IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 22, Issue 4, Ser. 1 (July. – August. 2026), PP 20-29
www.iosrjournals.org

Solution Of Wave Equations By The Weighted SNA Transform

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Abstract:
In this paper, the first-order wave equation and the second-order wave equation is solved by weighted SNA transformation[5], where the SNA transform is defined by

$$\mathcal{A}_\sigma[f](s) = \int_0^\infty e^{-st}\,(1+\sigma t) f(t)\,dt, \sigma \geq 0 \tag{1.1}$$

Also, we introduced a countably normed testing-function space so that the kernel $e^{-st}(1+\sigma t)$ is an admissible test function and functions of exponential order define regular generalized functions. The first and second derivative operational rules are established and combined with the Fourier transform in the spatial variable.[4] For the first-order model, the method produces the translated profile $u(x,t) = \phi(x-ct)$. For the second-order model, it yields the Fourier representation and the D' Alembert formula. Gaussian travelling and splitting pulses are studied in detail.

Key Word: *weighted SNA transform; wave equation; transport equation; Fourier transform; testing-function space; generalized function; D'Alembert solution.*

Date of Submission: 07-07-2026 Date of Acceptance: 17-07-2026

I. Introduction

Wave propagation in general appears in vibrating strings, acoustics, elastic media, transmission lines, traffic flow, and signal transport. Here, we take two equations repeatedly. The first is the first-order wave equation[3][8][9][10]

$$u_t + cu_x = 0, \qquad c > 0, \tag{1.2}$$

which corresponds to a wave travelling in one direction. The second is the classical two-way wave equation

$$u_{tt} - c^2 u_{xx} = 0, \qquad c > 0. \tag{1.3}$$

Equation (1.2) is a first-order partial differential equation involving only the first derivative with respect to time. Consequently, its solution propagates along a single family of characteristic curves, representing wave motion in one direction only. In contrast, Equation (1.3) is a second-order partial differential equation containing the second derivative with respect to time. Its solutions consist of both right-travelling and left-travelling wave components, allowing disturbances to propagate in opposite directions simultaneously. Thus, although both equations describe wave propagation[6] with speed (c), they differ significantly in their mathematical formulation and in the physical behavior of their solutions.[2][3][8][9]

The classical Laplace transform is one of the standard tools for solving initial-value problems. In the present work, we introduce a modified transform by replacing the conventional kernel e^{-st} with the weighted kernel[5]

$$K_\sigma(s,t) = e^{-st}\,(1+\sigma t) \tag{1.4}$$

The proposed transform is more than a simple modification of the classical Laplace transform. The additional weighting factor ((1+\sigma t)) incorporates the first-order moment of the function, thereby providing extra analytical information that is not available in the standard transform. A fundamental relationship between the proposed transform and the classical Laplace transform is given by and this identity serves as one of the principal tools for the analysis of the result in the paper.

$$\mathcal{A}_\sigma[f](s) = \mathcal{A}_0[f[(s) - \sigma \frac{d}{ds}\,\mathcal{A}_0[f](s), \tag{1.5}$$

For the transform technique to be mathematically rigorous, it is necessary to work within a suitable functional framework in which differentiation, integration, and limiting operations are well defined. Such a framework is particularly appropriate for the analysis of wave equations.

II. Testing-Function Space[1] For The $\mathcal{A}_\sigma -$ Transform

Let a be a positive real number, and let $I = [0, \infty)$ denote the half-line. $\mathbb{N}_0 = \{0,1,2, \dots\}$, let us define

$$\mathcal{Y}_\sigma(I) = \left\{\varphi \in C^\infty(I) : p_{m,n}^{(a)}(\varphi) < \infty \text{ for every } m, n \in \mathbb{N}_0\right\} \tag{2.1}$$

where

$$p_{m,n}^{(a)}(\varphi) = \sup_{t \geq 0} e^{at}(1+t)^m \, |D_t^n \varphi(t)| \tag{2.2}$$

and the notation $D_t^n \varphi$ means the nth derivative of φ with respect to t:

$$D_t^n \varphi(t) = \frac{d^n \varphi(t)}{dt^n}, \qquad D_t^0 \varphi(t) = \varphi(t). \tag{2.3}$$

The family in (2.2) is countable and separates points. Consequently, it determines a locally convex topology. A sequence $\{\varphi_j\}$ tends to zero in $\mathcal{Y}_a$ precisely when every seminorm in (2.2) tends to zero.

