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A study of Equivalence relations and Properties of Square (*n*,*m*) normal operators in Relation to Other Operators.

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Abstract

This paper presents new findings on square(n, m) normal operators in Hilbert spaces. We analyze the properties of this class of operators by establishing some theorems and proofs. Furthermore, we investigate unitary equivalence relations, with particular emphasis on cases where the unitary operator is either isometric or co-isometric.

Keywords: Adjoint, Normal operators, (n, m)-normal, Square normal operators, Square (n, m) normal operators, Equivalence relation.

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I. Introduction

Throughout this paper $B(H)$ denotes the algebra of all bounded linear operators on Hilbert space H. A linear operator T on a Hilbert space H is said to be bounded if there exist a constant $c > 0$ such that $\|Tx\| \leq c\,\|x\|\ \forall x \in H$. An operator T is called self-adjoint if $T = T^*$, invertible with inverse S if there exists $S \in B(H)$ such that $ST = I = TS$, where $I \in B(H)$ is the identity operator. An operator $T \in B(H)$ is called isometry if $\|Tx\| = \|x\|\ \forall x \in H$ or equivalently $T^*T = I$. An operator $T \in B(H)$ is called unitary if $TT^* = T^*T = I$. An operator $T \in B(H)$ is said to be normal if it commutes with its adjoint i.e ($T^*T = TT^*$), equivalently $T^*T - TT^* = 0$. An operator $T \in B(H)$ is said to be n-power normal if $T^nT^* = T^*T^n$ for $n \in \mathbb{N}$. An operator $T \in B(H)$ is square normal if $T^2(T^*)^2 = (T^*)^2T^2$. An operator $T \in B(H)$ is square(n, m) normal if $T^{2n}(T^*)^{2m} = (T^*)^{2m}T^{2n}$. Let A and B be bounded operators in a Hilbert space H, then A and B are said to be unitarily quasi-equivalent,if given unitary operator$U$, We have $A^*A = UB^*BU$ and $AA^* = UBB^*U$

2 Methodology

Alzuraiq (2010) introduced a new class of operators called the n-normal operators and gave some basic properties of this class.
Abood and Al-Loz (2015) generalized the class of normal operators and suggested the class of (n, m) powers operators where every n-normal operator is (n, m) powers (where $m = 1$)
Mahmood (2016) defined square normal operators and proved that every normal operator is a square normal operator. However, the converse of this statement is not true. The author gave an example to show that square normal operators are not necessarily normal and the conditions to get a normal operator from a square normal operator. Edith et al.(2024) explored the characterization of square normal operators and some results were examined.

Theorem 2.1: Edith et al. (2024)
If an operator $T \in B(H)$ is square normal, then

(i) T^* is a square normal operator.

(ii) If T is invertible, that is T^{-1} exist, then T^{-1} is a square normal operator.

(iii) Any operator $S \in B(H)$ that is unitarily equivalent to T is a square normal operator.

According to Wabuya et al (2023), every (n, m) normal operator is square (n, m) normal operator as seen in in theorem 2.2.

Theorem 2.2: Wabuya et al (2023)

Let $T \in B(H)$ be a (n, m) normal operator. Then T is a square (n, m) normal operator.

Proof
Since T is an (n, m) normal operator,
Then

$$T^nT^{*m} = T^{*m}T^n$$

We show that T is square (n, m) normal operator

$$\begin{aligned} T^{2n}T^{*2m} &= (T^2)^n(T^2)^{*m} \\ &= (TT)^n(TT)^{*m} \\ &= T^nT^nT^{*m}T^{*m} \\ &= T^nT^{*m}T^nT^{*m} \\ &= T^nT^{*m}T^nT^{*m} \\ &= T^{*m}T^nT^{*m}T^n \\ &= T^{*m}T^{*m}T^nT^n \\ &= T^{*2m}T^{2n} \end{aligned}$$

Hence T is square (n, m) normal.However, the converse of this theorem is not true for if we consider a matrix

$T = \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix}$, T is not $(1, 1)$ normal but it is square $(1, 1)$ normal.

The concept of unitarily equivalence was proved as seen in the Lemma 2.3 below.

