime, the star graph $\Gamma(Z_p)$ does not exist. Since Z_p has no zero divisors.

Theorem 3.2: The code C of a zero divisor graph $\Gamma(Z_{2p})$ is star graph and forms a perfect code, if $p > 2, prime$.
Proof: Let Z_n be the ring of integers modulo n and $\Gamma(Z_n)$ is the non-zero zero-divisor graph of Z_n, when n= $2p, p > 2,$ prime number.
Let the graph $\Gamma(Z_n)$ be star graph, when the order of non-zero zero divisors is odd and when n= $2p, p > 2.$
By the definition of Zero divisor graph, if u and v be in V($\Gamma(Z_n)$) , then u.v $\equiv 0 (mod\ n)$
The order of the graph $\Gamma(Z_n)$ = n-$\varphi(n) - 1$= m
The vertex set is defined as $V(\Gamma(Z_{2p})) = \{ u, v_1, v_2, \ldots, v_{p-1} \}$.
The edge set is defined as$\{ uv_i \,/\, 1\leq i \leq p-1$ and $uv_i \equiv 0 (mod\ n)\}$, every vertex is connected only to the central vertex u.
Therefore, |V $((\Gamma(Z_{2p}))|$ = m and $|E((\Gamma(Z_{2p})|$ = m-1



Star graph $K_{1,n}$

From the star graph $\Gamma(Z_{2p})$), a code of length n, d=2t+1 and t = [(d-1)/2]

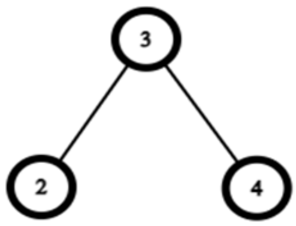
$|C| = \frac{2^m}{\binom{m}{0}+\binom{m}{1}+\cdots+\binom{m}{t}}$ = power of 2

Therefore, C is a perfect code■.

We consider Perfect code C for Z_6

Example 3.3: A zero divisor graph $\Gamma(Z_{2(3)}) = \Gamma(Z_6)$ is a star graph and forms a perfect code.
We know that
$Z_6=\{0,1,2,3,4,5\}$
The no.of non-zero zero divisors of Z_n= n-$\varphi(n) - 1$
$Here\ \ \varphi(n) = positive\ integer\ less\ than\ n\ and\ relatively\ prime\ to\ n.$
$= n \prod_{p/n} (1- 1/p)$
$\varphi(6)$= 6(1-1/2) (1-1/3) = 2

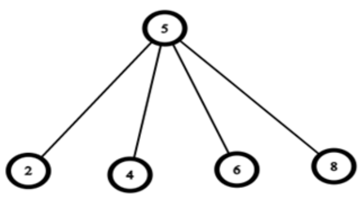
The no.of zero divisors of $Z_6 = 6 - 2 - 1 = 3$, odd
The set of non-zero zero divisors of Z_9 is $V(\Gamma(Z_6)) = \{2,3,4\}$.
The set of edges of $\Gamma(Z_6)$ is $E(\Gamma(Z_6))=\{(2,3), (3,4)\}$
Therefore, $|V((\Gamma(Z_6))| = 3$ and $|E((\Gamma(Z_6)| = 2$
A zero divisor graph $\Gamma(Z_6)$ is a star graph $K_{1,2}$



Star graph $\Gamma(Z_6)$, Fig 4.1.1

From the graph, the code length is 3 and d = 2t+1 odd distance; t =[(d-1)/2] = 3-1/2 =1
$|C| = \frac{2^3}{\binom{3}{0}+\binom{3}{1}} = \frac{2^3}{1+3} = \frac{2^3}{4} = 2^1$, is perfect code.
This code has minimum Hamming distance 3.
Since 3= 2(1)+1, this code can correct t = 1 bit error
Therefore, there are 2^3= 8 possible codewords, then C ={000, 100, 010, 001, 110, 101, 011, 111}, which is a perfect code.