Theorem 2.1 — Testing-space property

The space $\mathcal{Y}_a(I)$ satisfies the following conditions:
1. every element is a smooth complex-valued function on I;
2. $\mathcal{Y}_a(I)$ is a complete countably normed space;
3. if $\varphi_j \to 0$ in $\mathcal{Y}_a(I)$, then $D_t^k \varphi_j \to 0$ uniformly on every compact subset of I for every $k \in \mathbb{N}_0$.

Proof

The first statement follows directly from (2.1). For completeness, let $\{\varphi_j\}$ be Cauchy in all seminorms. Fix n and a compact interval $[0, R]$. Since

$$\sup_{0 \leq t \leq R} [D_t^n(\varphi_j - \varphi_l)(t) \leq p_{0,n}^{(a)}(\varphi_j - \varphi_l). \tag{2.4}$$

$\{D_t^n \varphi_j\}$ is uniformly Cauchy on $[0, R]$. It therefore converges uniformly to a continuous limit g_n. The standard compatibility theorem for uniformly convergent derivatives gives $g_n = D_t^n g_0$. Thus g_0 is smooth. Passing to the limit in each weighted estimate shows that $g_0 \in \mathcal{Y}_a$ and that $\varphi_j \to g_0$ in every seminorm. Hence the space is complete.

For the third statement, let $K \subset I$ be compact and choose R with $K \subset [0, R]$. Then

$$\sup_{t \in K} \left|D_t^n \varphi_j(t)\right| \leq p_{0,n}^{(a)}(\varphi_j) \longrightarrow 0. \tag{2.5}$$

This proves uniform convergence on compact subsets.

Proposition 2.2 — Admissibility of the kernel

If $\mathrm{Re}(s) > a$, then $K_\sigma(s, \mathrm{t}) \in \mathcal{Y}_a(I)$. Indeed,

$$D_t^n K_\sigma(s,t) = e^{-st}[(-s)^n(1+\sigma t) + n\sigma(-s)^{n-1}], \tag{2.6}$$

with the second term understood as zero for $n = 0$. After multiplication by the weight in (2.2), the result is a polynomial in t times $e^{-(\mathrm{Re}(s)-a)t}$, which is bounded.

The dual $\mathcal{Y}_a'(I)$ is the space of continuous linear functionals on $\mathcal{Y}_a$. If f is locally integrable and $|f(t)| \leq Me^{ct}$ with $c < a$, then

$$\langle T_f, \varphi \rangle = \int_0^\infty f(t)\varphi(t)\, dt \tag{2.6}$$

defines a regular generalized function. Its generalized $\mathcal{A}_\sigma$-transform is

$$\mathcal{A}_\sigma[T_f](s) = \langle T_f, K_\sigma(s,t) \rangle \tag{2,7}$$

For ordinary f, (2.7) reduces to (1.1).

III. Operational Properties Required For Wave Equations

Put

$$F_0(s) = \mathcal{A}_0\,[f](s) = \int_0^\infty e^{-st} f(t)\, dt. \tag{3.1}$$

Differentiating (3.1) with respect to s gives

$$F_0'(s) = -\int_0^\infty t e^{-st} f(t)\, dt. \tag{3.2}$$

Combining (3.1) and (3.2) proves (1.5). This identity also provides an inversion bridge. For $\sigma > 0$, if $F_0(s) \to 0$ as $s \to \infty$, then the equation $F_0 - \sigma F_0{}' = \mathcal{A}_\sigma[f]$ yields

$$F_0(s) = \frac{1}{\sigma}\, e^{\frac{s}{\sigma}} \int_s^{\infty} e^{-r/\sigma} \mathcal{A}_\sigma[f](r)\, dr. \tag{3.3}$$

Thus, the weighted SNA transform determines the Laplace transform and hence the original function under the usual uniqueness assumptions.

Theorem 3.1. First derivative

Let f be continuously differentiable, of exponential order, and suppose the boundary term at infinity vanishes. Then

$$\mathcal{A}_\sigma[f'](s) = s\, \mathcal{A}_\sigma[f](s) - f(0). \tag{3.4}$$

Proof

Integration by parts gives

$$\begin{aligned}\mathcal{A}_\sigma[f'](s) &= [e^{-st}(1+\sigma t)f(t)]_0^\infty - \int_0^\infty \frac{d}{dt}\{e^{-st}(1+\sigma t)\}f(t)\, dt \\ &= -f(0) + s\mathcal{A}_\sigma[f](s) - \sigma\mathcal{A}_0[f](s),\end{aligned} \tag{3.5}$$

which is (3.4).