Lemma 2.3: Wabuya et al (2023)

Let $T \in B(H)$ be a square (n, m) normal operator. If $U \in B(H)$ is a unitary operator such that $S \in B(H)$ is unitarily equivalent to T, then S is a square (n, m) normal operator.

Proof
Let $T \in B(H)$ be a (n, m) normal operator.
Since S is unitarily equivalent to T, we have
$S = UTU^*$
This implies $S^{*2n} = UT^{2n}U^*$ and $S^{*2m} = UT^{*2m}U^*$

We show that S is a square (n, m) normal operator.

$$\begin{aligned} S^{2n}S^{*2m} &= UT^{2n}U^*UT^{*2m}U^* \\ &= UT^{2n}T^{*2m}U^* \\ &= UT^{*2m}T^{2n}U^* \\ &= UT^{*2m}U^*UT^{2n}U^* \\ &= S^{*2m}S^{2n} \end{aligned}$$

Hence, S is a square (n, m) normal operator.

3 Results and Discussion

In this section, we discuss some of the properties of square (n, m) operators and their equivalence relations

3.1 Properties of square (n, m) operators

Theorem 3.1.1
Let $T \in B(H)$ be a square(n, m) normal operator. Then:

1. αT is square (n, m) normal for $\alpha \in \mathbb{C}$
2. T^{-1} is square (n, m) normal if it exists
3. T^* is square (n, m) normal

Proof 1

$$\begin{aligned} (\alpha T)^{2n}(\alpha T)^{*2m} &= \alpha^{2n}T^{2n}\alpha^{*2m}T^{*2m} \\ &= \alpha^n\alpha^nT^nT^n\alpha^{*n}\alpha^{*n}T^{*n}T^{*n} \\ &= \alpha^{2n}\alpha^{*2n}T^{2n}T^{*2m} \\ &= \alpha^{*2m}T^{*2m}\alpha^{2n}T^{2n} \\ &= (\alpha T)^{*2m}(\alpha T)^{2n} \end{aligned}$$

αT is square (n, m) normal.

Proof 2
Since T is a square (n, m) normal operator,we have,

$$\begin{aligned} (T^{-1})^{2n}(T^{-1})^{*2m} &= (T^{2n})^{-1}(T^{*2m})^{-1} \\ &= (T^{2n}T^{*2m})^{-1} \\ &= (T^{*2m}T^{2n})^{-1} \\ &= (T^{-1})^{2m}(T^{-1})^{2n} \end{aligned}$$

Hence T^{-1} is square (n, m) normal

Proof 3

Since T is square (n, m) normal operator, we have

$$\begin{aligned} (T^*)^{2n}(T^*)^{*2m} &= (T^{2n})^*(T^{*2m})^* \\ &= (T^{2n}T^{*2m})^* \\ &= (T^{*2m}T^{2n})^* \\ &= (T^{*2m})^*(T^{2n})^* \\ &= (T^*)^{*2m}(T^*)^{2n} \end{aligned}$$

Hence T^* is a square (n, m) normal.

Theorem 3.1.2
Suppose T is a self adjoint operator. Then T is square (n, m) normal operator.

Proof. Let T be such that $T^* = T$, then,

$$\begin{aligned} T^{2n}T^{*2m} &= T^nT^nT^{*m}T^{*m} \\ &= T^nT^nT^m)T^m \\ &= T^{2(n+m)} \\ &= T^{2(m+n)} \\ &= T^{*2(m+n)} \\ &= T^{*2m}T^{*2n} \\ &= T^{*2m}T^{2n} \end{aligned}$$

Hence T^* is a square (n, m) normal.

Theorem 3.1.3

Let $T \in B(H)$ be any operator. Then $T + T^*$ and TT^* are square (n, m) normal operators.

1. Let $S = T + T^*$, then,
   $S^* = (T + T^*)^* = T^* + T = S$.
   Hence S is self adjoint and by Theorem 3.2,every self adjoint operator is square (n, m) normal and so is $T + T^*$.

