

On the Nil Graph of $\mathbb{Z}_{p^a q^b}$: Chromatic Number and First Zagreb Index

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Abstract:

Let R be a non reduced commutative ring and let $N(R)$ be the set of nil- elements of R of index two. The nil graph of R , denoted by $\Gamma_N(R)$ is a graph whose vertex set is the set $Z_N(R)^* = \{x \in R \setminus \{0\} : xy \in N(R) \text{ for some } y \in R \setminus \{0\}\}$ and two vertices x and y are adjacent if and only if their product is a nil element of index two. The chromatic number of a graph is the assignment of minimum colors to the vertices of the graph so that no two adjacent vertices have assigned with the same color. The paper investigates the chromatic number of the graph $\Gamma_N(\mathbb{Z}_n)$, for $n = p^2 q^2, p^3 q^3, p^4 q^4$ and $p^2 q^3$. It is also of interest to calculate the degree based topological index, the first Zagreb index, sum of the squares of the degrees of the vertices of a graph. The authors evaluate the first Zagreb Index for the graph $\Gamma_N(\mathbb{Z}_n)$, for $n = p^2 q^2, p^3 q^3$ and $p^2 q^3$.

Key Word: Nil Graph; Graph Coloring; Clique; Topological Indices; Zagreb Index.

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I. Introduction

The interaction between algebra and graph theory has become an exciting research topic since last few decades. A fundamental approach in this area is to associate a graph with an algebraic structure in such a way that algebraic properties can be investigated through graph-theoretical techniques. Initiated by Beck [8] and later refined by Anderson and Livingston [5] the study of zero-divisor graphs have emerged a significant direction in this field. For a commutative ring R , the zero-divisor graph has the set of nonzero zero-divisors as its vertex set, where two distinct vertices are adjacent whenever their product is zero. The extensive developments of zero-divisor graphs (see [1, 2, 3, 4, 6, 7]) has subsequently motivated the construction of several other graph structures associated with rings.

Among the various graph structure associated with rings, a kind of graph introduced by P. W. Chen [9], which is determined by the nilpotent elements of a ring, have received considerable attention in algebraic graph theory. Chen defined the graph by considering all the elements of a ring R as the vertices of the graph and two distinct vertices are adjacent if and only if their product is a nil element. He mainly focused on the colorings of the graph. Motivated by this idea, Li and Li [13] constructed the graph $\Gamma_N(R)$ with the vertex set given by $Z_N(R)^* = \{x \in R \setminus \{0\} : xy \in N(R) \text{ for some } y \in R \setminus \{0\}\}$, where $N(R)$ denotes the set of nilpotent elements of R . Two distinct vertices x and y of $\Gamma_N(R)$ are adjacent if and only if $xy \in N(R)$. This new graph extends the usual zero-divisor graph, since the zero-divisor graph arises naturally as a subgraph of the graph $\Gamma_N(R)$. Their works provide a useful framework for investigating the interplay between ring-theoretic properties of the ring R and graph-theoretic properties of the graph $\Gamma_N(R)$. They studied several basic properties of $\Gamma_N(R)$ and established important results concerning connectivity, diameter, and girth for commutative rings and von Neumann regular rings. They also deal with finite commutative rings and obtained results related to edge coloring and graph isomorphism.

Subsequently, Nikmehr and Khojasteh [14] extended the work of Li and Li [13] and name the graph as Nilpotent graph and classified all finite rings whose nilpotent graphs have diameter at most three. Patra and Begum [16] adopted the definition of $\Gamma_N(R)$ by Li and Li and considered the nil elements of R of index two where R is a non-reduced commutative ring. By considering the vertex set same it is in [13], Patra and Begum defined the two distinct vertices x and y of $\Gamma_N(R)$ are adjacent if and only if xy is a nil element of R of index two and call the graph nil graph. The study considered here focuses on nil graphs associated with finite commutative rings of the form $\mathbb{Z}_n$, with particular emphasis on the determination of their chromatic numbers for suitable values of n , to the formation of cliques whose structures are essential in establishing the chromatic number of the corresponding graphs.