Theorem 3.2 Second derivative

Under the corresponding second-order regularity assumptions,

$$\mathcal{A}_\sigma[f''](s) = s^2\mathcal{A}_\sigma[f](s) - 2\sigma s\mathcal{A}_0[f](s) + (\sigma - s)f(0) - f'(0) \tag{3.6}$$

Proof

Apply (3.4) to f' and use the Laplace identity $\mathcal{A}_0[f'] = s\mathcal{A}_0[f] - f(0)$. The result is

$$\mathcal{A}_\sigma[f''] = s\{\, s\mathcal{A}_\sigma[f] - \sigma\mathcal{A}_0[f] - f(0)\} - \sigma\{s\mathcal{A}_\sigma[f] - f(0) - f'(0) \tag{3.7}$$

which simplifies to (3.6).

For functions of (x,t), the transform is applied in t only. When differentiation under the integral is justified

$$\mathcal{A}_\sigma[u_x](x,s) = \frac{\partial}{\partial x}\mathcal{A}_\sigma[u](x,s), \qquad \mathcal{A}_\sigma[u_{xx}](x,s) = \frac{\partial^2}{\partial x^2}\mathcal{A}_\sigma[u](x,s) \tag{3.8}$$

Equations (3.4), (3.6), and (3.8) are the operational core of the wave-equation derivations.

Proposition 3.3. Elementary exponential pair

For $\mathrm{Re}(s+\alpha) > 0$,

$$\mathcal{A}_\sigma[e^{-\alpha t}](s) = \frac{1}{s+\alpha} + \frac{\sigma}{(s+\alpha)^2}. \tag{3.9}$$

Proof

By direct integration,

$$\begin{aligned}\mathcal{A}_\sigma[e^{-\alpha t}](s) &= \int_0^\infty e^{-(s+\alpha)t}\,(1+\sigma t)\, dt \\ &= \int_0^\infty e^{-(s+\alpha)t}\, dt + \sigma\int_0^\infty t\, e^{-(s+\alpha)t}\, dt \\ &= \frac{1}{s+\alpha} + \frac{\sigma}{(s+\alpha)^2}.\end{aligned}$$

Theorem 3.4. Injectivity

Let f be a function of exponential order for which the ordinary Laplace uniqueness theorem applies. If$\mathcal{A}_\sigma[f](s) = 0$ for all sufficiently large s, then $f(t) = 0$ almost everywhere.

Proof

For $\sigma = 0$, the assertion is the ordinary Laplace uniqueness theorem. Let $\sigma > 0$ and set $F_0 = \mathcal{A}_0[f]$. By (3.3), condition (3.13) gives

$$F_0(s) - \sigma F_0'(s) = 0. \tag{3.14}$$

The solution of (3.14) is

$$F_0(s) = Ce^{s/\sigma}. \tag{3.15}$$

However, a Laplace transform of an exponential-order function tends to zero as $s \to \infty$. Hence $C = 0$, so $F_0 = 0$. Ordinary Laplace uniqueness now gives $f = 0$.

Theorem 3.3 permits the identification of a time function from a known $\mathcal{A}_\sigma$-transform pair.

IV. First-Order Wave Equation

The first-order wave equation is commonly written as

$$\begin{cases} u_t(x,t) + c\,u_x(x,t) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $c > 0$ is the propagation speed and $\phi(x) \in \mathcal{S}(\mathbb{R})$, the Schwartz space.

The purpose of this work is to solve the problem defined by (1.1) and (1.2) through the weighted SNA transform (1.1).

Define the Fourier transform in the spatial variable and inverse Fourier transform by

$$\hat{u}(\xi,t) = \int_{-\infty}^{\infty} e^{-i\xi x}\, u(x,t)\, dx, \qquad u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\, \hat{u}(\xi,t)\, d\xi. \tag{4.2}$$

Applying the derivative rule (3.4) to the first term of the first equation of (4.1) gives

$$\mathcal{A}_\sigma\left[\frac{\partial \hat{u}}{\partial t}\right](s) = sU_\sigma(\xi,s) - \sigma U_0(\xi,s) - \hat{\phi}(\xi). \tag{4.3}$$

The second term of the first equation of (4.1) transforms by linearity as

$$\mathcal{A}_\sigma[ic\xi\hat{u}] = ic\xi U_\sigma(\xi,s). \tag{4.4}$$

Substituting (4.3) and (4.4) into the transform of (4.1), we get

$$(s + ic\xi)U_\sigma(\xi,s) = \hat{\phi}(\xi) + \sigma U_0(\xi,s). \tag{4.5}$$

Equation (4.5) contains U_σ and U_0. To determine U_0, put $\sigma = 0$ in (4.5). This gives

$$(s + ic\xi)U_0(\xi,s) = \hat{\phi}(\xi).$$

Therefore,

$$U_0(\xi,s) = \frac{\hat{\phi}(\xi)}{s + ic\xi}. \tag{4.7}$$

Substituting (4.7) into (4.6),

$$(s + ic\xi)U_\sigma(\xi,s) = \hat{\phi}(\xi) + \frac{\sigma\hat{\phi}(\xi)}{s + ic\xi}.$$