2. Let $S = TT^*$,then
   $S^* = (TT^*)^* = TT^* = S$
   S is self adjoint and every self adjoint operator is square(n, m) normal operator.
   Thus TT^* is square (n, m) normal

Theorem 3.1.4

If $S, T \in B(H)$ are square (n, m) normal operators and S, T are doubly commuting operator $ST = TS$, $ST^* = T^*S$ then ST is square (n, m) normal operator.

Proof.

$$\begin{aligned}(ST)^{*2m}(ST)^{2n} &= T^{*2m}S^{*2n}S^{2n}T^{2n}\\ &= S^{*2m}T^{*2m}S^{2n}T^{2n}\\ &= S^{*2m}S^{2n}T^{*2m}T^{2n}\\ &= S^{2n}S^{*2m}T^{2n}T^{*2m}\\ &= S^{2n}T^{2n}S^{*2m}T^{*2m}\\ &= S^{2n}T^{2n}T^{*2m}S^{*2m}\\ &= (ST)^{2n}(ST)^{*2m}\end{aligned}$$

Hence ST is a square (n, m) normal operator.

3.3 Equivalence Relations

We now discuss equivalence relation of square (n, m) normal operators with respect to invertible and unitary operators.

Theorem 3.3.1

Let $T \in B(H)$ be a square (n, m) normal operator and $S \in B(H)$ be such that

1. $S = UTU^*$ with U being an isometry, then S is a square (n, m) normal operator.
2. $S = UTU^*$ with U being a co-isometry then S is also a square normal(n, m) operator.

Proof 1

Let $S \in B(H)$ be such that $S = UTU^*$ for U an isometry. Then,
$S^* = UT^*U^*$
$S^n = (UTU^*)^n$=UTU^*UTU^*... n times
$S^{2n} = UT^{2n}U^* = UT^nU^*UT^nU^*$
$S^{*m} = (UT^*U^*)^m$=$UT^*U^*UT^*U^*$... m times
$S^{*2m} = UT^{*2m}U^* = UT^{*m}U^*UT^{*m}U^*$

$$\begin{aligned}S^{2n}S^{*2m} &= UT^nU^*UT^nU^*UT^{*m}U^*UT^{*m}U^*\\ &= UT^nT^nT^{*m}T^{*m}U^*\\ &= UT^{2n}T^{*2m}U^*\\ &= UT^{*2m}T^{2n}U^*\\ &= UT^{*m}T^{*m}T^nT^nU^*\\ &= UT^{*m}U^*UT^{*m}U^*UT^nU^*UT^n\\ &= S^{*m}S^{*m}S^nS^n\\ &= S^{*2m}S^{2n}\end{aligned}$$

Hence S is a square(n, m) normal operator.

Proof 2

Let $S \in B(H)$ be such that $S = U^*TU$ for U a co-isometry operator. Then,
$S^* = (U^*TU)^* = U^*T^*U$
and
$S^n = (U^*TU)^n = U^*T^nU$
$S^{2n} = (U^*T^nU)^2 = U^*T^{2n}U = U^*T^nUU^*T^nU$
$S^{*2m} = U^*T^{*m}UU^*T^{*m}U = U^*T^{*2m}U$

$$
\begin{aligned}
S^{2n}S^{*2m} &= U^*T^nUU^*T^nUU^*T^{*m}UU^*T^{*m}U \\
&= U^*T^nT^nT^{*m}T^{*m}U \\
&= UT^{2n}T^{*2m}U^* \\
&= UT^{*2m}T^{2n}U^* \\
&= UT^{*m}U^*UT^{*m}U^*UT^nU^*UT^nU* \\
&= S^{*m}S^{*m}S^nS^n \\
&= S^{*2m}S^{2n}
\end{aligned}
$$

Hence S is a square(n, m) normal operator.

4 Conclusion

This paper has focused on investigating the square (n, m) normal operators in the Hilbert space. Some of its characteristics and equivalence relations are studied. We have shown that every self-adjoint operator is square (n, m)normal and if T is square (n, m) normal, then so is its inverse if it exists.The class of square (n, m) normal operators are closed under scalar multiplication. The sum $T + T^*$ and the product TT^* are square normal operators. We have also looked at unitary equivalence, where our unitary operator is either an isometry or a co-isometry operator.

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