A topological index provides a numerical approach that captures the structural characteristics of a graph, highlighting quantitative representation of properties such as connectivity, centrality, or branching patterns. These indices have found applications in various scientific domains including network analysis, cryptography, transportation, coding theory, and information systems [15, 17]. In chemistry, topological indices are particularly useful for representing molecular structures, in which atoms and chemical bonds are modeled by vertices and edges, respectively, enabling the prediction of molecular stability, boiling points, and biological activities. Among the earliest and most influential degree-based topological descriptors are the Zagreb indices, introduced by Ivan Gutman and Nenad Trinajstić in 1972 [11]. These indices are exclusively determined by the vertex degrees, which quantify the number of edges incident to each atom and helps in predicting molecular stability and reactivity. In particular, the first Zagreb Index (M_1) is obtained by summing the squares of the degrees of all vertices of the graph.

Motivated by these developments, the present study investigates nil graphs $\Gamma_N(R)$ associated with finite non reduced commutative rings R with the vertex set $\mathcal{Z}_N(R)^* = \{x \in R \setminus \{0\} : xy \in N(R) \text{ for some } y \in R \setminus \{0\}\}$. Any two vertices x and y of $\Gamma_N(R)$ are adjacent if and only if $xy \in N(R)$, where $N(R) = \{x \in R : x^2 = 0\}$. **Section 2** deals with some preliminaries and definitions which will be useful for the later sections. In **Section 3** we determine the chromatic number of the graph $\Gamma_N(\mathbb{Z}_n)$, for $n = p^2 q^2, p^3 q^3, p^4 q^4, p^2 q^3$. **Section 4** is concerned with the topological property, particularly the first Zagreb Index (M_1), for the graph $\Gamma_N(\mathbb{Z}_n)$, for $n = p^2 q^2, p^3 q^3$ and $p^2 q^3$.

II. Preliminaries

This section covers the essential definitions and notation of graph theory and ring theory that are needed for the subsequent sections. All the terminologies can be found in Singh & Zameeruddin [18] and Harary [10].

Definition 2.1: A non empty set R together with two binary operations $+$ and $\cdot$ is said to form a **ring** if it satisfies the following conditions:

- (a) $(R, +)$ is an abelian group.
- (b) $(R, \cdot)$ is a semi group.
- (c) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$, for all $a, b, c \in R$.

A ring is called **commutative** if $a \cdot b = b \cdot a$, for $a, b \in R$.

Definition 2.2: An element $x \in R$ (ring) is called nil element if $x^n = 0$, for a least positive integer n .

Definition 2.3: A ring is called **non-reduced** if there exists at least one non-zero nil element in the ring.

Definition 2.4: A simple **graph** G consists of a finite non empty set V of p vertices together with a prescribed set E of q unordered pairs of distinct vertices of V .

Definition 2.5: The **degree of a vertex** v , denoted by $\deg(v)$ is the number of edges incident with it.

Definition 2.6: If in a graph every pair of vertices are adjacent, then it is called a **complete** graph. A maximal complete subgraph is called a **clique**. A **bipartite** graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge joins a vertex of V_1 to a vertex of V_2 . A bipartite graph is said to be **complete** if it is maximal with respect to its bipartiteness.

Definition 2.7: A **proper coloring** or simply a coloring of a graph G is an assignment of k -colors to the vertices of G such that no two adjacent vertices have assigned same color. The **chromatic number** $\chi(G)$ is the minimum k for which G has k -coloring.

Definition 2.8: (Nil Graph) Let R be a commutative ring and R^* be the set of all non-zero elements of R . Let $N(R)$ be the set of nil elements of index two. i.e, $N(R) = \{x \in R : x^2 = 0\}$. The *nil graph* of R , denoted by $\Gamma_N(R)$, is a graph whose vertex set is the $\mathcal{Z}_N(R)^* = \{x \in R \setminus \{0\} : xy \in N(R) \text{ for some } y \in R \setminus \{0\}\}$, and any two distinct vertices x and y are adjacent if and only if $xy \in N(R)$.

For the nil graph $\Gamma_N(\mathbb{Z}_{p^a q^b})$, if $x, y \in V(\Gamma_N(\mathbb{Z}_{p^a q^b}))$, then $x \text{ adj. } y$ if and only if $xy \in N(\mathbb{Z}_{p^a q^b})$ i.e, if $(xy)^2 \equiv 0 \pmod{p^a q^b}$.