Dividing by $s + ic\xi$ gives

$$U_\sigma(\xi,s) = \frac{\hat{\phi}(\xi)}{s + ic\xi} + \frac{\sigma\hat{\phi}(\xi)}{(s + ic\xi)^2}.$$

Hence,

$$U_\sigma(\xi,s) = \hat{\phi}(\xi)\left\{\frac{1}{s + ic\xi} + \frac{\sigma}{(s + ic\xi)^2}\right\}. \tag{4.8}$$

Therefore, by the definition of $\mathcal{A}_\sigma$,

$$\mathcal{A}_\sigma\left[e^{-ic\xi t}\right](s) = \int_0^\infty e^{-(s+ic\xi)t}\,(1 + \sigma t)\, dt.$$

Splitting the integral,

$$\mathcal{A}_\sigma\left[e^{-ic\xi t}\right](s) = \int_0^\infty e^{-(s+ic\xi)t}\ dt + \sigma\int_0^\infty t\, e^{-(s+ic\xi)t}\, dt.$$

For $\mathrm{Re}(s) > 0$,

$$\int_0^\infty e^{-(s+ic\xi)t}\ dt = \frac{1}{s + ic\xi},$$

and

$$\int_0^\infty t\, e^{-(s+ic\xi)t}\, dt = \frac{1}{(s + ic\xi)^2}.$$

Thus,

$$\mathcal{A}_\sigma\left[e^{-ic\xi t}\right](s) = \frac{1}{s + ic\xi} + \frac{\sigma}{(s + ic\xi)^2}. \tag{4.9}$$

Comparing (4.8) and (4.9),

$$U_\sigma(\xi,s) = \mathcal{A}_\sigma\left[\hat{\phi}(\xi)e^{-ic\xi t}\right](s).$$

By injectivity of the transform on the admissible function space,

$$\hat{u}(\xi,t) = e^{-ic\xi t}\hat{\phi}(\xi). \tag{4.10}$$

Taking the inverse Fourier transform in (4.10),

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\, e^{-ic\xi t}\hat{\phi}(\xi)\, d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi(x-ct)}\, \hat{\phi}(\xi)\, d\xi. \end{aligned} \tag{4.11}$$

The last integral in (4.11) is the inverse Fourier representation of ϕ evaluated at $x - ct$. Hence

$$u(x,t) = \phi(x - ct). \tag{4.12}$$

(4.12) is complete derivation of (4.1)
Therefore, the initial profile moves to the right with speed c and does not change its form.
Differentiating (4.12) gives

$$u_t(x,t) = -c\phi'(x - ct),$$

and

$$u_x(x,t) = \phi'(x - ct).$$

Therefore,

$$u_t(x,t) + cu_x(x,t) = -c\phi'(x - ct) + c\phi'(x - ct) = 0.$$

Also,

$$u(x,0) = \phi(x).$$

Thus (4.11) satisfies both the differential equation and the initial condition.

4.1. Travelling Gaussian Pulse

In this section we establish a Gaussian pulse governed by the one-dimensional first-order transport equation.[7][8] The spatial Fourier transform reduces the partial differential equation to a first-order ordinary differential equation[4] in time, while the weighted transform (1.1) provides the corresponding representation in the transform variable. The calculation shows that the weighted kernel introduces a repeated pole in addition to the usual simple pole, but both terms invert to the same exponential time factor. Consequently, the physical solution is independent of the auxiliary parameter σ and is a rigid translation of the initial Gaussian profile.

The constant-coefficient transport equation is the simplest hyperbolic model for the undistorted translation of a spatial profile. Its characteristic and transform formulations are standard in the analysis of first-order partial differential equations [2][3][7][11]. Consider

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \qquad x \in R, \quad t > 0. \tag{4.13}$$

The propagation speed in (4.13) is $c = 1$. The initial profile is prescribed by

$$u(x,0) = \phi(x) = e^{-x^2}, \qquad x \in R. \tag{4.14}$$

The function ϕ in (4.14) is smooth, positive, rapidly decreasing, and belongs to the Schwartz space. It is therefore admissible for both the Fourier transform and the testing-function framework used in generalized transform theory [1][4]. The objective is to solve (4.13)–(4.14) by combining the spatial Fourier transform with the time-domain $\mathcal{A}_\sigma$-transform.