Example 3.4: Consider a star graph $\Gamma(Z_{2(5)}) = \Gamma(Z_{10})$, which forms a perfect code.
The no.of non-zero zero divisors of $Z_{10} = 10 - 4 - 1 = 5$, odd
The set of non-zero zero divisors of Z_{10} is $V(\Gamma(Z_{10})) = \{2,4,5,6,8\}$.
The set of edges of $\Gamma(Z_{10})$ is $E(\Gamma(Z_{10}))=\{(2,5), (4,5), (6,5),(8,5)\}$
Therefore, $|V((\Gamma(Z_{10})| = 5$ and $|E((\Gamma(Z_{10})| = 4$



Star graph Γ (Z_{10}), 4.1.2

From the graph, the code length is 5 and d = 2t+1, odd distance; t =[(d-1)/2] = 4/2 =2

$|C| = \frac{2^5}{\binom{5}{0}+\binom{5}{1}+\binom{5}{2}} = \frac{2^5}{1+5+10} = \frac{2^5}{16} = 2^1$, is a perfect code.

This code has minimum Hamming distance 5.

Since 5= 2(2)+1, this code can correct t = 2 bit errors

Therefore, there are 2^5= 32 possible codewords, C ={00000, 10000, 01000, 001000,….,11111}, is a perfect code.

Example 3.4: Consider a star graph $\Gamma(Z_{2(7)}) = \Gamma(Z_{14})$, which forms a perfect code.

The no.of non-zero zero divisors of $Z_{14} = 14 - 6 - 1 = 7$, odd

Therefore, $|V((\Gamma(Z_{14})| = 7$ and $|E((\Gamma(Z_{14})| = 6$

Star graph Γ (Z_{14}), Fig 4.1.3

From the graph, the code length is 7 and d = 2t+1, odd distance; t =[(7-1)/2] =3

$|C| = \frac{7}{\binom{7}{0}+\binom{7}{1}+\binom{7}{2}+\binom{7}{3}} = \frac{2^7}{1+7+21+35} = \frac{2^7}{64} = 2^1$, is a perfect code.

This code has minimum Hamming distance 7.

Since 5= 2(3)+1, this code can correct t = 3 bit errors.

Therefore, there are 2^7= 128 possible codewords, C ={0000000, 1000000, 0100000, 00100000,….,1111111}, is a perfect code.

$\Gamma(Z_{2p})$, is star graph and it forms a perfect code, when $p > 2$.

Construction of perfect codes for join zero divisor graphs of a ring z_n :

The **join of two graphs** $G_1$and G_2 with disjoint vertex sets is a new graph formed by taking the union of $G_1$and G_2 and adding an edge between every vertex of $G_1$to every vertex of G_2.

i.e, $V(G_1)$ and $V(G_2)$ are the vertex sets and $E(G_1)$ and $E(G_2)$are the edge sets, then the join graph G_1+G_2 has:

Vertex set: $V(G_1) \cup V(G_2)$ and edge set: $E(G_1) \cup E(G_2) \cup \{uv| \; u \in V(G_1), v \in V(G_2)\}$

The code C of the join of zero divisor graph $\Gamma(Z_{2p})+ \Gamma(Z_4)$ does not form a perfect code, if $p = 2, prime.$

(The order of the graph $|\Gamma(Z_{2p})+ \Gamma(Z_4)|$ is even number of vertices, C does not form a Perfect code).

Theorem 3.5: The code C of the join of zero divisor graph $\Gamma(Z_{2p})+ \Gamma(Z_9)$ is a perfect code, if $p > 2, prime$

Proof: Let the graph G = $\Gamma(Z_{2p})+ \Gamma(Z_9)$

The order of the graph $|\Gamma(Z_{2p})+ \Gamma(Z_9)|$ is odd number of vertices.

We know that $V(\Gamma(Z_{2p})) = \{2, 4, \ldots, 2(p-1), p\} = \{u_1, u_2, \ldots, u_{p-1}\}$ and $V(\Gamma(Z_9)) = \{3, 6\} = \{v_1, v_2\}$.

Now, the vertex set is defined as $V(G) = \{u_0, u_1, u_2, \ldots, u_{p-1}, v_1, v_2\}$

The edge set of $E(G) = \{u_0u_i; \; 1 \le i \le p-1\} \cup \{u_iv_j; \; 0 \le i \le p-1, j = 1,2\} \cup \{v_1v_2\}$

Therefore, $|V((\Gamma(Z_{2p})+ \Gamma(Z_9))| = p+1$ and $|E((\Gamma(Z_{2p}) + \Gamma(Z_9))| = 3p$

From Theorem, 4.1.1, C is a Perfect code.