III. Chromatic number of the graph

In this section, a complete classification of the vertex sets of graph $\Gamma_N(\mathbb{Z}_n)$, for some specific values of n have been analyzed. Based on these classification we determine the chromatic number of the graph $\Gamma_N(\mathbb{Z}_n)$, for $n = p^2 q^2, p^3 q^3, p^4 q^4, p^2 q^3$.

Theorem 3.1: Let $\Gamma_N(\mathbb{Z}_{p^2 q^2})$ be the nil graph of the commutative ring $\mathbb{Z}_{p^2 q^2}$, where p, q are primes, then $\chi(\Gamma_N(\mathbb{Z}_{p^2 q^2})) = pq + 1$.

Proof: The vertex of the graph $\Gamma_N(\mathbb{Z}_{p^2 q^2})$ can be partitioned into the following subsets:

$$\begin{array}{ll} V_{p^2 q} = \{mp^2 q : q \nmid m, 1 \leq m \leq q-1\} & |V_{p^2 q}| = q-1 \\ V_{pq^2} = \{mpq^2 : p \nmid m, 1 \leq m \leq p-1\} & |V_{pq^2}| = p-1 \\ V_{pq} = \{mpq : p, q \nmid m, 1 \leq m \leq pq-1\} & |V_{pq}| = (p-1)(q-1) \\ V_{p^2} = \{mp^2 : q \nmid m, 1 \leq m \leq q^2-1\} & |V_{p^2}| = q(q-1) \\ V_{q^2} = \{mq^2 : p \nmid m, 1 \leq m \leq p^2-1\} & |V_{q^2}| = p(p-1) \\ V_p = \{mp : p, q \nmid m, 1 \leq m \leq pq^2-1\} & |V_p| = q(p-1)(q-1) \\ V_q = \{mq : p, q \nmid m, 1 \leq m \leq p^2 q-1\} & |V_q| = p(p-1)(q-1) \\ U = \{m : p, q \nmid m, 1 \leq m \leq p^2 q^2-1\} & |U| = pq(p-1)(q-1) \end{array}$$

The elements of the set V_{pq} are adjacent to each other and so they form a complete subgraph of order $pq - 1$ and these elements can be colored with $pq - 1$ colors.

The elements of $V_{p^2} \cup V_p$ and $V_{q^2} \cup V_q$ are not adjacent among themselves but are adjacent to each other and also adjacent to each element of V_{pq} . Hence all these adjacent vertices will induce a clique of order $pq - 1 + 2 = pq + 1$. Therefore $\chi(\Gamma_N(\mathbb{Z}_{p^2 q^2})) \geq pq + 1$.

The elements of the set U are not adjacent among themselves and not adjacent every vertex of the clique. So, they can be colored with any color from the clique. Therefore $\chi(\Gamma_N(\mathbb{Z}_{p^2 q^2})) \leq pq + 1$. Hence $\chi(\Gamma_N(\mathbb{Z}_{p^2 q^2})) = pq + 1$.

Theorem 3.2: Let $\Gamma_N(\mathbb{Z}_{p^3 q^3})$ be the nil graph of the commutative ring $\mathbb{Z}_{p^3 q^3}$, where p, q are primes, then $\chi(\Gamma_N(\mathbb{Z}_{p^3 q^3})) = p^2 q^2 + 1$.

Proof: The vertex of the graph $\Gamma_N(\mathbb{Z}_{p^3 q^3})$ can be partitioned into the following subsets:

$$\begin{array}{ll} V_{p^3 q^2} = \{mp^3 q^2 : q \nmid m, 1 \leq m \leq q-1\} & |V_{p^3 q^2}| = q-1 \\ V_{p^2 q^3} = \{mp^2 q^3 : p \nmid m, 1 \leq m \leq p-1\} & |V_{p^2 q^3}| = p-1 \\ V_{p^2 q^2} = \{mp^2 q^2 : p, q \nmid m, 1 \leq m \leq pq-1\} & |V_{p^2 q^2}| = (p-1)(q-1) \\ V_{p^3 q} = \{mp^3 q : q \nmid m, 1 \leq m \leq q^2-1\} & |V_{p^3 q}| = q(q-1) \\ V_{pq^3} = \{mpq^3 : p \nmid m, 1 \leq m \leq p^2-1\} & |V_{pq^3}| = p(p-1) \\ V_{p^2 q} = \{mp^2 q : p, q \nmid m, 1 \leq m \leq pq^2-1\} & |V_{p^2 q}| = q(p-1)(q-1) \\ V_{pq^2} = \{mpq^2 : p, q \nmid m, 1 \leq m \leq p^2 q-1\} & |V_{pq^2}| = p(p-1)(q-1) \\ V_{pq} = \{mpq : p, q \nmid m, 1 \leq m \leq p^2 q^2-1\} & |V_{pq}| = pq(p-1)(q-1) \\ V_{p^3} = \{mp^3 : p, q \nmid m, 1 \leq m \leq q^3-1\} & |V_{p^3}| = q^2(q-1) \\ V_{q^3} = \{mq^3 : p, q \nmid m, 1 \leq m \leq p^3-1\} & |V_{q^3}| = p^2(p-1) \\ V_{p^2} = \{mp^2 : p, q \nmid m, 1 \leq m \leq pq^3-1\} & |V_{p^2}| = q^2(p-1)(q-1) \\ V_{q^2} = \{mq^2 : p, q \nmid m, 1 \leq m \leq p^3 q-1\} & |V_{q^2}| = p^2(p-1)(q-1) \\ V_p = \{mp : p, q \nmid m, 1 \leq m \leq p^2 q^3-1\} & |V_p| = pq^2(p-1)(q-1) \\ V_q = \{mq : p, q \nmid m, 1 \leq m \leq p^3 q^2-1\} & |V_q| = p^2 q(p-1)(q-1) \\ U = \{m : p, q \nmid m, 1 \leq m \leq p^3 q^3-1\} & |U| = p^2 q^2(p-1)(q-1) \end{array}$$

All the elements of the set $X = \bigcup_{i,j=1}^2 V_{p^i q^j} \cup \bigcup_{j=1}^2 V_{p^3 q^j} \cup \bigcup_{i=1}^2 V_{p^i q^3}$ are adjacent with each other and induce a complete sub graph of order $p^2 q^2 - 1$. The elements of the sets $V_{p^2} \cup V_{p^3}$ and $V_{q^2} \cup V_{q^3}$ are not adjacent among themselves but are adjacent to each other and hence will form a complete bipartite sub graph. These elements are also adjacent to every vertex of the complete sub graph induced by X and together induce a clique of order $p^2 q^2 - 1 + 2 = p^2 q^2 + 1$. Therefore $\chi(\Gamma_N(\mathbb{Z}_{p^3 q^3})) \geq p^2 q^2 + 1$.

The elements of both the sets V_p , V_q and U are adjacent only to the elements of the set $V_{p^2 q^2} \cup \bigcup_{j=1}^2 V_{p^3 q^j} \cup \bigcup_{i=1}^2 V_{p^i q^3}$. These elements are not adjacent among themselves and also not adjacent to any member of $\bigcup_{i=2}^3 V_{p^i}$ or $\bigcup_{j=2}^3 V_{q^j}$. These elements can be assigned with some colors that are assigned to the elements of $V_{pq} \cup \bigcup_{i=2}^3 V_{p^i} \cup \bigcup_{j=2}^3 V_{q^j}$. So, all the colors assigned to the clique induced by X can give a proper coloring of $\Gamma_N(\mathbb{Z}_{p^3 q^3})$. Therefore $\chi(\Gamma_N(\mathbb{Z}_{p^3 q^3})) \leq p^2 q^2 + 1$. Hence $\chi(\Gamma_N(\mathbb{Z}_{p^3 q^3})) = p^2 q^2 + 1$.

Theorem 3.3: Let $\Gamma_N(\mathbb{Z}_{p^4 q^4})$ be the nil graph of the commutative ring $\mathbb{Z}_{p^4 q^4}$, where p, q are primes, then $\chi(\Gamma_N(\mathbb{Z}_{p^4 q^4})) = p^3 q^3 + 1$.