We use the Fourier-transform convention

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} \hat{f}(\xi)\, d\xi. \tag{4.15}$$

The convention in (4.15) is commonly used in Fourier analysis and the transform treatment of partial differential equations [4] [9]. For a time-dependent function $g(t)$, let

$$G(s) = \mathcal{L}[g](s) = \int_0^{\infty} e^{-st} g(t)\, dt.$$

Differentiating under the integral sign gives

$$G_s(s) = -\int_0^{\infty} t\, e^{-st} g(t)\, dt.$$

It follows that

$$W_\sigma[g](s) = G(s) - \sigma G_s(s). \tag{4.15a}$$

Thus, the weighted transform consists of the ordinary Laplace transform together with the first time moment of the function. This structural relation is the main operational identity required below; analogous weighted-transform constructions are used to enlarge the operational content of classical Laplace-type transforms [6][10] [11][17].

Taking the Fourier transform of (4.13) with respect to x and using

$$\mathcal{F}_x[u_x](\xi,t) = i\xi\hat{u}(\xi,t),$$

we obtain

$$\frac{\partial \hat{u}}{\partial t}(\xi,t) + i\xi\hat{u}(\xi,t) = 0, \qquad \hat{u}(\xi,0) = \hat{\phi}(\xi). \tag{4.15b}$$

For the Gaussian profile in (4.14), completion of the square in the classical Gaussian integral yields [4]

$$\hat{\phi}(\xi) = \int_{-\infty}^{\infty} e^{-x^2} e^{-i\xi x}\, dx = \sqrt{\pi}\, e^{-\frac{\xi^2}{4}}. \tag{4.15c}$$

Define the ordinary Laplace image and the weighted image by

$$V(\xi,s) = \mathcal{L}_t[\hat{u}(\xi,t)](s), \qquad U_\sigma(\xi,s) = W_{\sigma,t}[\hat{u}(\xi,t)](s).$$

Applying the ordinary Laplace transform to (4.15b) gives

$$(s+i\xi)V(\xi,s) = \hat{\phi}(\xi),$$

and hence

$$V(\xi,s) = \frac{\hat{\phi}(\xi)}{s+i\xi}.$$

Differentiation with respect to s gives

$$V_s(\xi,s) = -\frac{\hat{\phi}(\xi)}{(s+i\xi)^2}.$$

Substitution of these two expressions into (4.15a), followed by the use of (4.15c), leads to

$$U_\sigma(\xi,s) = \sqrt{\pi}\, e^{-\xi^2/4}\left\{\frac{1}{s+i\xi} + \frac{\sigma}{(s+i\xi)^2}\right\}. \tag{4.16}$$

Equation (4.16) is the $\mathcal{A}_\sigma$-domain representation of the Gaussian pulse. To invert it, calculate directly

$$\mathcal{A}_\sigma\left[e^{-i\xi t}\right](s) = \int_0^\infty e^{-(s+i\xi)t}\,(1+\sigma t)\,dt.$$

Separating the two terms in the weight gives

$$\mathcal{A}_\sigma\left[e^{-i\xi t}\right](s) = \int_0^\infty e^{-(s+i\xi)t}\,dt + \sigma\int_0^\infty t\,e^{-(s+i\xi)t}\,dt.$$

Evaluation of these elementary integrals yields

$$\mathcal{A}_\sigma\left[e^{-i\xi t}\right](s) = \frac{1}{s+i\xi} + \frac{\sigma}{(s+i\xi)^2}.$$

provided $\mathrm{Re}(s) > 0$. Comparison with (4.16) therefore gives

$$\hat{u}(\xi,t) = \sqrt{\pi}\, e^{-\xi^2/4} e^{-i\xi t}. \tag{4.17}$$

The multiplier $e^{-i\xi t}$ in (4.17) is the Fourier signature of translation. Applying the inverse transform in (4.15), or equivalently the Fourier shifting theorem [4, 5], yields

$$u(x,t) = \phi(x-t) = e^{-(x-t)^2}. \tag{4.18}$$

Formula (4.18) shows that the Gaussian profile is transported to the right with unit velocity.