Example 3.6: Consider a join zero divisor graph $\Gamma(Z_{2(3)})+ \Gamma(Z_9)$

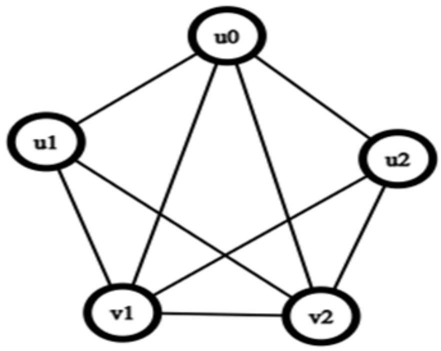
Vertex set of $\Gamma(Z_6)$ be $\{u_0, u_1, u_2\}$and edge set of $\Gamma(Z_9)$ be $\{v_1, v_2\}$

Now, the vertex set is defined as $V(\Gamma(Z_6)+ \Gamma(Z_9)) = \{u_0, u_1, u_2, v_1, v_2\}$

The edge set of $E(\Gamma(Z_6)+ \Gamma(Z_9)) = \{u_0 u_1, u_0u_2\} \cup \{u_ov_1, u_0v_2, u_1v_2, u_2v_1\} \cup \{v_1v_2\}$

Therefore, $|V((\Gamma(Z_{2p})+ \Gamma(Z_9))| = 5$ and $|E((\Gamma(Z_{2p}) + \Gamma(Z_9))| = 9$

The join zero divisor graph $\Gamma(Z_6)+ \Gamma(Z_9)$ is



Join graph $\Gamma(Z_6)+ \Gamma(Z_9)$, Fig 4.1.4

From the graph, the code length is 5 and d = 2t+1, odd distance; t =[(d-1)/2] = 4/2 =2

$|C| = \frac{2^5}{\binom{5}{0}+\binom{5}{1}+\binom{5}{2}} = \frac{2^5}{1+5+10} = \frac{2^5}{16} = 2^1$, is a perfect code.

This code has minimum Hamming distance 5.

Since 5= 2(2)+1, this code can correct t = 2 bit errors

Therefore, there are 2^5= 32 possible codewords, C ={00000, 10000, 01000, 001000,….,11111}, is a perfect code.

Example 3.7: Consider a join zero divisor graph $\Gamma(Z_{10})+\Gamma(Z_9)$

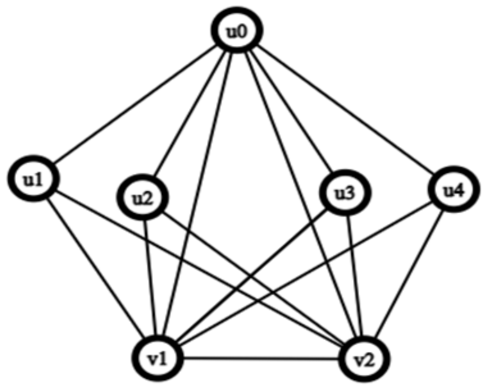
Vertex set of $\Gamma(Z_{10})$ be $\{u_0, u_1, u_2, u_3, u_4\}$and edge set of $\Gamma(Z_9)$ be $\{v_1, v_2\}$

Now, the vertex set is defined as $V(\Gamma(Z_{10})+\Gamma(Z_9)) = \{u_0, u_1, u_2, u_3, u_4, v_1, v_2\}$

The edge set of $E(\Gamma(Z_{10})+\Gamma(Z_9)) = \{u_0u_1, u_0u_2, u_0 u_3, u_0u_4\} \cup \{u_ov_1, u_0v_2, u_1v_1, u_1v_2, u_2v_1, u_2v_2, u_3v_1, u_3v_2, u_3v_1, u_4v_2\} \cup \{v_1v_2\}$

Therefore, $|V((\Gamma(Z_{2p})+\Gamma(Z_9))| = 7$ and $|E((\Gamma(Z_{2p})+\Gamma(Z_9))| = 15$

The join zero divisor graph $\Gamma(Z_{10})+\Gamma(Z_9)$ is



Join graph $\Gamma(Z_{10})+\Gamma(Z_9)$, Fig 4.1.5

From the graph, the code length is 7 and d = 2t+1, odd distance; t =[(7-1)/2] =3

$|C| = \frac{2^7}{\binom{7}{0}+\binom{7}{1}+\binom{7}{2}+\binom{7}{3}} = \frac{2^7}{1+7+21+35} = \frac{2^7}{64} = 2^1$, is a perfect code.