Proof: The vertex of the graph $\Gamma_N(\mathbb{Z}_{p^4 q^4})$ can be partitioned into the following subsets:

$V_{p^4 q^3} = \{mp^4 q^3 : q \nmid m, 1 \leq m \leq q - 1\}$	$ V_{p^4 q^3} = q - 1$
$V_{p^3 q^4} = \{mp^3 q^4 : p \nmid m, 1 \leq m \leq p - 1\}$	$ V_{p^3 q^4} = p - 1$
$V_{p^3 q^3} = \{mp^3 q^3 : p, q \nmid m, 1 \leq m \leq pq - 1\}$	$ V_{p^3 q^3} = (p - 1)(q - 1)$
$V_{p^4 q^2} = \{mp^4 q^2 : q \nmid m, 1 \leq m \leq q^2 - 1\}$	$ V_{p^4 q^2} = q(q - 1)$
$V_{p^3 q^2} = \{mp^3 q^2 : p, q \nmid m, 1 \leq m \leq pq^2 - 1\}$	$ V_{p^3 q^2} = q^2(p - 1)$
$V_{p^2 q^4} = \{mp^2 q^4 : p \nmid m, 1 \leq m \leq p^2 - 1\}$	$ V_{p^2 q^4} = p(p - 1)$
$V_{p^2 q^3} = \{mp^2 q^3 : p, q \nmid m, 1 \leq m \leq p^2 q - 1\}$	$ V_{p^2 q^3} = p(p - 1)(q - 1)$
$V_{p^2 q^2} = \{mp^2 q^2 : p, q \nmid m, 1 \leq m \leq p^2 q^2 - 1\}$	$ V_{p^2 q^2} = pq(p - 1)(q - 1)$
$V_{p^4 q} = \{mp^4 q : q \nmid m, 1 \leq m \leq q^3 - 1\}$	$ V_{p^4 q} = q^2(q - 1)$
$V_{p^3 q} = \{mp^3 q : p, q \nmid m, 1 \leq m \leq pq^3 - 1\}$	$ V_{p^3 q} = pq^2(q - 1)$
$V_{p^2 q} = \{mp^2 q : p, q \nmid m, 1 \leq m \leq p^2 q^3 - 1\}$	$ V_{p^2 q} = p^2 q^2(q - 1)$
$V_{pq^4} = \{mpq^4 : p \nmid m, 1 \leq m \leq p^3 - 1\}$	$ V_{pq^4} = p^2(p - 1)$
$V_{pq^3} = \{mpq^3 : p, q \nmid m, 1 \leq m \leq p^3 q - 1\}$	$ V_{pq^3} = p^2 q(p - 1)$
$V_{pq^2} = \{mpq^2 : p, q \nmid m, 1 \leq m \leq p^3 q^2 - 1\}$	$ V_{pq^2} = p^2 q^2(p - 1)$
$V_{pq} = \{mpq : p, q \nmid m, 1 \leq m \leq p^3 q^3 - 1\}$	$ V_{pq} = p^2 q^2(p - 1)(q - 1)$
$V_{p^4} = \{mp^4 : p, q \nmid m, 1 \leq m \leq q^4 - 1\}$	$ V_{p^4} = q^3(q - 1)$
$V_{q^4} = \{mq^4 : p, q \nmid m, 1 \leq m \leq p^4 - 1\}$	$ V_{q^4} = p^3(p - 1)$
$V_{p^3} = \{mp^3 : p, q \nmid m, 1 \leq m \leq pq^4 - 1\}$	$ V_{p^3} = q^3(p - 1)(q - 1)$
$V_{q^3} = \{mq^3 : p, q \nmid m, 1 \leq m \leq p^4 q - 1\}$	$ V_{q^3} = p^3(p - 1)(q - 1)$
$V_{p^2} = \{mp^2 : p, q \nmid m, 1 \leq m \leq p^2 q^4 - 1\}$	$ V_{p^2} = pq^3(p - 1)(q - 1)$
$V_{q^2} = \{mq^2 : p, q \nmid m, 1 \leq m \leq p^4 q^2 - 1\}$	$ V_{q^2} = p^3 q(p - 1)(q - 1)$
$V_p = \{mp : p, q \nmid m, 1 \leq m \leq p^3 q^4 - 1\}$	$ V_p = p^3 q^2(p - 1)(q - 1)$
$V_q = \{mq : p, q \nmid m, 1 \leq m \leq p^4 q^3 - 1\}$	$ V_q = p^3 q^2(p - 1)(q - 1)$
$U = \{m : p, q \nmid m, 1 \leq m \leq p^4 q^4 - 1\}$	$ U = p^3 q^3(p - 1)(q - 1)$