4.1.1. Verification of the solution

Differentiating (4.18) gives

$$u_x(x,t) = -2(x-t)e^{-(x-t)^2}$$

and

$$u_t(x,t) = 2(x-t)e^{-(x-t)^2}.$$

Therefore,

$$u_t(x,t) + u_x(x,t) = 0,$$

which verifies (4.13). Furthermore,

$$u(x,0) = e^{-x^2} = \phi(x),$$

so, the initial condition (4.14) is satisfied. Hence, (4.18) is a classical solution of the initial-value problem.[2] The exponent in (4.18) attains its largest value when

$$x - t = 0.$$

Consequently, the crest position and maximum height are

$$x_{\max}(t) = t, \qquad \max_{x\in\mathbb{R}} u(x,t) = u(t,t) = 1. \tag{4.18a}$$

The half-maximum points satisfy

$$e^{-(x-t)^2} = \frac{1}{2},$$

so that

$$(x-t)^2 = \ln 2.$$

Therefore, the full width at half maximum is

$$\mathrm{FWHM} = 2\sqrt{\ln 2}. \tag{4.18b}$$

The quantities in (4.18a)–(4.18b) show that the pulse translates without alteration: its crest moves linearly, while its height and width remain fixed.

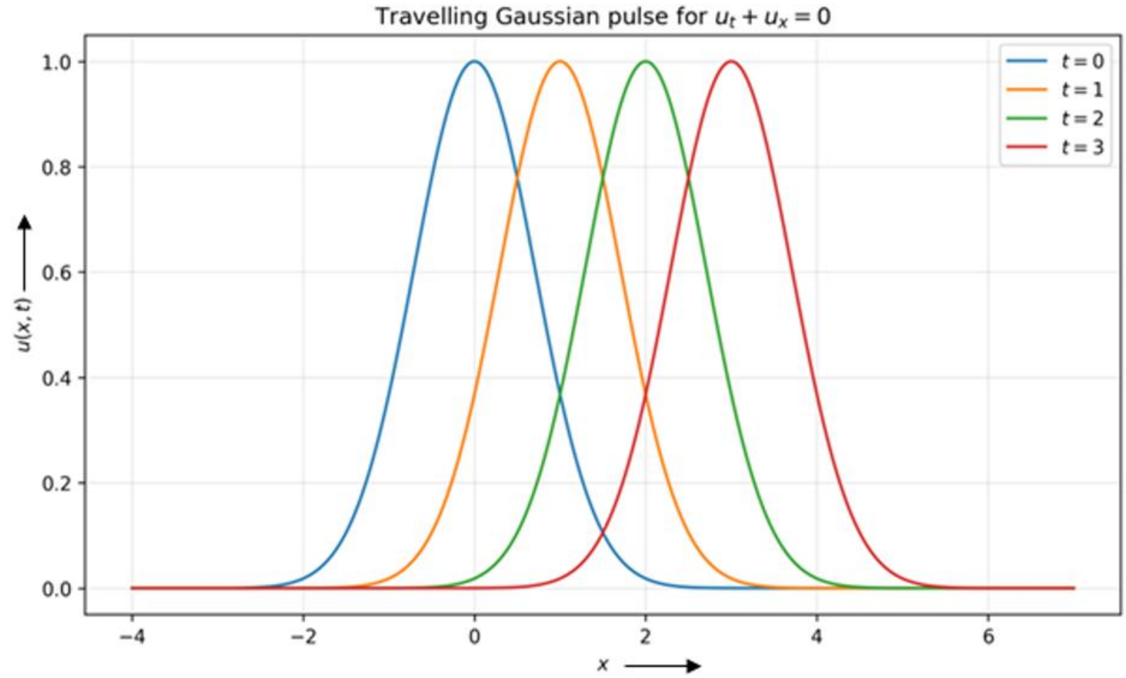

Figure 1. Gaussian travelling pulse at $t = 0,1,2,3$.

Figure 1 shows the same bell-shaped curve at four successive positions. The horizontal separation of the peaks is exactly equal to the elapsed time because $c = 1$. The curves have identical height and width, which illustrates non-dispersive and non-dissipative transport. No secondary oscillation is created because the governing equation contains only first derivatives and a constant velocity.

4.1.2. Pole structure and the role of σ

The transformed expression (4.16) contains the simple and repeated poles

$$\frac{1}{s+i\xi}, \qquad \frac{\sigma}{(s+i\xi)^2}.^{\mathrm{i}}$$

The repeated pole arises solely from the factor $1 + \sigma t$ in the kernel of $\mathcal{A}_\sigma$. Indeed, multiplication by t in the time variable corresponds to differentiation with respect to s, which increases the pole order by one. Nevertheless, both pole terms form the single transform pair

$$\mathcal{A}_\sigma[e^{-i\xi t}](s) = \frac{1}{s+i\xi} + \frac{\sigma}{(s+i\xi)^2}.$$

Their inverse is therefore the one-time factor $e^{-i\xi t}$. The parameter σ changes the transform-domain representation but does not change the physical solution. In particular, it does not affect the velocity, height, width, or spatial norms of the pulse. This independence is an essential consistency requirement for the weighted transform method.

