IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 22, Issue 4, Ser. 1 (July. – August. 2026), PP 37-44
www.iosrjournals.org

Properties Of Quasi-Similarity On Isometries, Co-Isometries And Partial Isometries In Hilbert Spaces

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Abstract
Classes of operators in Hilbert spaces have recently been studied in relation to quasi-similarity. Quasi-similarity has been shown to be an equivalence relation. Further, it has been established that normality, hyponormality, -hyponormality, -hyponormality, -hyponormality and -quasihyponormality are all preserved by quasi-similarity. It has also been demonstrated how quasi-similarity relates to other equivalence relations such as similarity and almost similarity. Specifically, similarity implies quasi-similarity which in turn implies almost similarity under certain conditions. That is, if are normal operators where is an infinite dimensional Hilbert space such that and are quasi-similar, then and are almost similar. However, there exist other operators whose quasi-similarity properties have not been established. In this study, we focus on isometries, co-isometries and partial isometries operators and determine their quasi-similarity properties.

Date of Submission: 20-07-2026 Date of Acceptance: 30-07-2026

I. Introduction

Let denote a complex Hilbert space and denote a banach algebra of bounded linear operators on a Hilbert space . An operator is a structure preserving map or a symbol that represents a transformation that can be performed on a function. If , then denotes the adjoint of an operator and denotes the inverse of . If then . Sitati *et al*. (2012) demonstrated that quasi-similarity is an equivalence relation on the classes of all operators. Sitati *et al*. (2015) gave the condition under which similarity implies quasi-similarity which in turn implies almost similarity. Kamau *et al*. (2024) noted that quasi-similarity preserves normality, hyponormality and -hyponormality of an operator. Properties of partial isometries have also been determined on other classes of equivalence relations such as almost similarity, unitary equivalence and unitary quasi-equivalence. Luketero & Khalagai (2020) proved that unitary equivalence preserves partial isometries of operators. Jibril (1996) established that almost similar operators preserve isometries and partial isometries. Lilian *et al*. (2023) established that unitary quasi-equivalence operators preserves isometry, co-isometry and partial isometry. Lilian *et al*. (2023) also established that if are self-adjoint and unitary quasi-equivalent operators and is partial isometric, then is also partial isometric.

We also recall the following key definitions that are useful in the subsequent sections.
Definition 1.1: A Hilbert space (Halmos, 2017). A Hilbert space is a complete inner product space.
Definition 1.2: Quasi-affinity operator (Nzimbi *et al*., 2016). An operator is said to be quasi-affinity or quasi-invertible if it is injective with dense range.
Definition 1.3: Quasi-similar operators (Nagy *et al*., 2010). Two operators and of bounded linear operators and on Hilbert spaces and respectively are said to be quasi-similar if there exist quasi-affinity operators and satisfying the equations and .
Definition 1.4: (Nzimbi & Wanyonyi, 2020). An operator is said to be
(i) Binormal if
(ii) Isometry if .
(iii) Co-Isometry if .
(iv) Unitary if .
Definition 1.5: A partial isometry (Halmos, 2017). An operator is said to be partial isometry if . That is, and are projections.
Definition 1.6: An orthogonal projection (Halmos, 2017). An operator is said to be an orthogonal projection if and . That is, is idempotent and self adjoint.

II. Methodology

In order to achieve the desired results, the following outcomes were useful:
Theorem 2.1: (Helmberg, 2008). Let and be bounded linear operators on a Hilbert space . Then
(i) .
(ii) .
(iii)
(iv)
(v) If is invertible, then so is and .

Theorem 2.2: (Salhi & Zerouali, 2020). For , the following statements are equivalent;
(i) is a partial isometry,
(ii) is a partial isometry,
(iii) is a projection,
(iv) is a projection,
(v) ,
(vi) .

Theorem 2.3: (Luketero &Khalagai, 2020). Let be a partial isometry and be any other operator such that either,
i) where is an isometry or
ii) where is a co-isometry
Then is a partial isometry.
Lemma 2.4: (Luketero & Khalagai, 2020). Let be a partial isometry operator and be any other operator such that, either or where is unitary. Then is also a partial isometry operator.
Lemma 2.5: (Lilian et al., 2023). If are self-adjoint and unitary quasi-equivalent operators and is partial isometric, then is also partial isometric.

III. Main Results

In Theorem 3.1, we present the equivalence property of quasi-similarity on isometric operators.
Theorem 3.1
Let be quasi-similar operators, then is an isometry if and only if is an isometry.