The elements of the set $\bigcup_{i,j=1}^3 V_{p^i q^j} \cup \bigcup_{i=1}^4 V_{p^i q^4} \cup \bigcup_{j=1}^4 V_{p^4 q^j}$ form a complete sub graph of order $p^3 q^3 - 1$. The elements of the set $\bigcup_{i=2}^4 V_{p^i}$ and $\bigcup_{j=2}^4 V_{q^j}$ are not adjacent among themselves but are adjacent to each other and to every member of the complete sub graph induced by $\bigcup_{i,j=1}^3 V_{p^i q^j} \cup V_{p^3 q^4} \cup V_{p^4 q^3}$. Therefore the set $\bigcup_{i,j=1}^3 V_{p^i q^j} \cup V_{p^3 q^4} \cup V_{p^4 q^3} \cup \{a\} \cup \{b\}$, $a \in \bigcup_{i=2}^4 V_{p^i}$ and $b \in \bigcup_{j=2}^4 V_{q^j}$ which will induce a clique of order $p^3 q^3 - 1 + 2 = p^3 q^3 + 1$. Therefore $\chi(\Gamma_N(\mathbb{Z}_{p^4 q^4})) \geq p^3 q^3 + 1$.

The elements of the set U , V_p and V_q are not adjacent among themselves but are adjacent to some members of the clique. So, they can be colored with some colors used in clique which are not adjacent with them. Therefore $\chi(\Gamma_N(\mathbb{Z}_{p^4 q^4})) \leq p^3 q^3 + 1$. Hence, $\chi(\Gamma_N(\mathbb{Z}_{p^4 q^4})) = p^3 q^3 + 1$.

Theorem 3.4: Let $\Gamma_N(\mathbb{Z}_{p^2 q^3})$ be the nil graph of the commutative ring $\mathbb{Z}_{p^2 q^3}$, where p, q are primes, then $\chi(\mathbb{Z}_{p^2 q^3}) = pq^2 + 1$.

Proof: The vertex set of the graph $\Gamma_N(\mathbb{Z}_{p^2 q^3})$ can be partitioned into the following subsets:

$V_{p^2 q^2} = \{mp^2 q^2 : q \nmid m, 1 \leq m \leq q - 1\}$	$ V_{p^2 q^2} = q - 1$
$V_{p^2 q} = \{mp^2 q : q \nmid m, 1 \leq m \leq q^2 - 1\}$	$ V_{p^2 q} = q(q - 1)$
$V_{pq^3} = \{mpq^3 : p \nmid m, 1 \leq m \leq p - 1\}$	$ V_{pq^3} = p - 1$

$$\begin{aligned}
 V_{pq^2} &= \{mpq^2: p, q \nmid m, 1 \leq m \leq pq - 1\} & |V_{pq^2}| &= (p-1)(q-1) \\
 V_{pq} &= \{mpq: p, q \nmid m, 1 \leq m \leq pq^2 - 1\} & |V_{pq}| &= q(p-1)(q-1) \\
 V_{q^3} &= \{mq^3: p \nmid m, 1 \leq m \leq p^2 - 1\} & |V_{q^3}| &= p(p-1) \\
 V_{p^2} &= \{mp^2: q \nmid m, 1 \leq m \leq q^3 - 1\} & |V_{p^2}| &= q^2(q-1) \\
 V_{q^2} &= \{mq^2: p, q \nmid m, 1 \leq m \leq p^2q - 1\} & |V_{q^2}| &= p(p-1)(q-1) \\
 V_p &= \{mp: p, q \nmid m, 1 \leq m \leq pq^3 - 1\} & |V_p| &= q^2(p-1)(q-1) \\
 V_q &= \{mq: p, q \nmid m, 1 \leq m \leq p^2q^2 - 1\} & |V_q| &= pq(p-1)(q-1) \\
 U &= \{m: p, q \nmid m, 1 \leq m \leq p^2q^3 - 1\} & |U| &= pq^2(p-1)(q-1)
 \end{aligned}$$

All the elements of the set $\bigcup_{i,j=1}^2 V_{p^i q^j} \cup V_{pq^3}$ are adjacent with each other and induce a complete sub graph of order $pq^2 - 1$. The elements of the sets V_{p^2} and $V_{q^2} \cup V_{q^3}$ are not adjacent among themselves but are adjacent to each other and hence will form a complete bipartite sub graph. All the vertices of the bipartite graph are adjacent to every vertex of the complete sub graph and induce a clique of order $pq^2 + 1$. Therefore $\chi(\Gamma_N(\mathbb{Z}_{p^2 q^3})) \geq pq^2 + 1$.