V. Second-Order Wave Equation

Consider

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, x \in \mathbb{R}, t > 0, \\ u(x,0) \quad = \phi(x), \qquad\qquad c > 0 \\ u_t(x,0) \quad = \psi(x), \end{cases} \tag{5.1}$$

Here $u(x,t)$ denotes the displacement, c is the propagation speed, ϕ is the initial displacement, and ψ is the initial velocity. For the classical derivation[2], assume $\phi, \psi \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space. This assumption ensures that the spatial Fourier transforms and all differentiations used below are justified. The formulas later extend to broader classes of data by density and generalized-function arguments.[4]

Taking the Fourier transform in x gives

$$\hat{u}_{tt}(\xi,t) + c^2\xi^2\hat{u}(\xi,t) = 0, \tag{5.2}$$

with

$$\hat{u}(\xi,0) = \hat{\phi}(\xi), \hat{u}_t(\xi,0) = \hat{\psi}(\xi). \tag{5.3}$$

Put $\omega = c|\xi|$ and define U_σ and U_0 as in (4.4). Applying (3.6) to (5.2) yields

$$(s^2 + \omega^2)U_\sigma = 2\sigma s U_0 + (s-\sigma)\hat{\phi} + \hat{\psi} \tag{5.4}$$

For $\sigma = 0$, equation (5.4) reduces to

$$U_0(\xi, s) = \frac{s\hat{\phi}(\xi) + \hat{\psi}(\xi)}{s^2 + \omega^2}. \tag{5.5}$$

Substituting (5.5) into (5.4) and simplifying gives

$$U_\sigma(\xi, s) = \hat{\phi}(\xi)\left\{\frac{s}{s^2 + \omega^2} + \sigma\,\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}\right\} + \hat{\psi}(\xi)\left\{\frac{1}{s^2 + \omega^2} + \frac{2\sigma s}{(s^2 + \omega^2)^2}\right\} \tag{5.6}$$

The required elementary transform pairs are

$$\mathcal{A}_\sigma[\cos\omega t](s) = \frac{s}{s^2 + \omega^2} + \sigma\,\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}. \tag{5.7}$$

and

$$\mathcal{A}_\sigma\left[\frac{\sin\omega t}{\omega}\right](s) = \frac{1}{s^2 + \omega^2} + \frac{2\sigma s}{(s^2 + \omega^2)^2}. \tag{5.8}$$

Consequently,

$$\hat{U}(\xi, s) = \hat{\phi}(\xi)\cos(c|\xi|t) + \hat{\psi}(\xi)\frac{(\sin(c|\xi|t))}{c|\xi|} \tag{5.9}$$

where the second quotient is interpreted by continuity at $\xi = 0$.

5.1. Inversion and the D'Alembert Formula[1][12]

Theorem 5.1

For sufficiently regular and decaying initial data, the solution of (5.1) is

$$u(x, t) = \frac{1}{2}\{\phi(x - ct) + \phi(x + ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy. \tag{5.10}$$

Proof

The Fourier multiplier $\cos(c|\xi|t)$ is even in ξ. Since $\cos(c|\xi|t) = \cos(c\xi t)$, the first term of (5.9) may be written as

$$\hat{\phi}(\xi)\cos(c\xi t) = \frac{1}{2}\hat{\psi}(\xi)\left\{e^{ic\xi t} + e^{-ic\xi t}\right\}. \tag{5.11}$$

The inverse Fourier translation rule transforms (5.11) into

$$\frac{1}{2}\{\phi(x + ct) + \phi(x - ct)\}. \tag{5.12}$$

For the second term, define

$$q_t(x) = \begin{cases} \dfrac{1}{2c}, & |\, x \,| \le ct, \\ 0, & |\, x \,| > ct. \end{cases} \tag{5.13}$$

Its Fourier transform is

$$\hat{q_t}(\xi) = \frac{\sin c\xi t}{c\xi}, \tag{5.14}$$

with value t at $\xi = 0$. Hence the inverse transform of the second term in (5.9) is the convolution

$$(q_t * \psi)(x) = \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy. \tag{5.15}$$

Combining (5.12) and (5.15) gives (5.10).

Formula (5.10) displays finite propagation speed. The value $u(x, t)$ depends only on the initial data in the interval $[x - ct, x + ct]$. No disturbance can travel faster than c.