This code has minimum Hamming distance 7.

Since 7= 2(3)+1, this code can correct t = 3 bit errors

Therefore, there are 2^7= 128 possible codewords, C ={0000000, 1000000, 0100000, 0010000,….,1111111}, is a perfect code.

$\Gamma(Z_{2p}) + \Gamma(Z_{p^2})$, is join graphs and form a perfect code, when $p > 2$.

Theorem 3.8: $\Gamma(Z_{2p}) + \overline{\Gamma(Z_9)}$, is join graphs and form a perfect code, when $p > 2$.

Proof: Let the graph $G = \Gamma(Z_{2p})+\overline{\Gamma(Z_9)}$

The order of the graph $|\Gamma(Z_{2p})+\overline{\Gamma(Z_9)}|$ is odd

We know that $V(\Gamma(Z_{2p})) = \{2, 4, \ldots, 2(p-1), p\} = \{u_1, u_2, \ldots, u_{p-1}\}$ and $V(\Gamma(Z_9) = \{3, 6\} = \{v_1, v_2\}$.

Now, the vertex set is defined as $V(G) = \{u_0, u_1, u_2,\ldots,u_{p-1}, v_1, v_2\}$

The edge set of $E(G) = \{u_0u_i;\ 1 \le i \le p-1\} \cup \{u_iv_j;\ 0 \le i \le p-1, j = 1,2\}$

Therefore, $|V((\Gamma(Z_{2p})+\overline{\Gamma(Z_9)})| = p+1$ and $|E((\Gamma(Z_{2p})+\overline{\Gamma(Z_9)})| = 3p-1$

From Theorem, 4.1.1, C is a Perfect code.

Example 3.9: Consider a join zero divisor graph $\Gamma(Z_{2(3)})+\overline{\Gamma(Z_9)}$

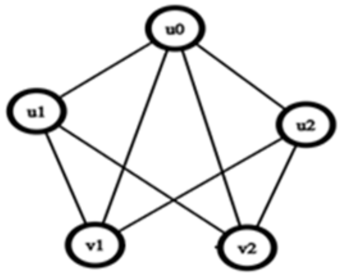
Vertex set of $\Gamma(Z_6)$ be $\{u_0, u_1, u_2\}$and edge set of $\Gamma(Z_9)$ be $\{v_1, v_2\}$

Now, the vertex set is defined as $V(\Gamma(Z_6)+\overline{\Gamma(Z_9)}) = \{u_0, u_1, u_2, v_1, v_2\}$

The edge set of $E(\Gamma(Z_6)+\overline{\Gamma(Z_9)}) = \{u_0 u_1, u_0u_2\} \cup \{u_ov_1, u_0v_2, u_1v_2, u_2v_1\} \cup \{v_1v_2\}$

Therefore, $|V((\Gamma(Z_{2p})+\overline{\Gamma(Z_9)})| = 5$ and $|E((\Gamma(Z_{2p})+\overline{\Gamma(Z_9)})| = 9$

The join zero divisor graph $\Gamma(Z_6)+\overline{\Gamma(Z_9)}$ is



Join graph $\Gamma(Z_6)+\overline{\Gamma(Z_9)}$, Fig 4.1.4

From the graph, the code length is 5 and d = 2t+1, odd distance; t =[(d-1)/2] = 4/2 =2

$|C| = \frac{2^5}{\binom{5}{0}+\binom{5}{1}+\binom{5}{2}} = \frac{2^5}{1+5+10} = \frac{2^5}{16} = 2^1$, is a perfect code.

This code has minimum Hamming distance 5.

Since 5= 2(2)+1, this code can correct t = 2 bit errors

Therefore, there are 2^5= 32 possible codewords, C ={00000, 10000, 01000, 001000,….,11111}, is a perfect code.

IV. Conclusions

Constructed perfect codes on star graphs and join graphs. If exists,

1. $\Gamma(Z_{2p})$, is star graph and it forms a perfect code, when $p > 2$.
2. $\Gamma(Z_{2p}) + \Gamma(Z_{p^2})$, is a join graphs forms a perfect code, when $p > 2$.
3. $\Gamma(Z_{2p}) + \overline{\Gamma(Zp^2)}$, is a join graph and forms a perfect code, when $p > 2$.

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