The elements of the sets V_p , V_q and U are not adjacent among themselves but are adjacent to some elements of the clique. So, they can be colored with some colors used in clique which are not adjacent with them. Therefore $\chi(\Gamma_N(\mathbb{Z}_{p^2 q^3})) \leq pq^2 + 1$. Hence $\chi(\Gamma_N(\mathbb{Z}_{p^2 q^3})) = pq^2 + 1$.

IV. First Zagreb Index

In this section, we compute the first Zagreb index of the graph $\Gamma_N(\mathbb{Z}_n)$, for $n = p^2 q^2$, $p^3 q^3$ and $p^2 q^3$. The Zagreb indices are degree-based topological indices and are widely used in chemical graph theory [11]. For a graph G with vertex degree $\deg(v)$, the first Zagreb index is defined by

$$M_1(G) = \sum_{v \in V(G)} \deg(v)^2$$

Theorem 4.1: The first Zagreb number of the graph $\Gamma_N(\mathbb{Z}_{p^2 q^2})$ is given by

$$p^5 q^3 (-1 + q + q^2) + p^4 q^3 (1 - 4q + q^2) + p^3 q^2 (8 - 9q + q^2 - q^3) + p^2 (-2 - 5q^2 + 8q^3) + p(2 + 3q) - 2(2 - q + q^2)$$

Proof: Considering the vertex set given in proof of the **Theorem 3.1**

$$\deg v = \begin{cases} p^2 q^2 - 2, & v \in V_{pq} \cup V_{p^2 q} \cup V_{pq^2} \\ p^2 q - 1, & v \in V_{p^2} \cup V_p \\ pq^2 - 1, & v \in V_{q^2} \cup V_q \\ pq - 1, & v \in U \end{cases}$$

Therefore, the first Zagreb index of $\Gamma_N(\mathbb{Z}_{p^2 q^2})$ is

$$\begin{aligned}
 M_1(\Gamma_N(\mathbb{Z}_{p^2 q^2})) &= \sum_{v \in V(\Gamma_N(\mathbb{Z}_{p^2 q^2}))} \deg(v)^2 \\
 &= (pq - 1)(p^2 q^2 - 2)^2 + (q(p - 2)(q - 1))(p^2 q - 1)^2 + (p(p - 1)(q - 2))(pq^2 - 1)^2 + (pq(p - 1)(q - 1))(pq - 1)^2 \\
 &= p^5 q^3 (-1 + q + q^2) + p^4 q^3 (1 - 4q + q^2) + p^3 q^2 (8 - 9q + q^2 - q^3) + p^2 (-2 - 5q^2 + 8q^3) + p(2 + 3q) - 2(2 - q + q^2).
 \end{aligned}$$

Theorem 4.2: The first Zagreb number of the graph $\Gamma_N(\mathbb{Z}_{p^3 q^3})$ is given by

$$p^6 q^4 (-1 + q^3) + p^7 q^4 (-1 + q^2 + q^3) - p^5 q^7 + p^2 q (1 + 4q + 2q^4) + p^4 q^3 (16 - 23q - q^3 + q^5) - p^3 (q + 5q^3 - 14q^4 + 2q^5 + q^8) - pq^2 - 4$$

Proof: Considering the vertex set given in Proof of **Theorem 3.2**

$$\deg v = \begin{cases} p^3 q^3 - 2, & v \in V_{p^2 q^2} \cup V_{p^2 q^3} \cup V_{p^3 q^2} \\ p^3 q^2 - 2 & v \in V_{p^2 q} \cup V_{p^3 q} \\ p^2 q^3 - 2 & v \in V_{p q^2} \cup V_{p q^3} \\ p^2 q^2 - 2 & v \in V_{p q} \\ p^3 q - 1, & v \in V_{p^2} \cup V_{p^3} \\ p q^3 - 1, & v \in V_{q^2} \cup V_{q^3} \\ p^2 q - 1, & v \in V_p \\ p q^2 - 1, & v \in V_q \\ p q - 1, & v \in U \end{cases}$$