If $\psi = 0$, the initial displacement divides into two equal waves:

$$u(x, t) = \frac{1}{2}\phi(x - ct) + \frac{1}{2}\phi(x + ct) \tag{5.16}$$

If $\phi = 0$, the initial velocity produces an averaged signal over the backward characteristic interval:

$$u(x, t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy. \tag{5.17}$$

The transform calculation therefore recovers not only an algebraic solution but also the complete causal geometry of the second-order equation.

5.2. Splitting of a Gaussian Displacement

This describes the motion of an unbounded string when its initial displacement is Gaussian and it is released from rest.

Let $c = 1$ and choose

$$\phi(x) = e^{-x^2}, \qquad \psi(x) = 0. \tag{5.18}$$

Equation (5.6) becomes

$$U_\sigma(\xi, s) = \sqrt{\pi}\, e^{\xi^2/4}\left\{\frac{s}{s^2+\xi^2} + \sigma\,\frac{s^2-\xi^2}{(s^2+\xi^2)^2}\right\}. \tag{5.19}$$

Using (5.7) and then (5.10),

$$u(x,t) = \frac{1}{2}e^{-(x-t)^2} + \frac{1}{2}e^{-(x+t)^2} \tag{5.20}$$

At $t = 0$, the two half-amplitude components overlap and give the original unit-amplitude Gaussian. As t increases, their centres move along $x = t$ and $x = -t$. When their separation becomes large, each visible pulse has height approximately $1/2$.

The initial-velocity condition can be checked directly. Differentiating (5.20),

$$u_t(x,t) = (x-t)e^{-(x-t)^2} - (x+t)e^{-(x+t)^2}. \tag{5.21}$$

and therefore $u_t(x,0) = 0$. A second differentiation confirms $u_{tt} = u_{xx}$.

For sufficiently decaying solutions, the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty}\{u_t(x,t)^2 + u_x(x,t)^2\}\,dx \tag{5.22}$$

is constant. In the present example,

$$E(0) = \frac{1}{2}\int_{-\infty}^{\infty}\{\phi'(x)\}^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}. \tag{5.23}$$

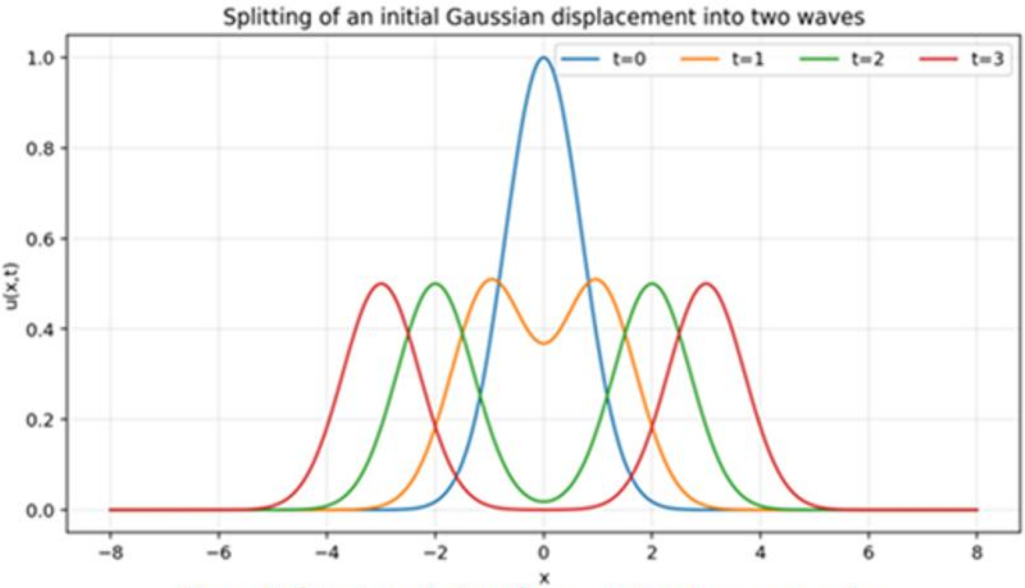

Figure 2. Gaussian solution of the second-order wave equation.

At $t = 0$ the two components coincide. By $t = 1$ they begin to separate, and by $t = 2$ and $t = 3$ two distinct pulses are visible. The graph demonstrates the central distinction between (1.2) and (1.3): the first-order equation transports one profile in one direction[7], whereas the second-order equation creates two counter-propagating components from a pure initial displacement.

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[i] For a measurable function $f(x)$, the essential supremum is defined by

$$\operatorname*{ess\,sup}_{x\in\Omega} | f(x) | = \inf\{M > 0 : | f(x) | \leq M \text{ for almost every } x \in \Omega\}.$$

Here, **almost every** means that exceptions are allowed on a set of **measure zero**.