Proof. Suppose that an operator is an isometry. Since and are quasi-similar, then there exist two operators and that are quasi-affine/ quasi-invertible such that
(1)
and
(2)
Consider equation (1), since is quasi-invertible, then multiplying both sides from the right by we have

Consequently, we have

and

Since is quasi-invertible, then and hence

But is an isometry, thus by definition . Hence
(3)
Since is invertible then and equation (3) becomes

Thus which implies that is an isometry.
Also considering equation (2), since is quasi-invertible, then multiplying both sides of equation (2) from the left by we have

Taking the adjoint of , we have

and consequently,

Since is an isometry, then by definition

Hence

But is quasi-invertible, hence

Thus
which implies that is an isometry.
Conversely, suppose that is an isometry. Since and are quasi-similar, then there exist two operators and that are quasi-affine/ quasi-invertible such that
(4)
and
(5)
Considering equation (4), since is quasi-invertible, then multiplying both sides of equation (4) from the left by we have

Also,

Thus,

But , hence

But is an isometry, thus by definition and hence

But is quasi-invertible, hence

Therefore

which implies that is an isometry.
Also considering equation (5), since is quasi-invertible, then multiplying both sides of equation (5) from the right by we have

Consequently,

and

But

Hence

That is,

implying that is an isometry.

Having considered the equivalence property of quasi-similarity on isometric operators in Theorem 3.1, and utilizing the relationship between isometry and co-isometry we explore results of theorem 3.1 in Theorem 3.2.

Theorem 3.2
Let be quasi-similar operators, then is co-isometry if and only if is co-isometry.

Proof. Suppose that an operator is co-isometry. Since and are quasi-similar, then there exist two operators and that are quasi-affine or quasi-invertible such that
(6)
and
(7)
Considering equation (6), since is quasi-invertible, then multiplying both sides of equation (6) from the right by we have

and the adjoint of becomes

Consequently,

But is a co-isometry, that is, and hence

Therefore

implying that is a co-isometry.
Also considering equation (7), since is quasi-invertible, then multiplying both sides of equation (7) from the left by we get

Hence

and

But is co-isometry, thus by definition . Hence

That is,

and therefore is also co-isometry.
Conversely, suppose that is a co-isometry. Since and are quasi-similar, then by definition there exist two operators and that are quasi-affine/ quasi-invertible such that

and

(9)

Considering equation (8), since is quasi-invertible, then by multiplying both sides of equation (8) from the left by we have

Taking the adjoint of , we have

As a result,

But is a co-isometry, hence . Therefore

That is,

which implies that is a co-isometry.
Also considering equation (9), since is quasi-invertible, then multiplying both sides of equation (9) from the right by we have

Consequently,

and therefore

But is a co-isometry, thus by definition . Hence

Thus

implying that is also a co-isometry.

Similarly, the results of Theorem 3.1 and Theorem 3.2 are considered in the case of partial isometric operators in Theorem 3.3.

Theorem 3.3
Let be quasi-similar operators and let be a partial isometry, then the operator is also a partial isometry operator.

Proof. Since and are quasi-similar operators, then there exist two operators and that are quasi-affine/ quasi-invertible such that

(10)

and

(11)

Considering equation (10), since is quasi-invertible then multiplying both sides of equation (10) from the left by we have

which follows that,

Since is a partial isometry, then by definition is a projection, thus by definition of a projection it implies that

This implies that

Also,
=
Thus

(12)

Multiplying both sides of equation (12) from the left and from the right by and respectively, we get

But and .
Thus

That is,

This implies that is a projection and thus by definition of partial isometry, is also a partial isometry operator.
Also considering equation (11), since is quasi-invertible, then multiplying both sides of equation (11) from the right by we get

which follows that

Since is a partial isometry, then by definition is a projection, thus by definition of a projection it implies that

Consequently, we have

DOI: 10.9790/5728-2204013744

and

Thus

(13)

Multiplying both sides of equation (13) from the left and from the right by and respectively, we get

That is,

This implies that is a projection, thus by definition of partial isometry it means that is a partial isometry operator.

As an immediate consequence of Theorem 3.3, as well as applying the concept of symmetric property, this study established that the equivalence relation of quasi-similarity to square operators of partial isometry mappings as illustrated in Theorem 3.4.

Theorem 3.4
Let be symmetry and quasi-similar operators. If is a partial isometry, then is also a partial isometry operator.

Proof. Since and are quasi-similar, then there exist two operators and that are quasi-affine/quasi-invertible such that

(14)

and

(15)

Since is a partial isometry, we have

and by projection property we also have

Considering equation (14), since is quasi-invertible, then multiplying both sides of equation (14) by from the left we have

which implies that

and

But

and

Thus

But

Hence is also a partial isometry operator.
Also considering equation (15), since is quasi-invertible, then multiplying both sides of equation (15) by from the right we have

which follows that

and

But
Note that

Thus

and consequently

Thus

Therefore is a partial isometry operator.

IV. Conclusion

From the above results, it is established that quasi-similar operators preserve isometries, co-isometries and partial isometries.

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