Therefore, the first Zagreb index is

$$\begin{aligned} M_1(\Gamma_N(\mathbb{Z}_{p^3 q^3})) &= \sum_{u \in V(\Gamma_N(\mathbb{Z}_{p^2 q^2}))} \deg(u)^2 \\ &= (pq - 1)(p^3 q^3 - 2)^2 + (pq(q - 1))(p^3 q^2 - 2)^2 + (pq(p - 1))(p^2 q^3 - 2)^2 \\ &\quad + (pq(p - 1)(q - 1))(p^2 q^2 - 2)^2 + (pq^2(q - 1))(p^3 q - 1)^2 + (pq^2(p - 1))(pq^3 - 1)^2 \\ &\quad + (pq^2(p - 1)(q - 1))(p^2 q - 1)^2 + (p^2 q(p - 1)(q - 1))(pq^2 - 1)^2 \\ &\quad + (p^2 q^2(p - 1)(q - 1))(pq - 1)^2 \\ &= p^6 q^4(-1 + q^3) + p^7 q^4(-1 + q^2 + q^3) - p^5 q^7 + p^2 q(1 + 4q + 2q^4) + p^4 q^3(16 - 23q - q^3 + q^5) \\ &\quad - p^3(q + 5q^3 - 14q^4 + 2q^5 + q^8) - pq^2 - 4 \end{aligned}$$

Theorem 4.3: The first Zagreb number of the graph $\Gamma_N(\mathbb{Z}_{p^2 q^3})$ is given by

$$p^5 q^4(-1 + q^2 + q^3) + p^4 q^4(-1 + q^3) - p^3 q^3(-10 + 15q + q^3 + q^4) + p^2 q^3(1 + 6q) + 3pq^2 - 4$$

Proof: Considering the vertex set given in proof of the **Theorem 3.4**

$$\deg v = \begin{cases} p^2 q^3 - 2, & v \in V_{p^2 q^2} \cup V_{p q^3} \cup V_{p q^2} \\ p^2 q^2 - 2 & v \in V_{p q} \cup V_{p^2 q} \\ p^2 q - 1 & v \in V_{p^2} \cup V_p \\ p q^3 - 1 & v \in V_{q^2} \cup V_{q^3} \\ p q^2 - 1, & v \in V_q \\ p q - 1, & v \in U \end{cases}$$

Therefore, the first Zagreb index is

$$\begin{aligned} M_1(\Gamma_N(\mathbb{Z}_{p^2 q^3})) &= \sum_{u \in V(\Gamma_N(\mathbb{Z}_{p^2 q^2}))} \deg(u)^2 \\ &= (pq - 1)(p^2 q^3 - 2)^2 + (pq(q - 1))(p^2 q^2 - 2)^2 + (pq^2(q - 1))(p^2 q - 1)^2 + (pq(p - 1))(pq^3 - 1)^2 \\ &\quad + (pq(p - 1)(q - 1))(pq^2 - 1)^2 + (pq^2(p - 1)(q - 1))(pq - 1)^2 \\ &= p^5 q^4(-1 + q^2 + q^3) + p^4 q^4(-1 + q^3) - p^3 q^3(-10 + 15q + q^3 + q^4) + p^2 q^3(1 + 6q) + 3pq^2 - 4. \end{aligned}$$

V. Conclusion

The results and findings discussed above lead to the following conclusions:

- (i) In this work, we studied the complete classification of the vertex set of the graph $\Gamma_N(\mathbb{Z}_n)$. Based on these structural characteristics, we established that the chromatic number of the graph is closely related to the prime factorization of n and the clique of the graph, which enabled us to determine the chromatic number of the graph $\Gamma_N(\mathbb{Z}_n)$ for specific values for $n = p^2 q^2, p^3 q^3, p^4 q^4, p^2 q^3$.
- (ii) Furthermore, the degree-based topological index, namely the **first Zagreb index**, was computed for the graph $\Gamma_N(\mathbb{Z}_n)$ for different forms of $n = p^2 q^2, p^3 q^3$ and $p^2 q^3$. The derived expressions illustrates how the degree distribution and structural properties of the graph vary with the prime factorization of n .

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