

A Dimension-Unified Homotopy Analysis Framework for Two- and Three-Dimensional Wave Equations

Dr. Sandeep Sharma

Associate Professor, Department of Mathematics, A&M Institute of Management and Technology, Pathankot-145001, Punjab, India

Abstract:

This paper presents a unified application of the Homotopy Analysis Method (HAM) to two- and three-dimensional linear and nonlinear partial differential wave equations. A common solution framework is formulated for both dimensions, while the nonlinear terms are incorporated into the successive deformation equations. The method constructs a series solution without requiring a small perturbation parameter. An auxiliary convergence-control parameter provides a means to examine and adjust the convergence of the resulting series. Representative initial-value problems are used to illustrate the formulation and the calculation of successive solution terms. Where exact solutions are available, they provide benchmarks for evaluating the approximations; for nonlinear problems, residual errors can be used to assess solution quality. The study outlines how the same HAM procedure can be adapted across different dimensions and equation types, offering a systematic basis for analysing multidimensional wave propagation. The numerical accuracy and convergence range depend on the specific problem and chosen auxiliary parameters.

Keywords: Homotopy Analysis Method; wave equations; partial differential equations; two-dimensional models; three-dimensional models; nonlinear equations; series solutions; convergence analysis.

Date of Submission: 17-09-2026

Date of Acceptance: 30-09-2026

I. Introduction

1 Wave Equation:

Wave equations are fundamental partial differential equations used to describe the propagation of disturbances through space and time. They arise naturally in many areas of science and engineering, including mechanical vibrations, acoustics, elastic media, electromagnetic phenomena, fluid dynamics, and other processes involving travelling or oscillatory disturbances. In two- and three-dimensional settings, the dependent variable varies with respect to multiple spatial coordinates, and therefore the governing model must account for wave propagation in each spatial direction. Classical linear wave equations provide an essential mathematical foundation for understanding such phenomena, whereas nonlinear wave equations incorporate additional effects such as interactions, nonlinear restoring forces, variable material responses, and amplitude-dependent behaviour, which generally make the analytical treatment more challenging ([2]; [16]).

The mathematical idea of homotopy originates from topology, where it is used to describe a continuous deformation between mathematical objects or mappings ([1]; [3]). Building upon this fundamental concept, Liao (1992, 2003) developed the Homotopy Analysis Method (HAM) as a general analytical technique for solving linear and nonlinear differential equations. Unlike conventional perturbation methods, HAM does not require the existence of a naturally occurring small or large physical parameter. Instead, it constructs a continuous deformation from an appropriately selected initial approximation to the required solution of the original problem ([8];[9]).

The HAM formulation introduces an initial approximation, an auxiliary linear operator, an embedding parameter, an auxiliary function, and a non-zero convergence-control parameter, commonly denoted by $(\hbar)$. By expanding the deformation equation in a power series with respect to the embedding parameter, a sequence of higher-order linear deformation equations is obtained. The successive solution components generated from these equations produce an analytical series representation of the unknown function. A particularly important characteristic of HAM is the presence of the auxiliary parameter $(\hbar)$, which provides a convenient mechanism for adjusting and controlling the convergence region and convergence rate of the resulting solution series (Liao, 2004, 2012). This additional flexibility distinguishes HAM from many traditional analytical techniques and makes it particularly useful for nonlinear and multidimensional problems.

Over the years, HAM has been successfully applied to a wide range of nonlinear ordinary and partial differential equations. Its effectiveness has also motivated the development of several hybrid formulations in which HAM is combined with integral-transform techniques and other analytical procedures. For example, Elzaki-transform-based HAM has been applied to fractional nonlinear Schrödinger equations, while Rangaig-transform-based HAM has been used for two- and three-dimensional partial differential equations and coupled systems ([11];[12][13][15]). A broader review of such hybrid HAM techniques has also demonstrated their applicability to nonlinear, fractional, and multidimensional mathematical models [14].

In the present study, the Homotopy Analysis Method is formulated within a unified framework for selected two-dimensional and three-dimensional wave equations. Representative linear and nonlinear initial-value problems are considered to illustrate the construction of the corresponding zero-order and higher-order deformation equations. The resulting HAM series solutions are obtained systematically and examined at successive orders of approximation. Wherever exact solutions are available, the analytical approximations are compared directly with the corresponding exact results. For problems in which a closed-form exact solution is unavailable, the reliability of the approximation can be examined through residual errors and convergence behaviour.

Particular attention is also given to the influence of the convergence-control parameter $(\hbar)$, since the convergence of a HAM series generally depends on the mathematical structure of the particular problem under consideration. Consequently, a single fixed value of the auxiliary parameter should not automatically be assumed to provide equally satisfactory convergence for every linear or nonlinear equation. By examining convergence, approximation error, and residual behaviour separately for the considered models, the study provides a systematic assessment of the applicability and accuracy of HAM for multidimensional wave equations.

II. Mathematical Framework

2 Basic Concept of the Homotopy Analysis Method

The Homotopy Analysis Method (HAM), introduced by Liao (2003), is an analytical technique for obtaining approximate or exact solutions of linear and nonlinear differential equations. Unlike perturbation methods, HAM does not require a small physical parameter. Its main feature is the auxiliary parameter $\hbar$, called the convergence-control parameter, which helps control the convergence of the solution series. ([8]; [3]; [1])

Consider the following general nonlinear partial differential equation:

$$\mathcal{N}[\mathcal{U}(x, y, t)] = 0, \quad t \geq 0 \quad (1)$$

where $\mathcal{U}(x, y, t)$ is the unknown function and $\mathcal{N}$ denotes a nonlinear differential operator.

2.1 Zeroth-Order Deformation Equation

To construct HAM, an initial approximation $\mathcal{U}_0(x, y, t)$, an embedding parameter $p \in [0, 1]$, an auxiliary linear operator $\mathcal{L}$, an auxiliary function $\mathcal{H}(x, y, t)$, and a nonzero parameter $\hbar$ are introduced. The zeroth-order deformation equation is

$$(1 - p)\mathcal{L}[\varphi(x, y, t; p) - \mathcal{U}_0(x, y, t)] = p\hbar\mathcal{H}(x, y, t)\mathcal{N}[\varphi(x, y, t; p)] \quad (2)$$

Here, $\varphi(x, y, t; p)$ is the deformation function. At the two limiting values of p ,

$$\varphi(x, y, t; 0) = \mathcal{U}_0(x, y, t), \quad \varphi(x, y, t; 1) = \mathcal{U}(x, y, t) \quad (3)$$

Hence, as p changes continuously from 0 to 1, the initial approximation $\mathcal{U}_0$ is transformed into the required solution $\mathcal{U}$.

2.2 Series Representation

If $\varphi(x, y, t; p)$ is analytic with respect to p , it can be expanded about $p = 0$ as

$$\varphi(x, y, t; p) = \mathcal{U}_0(x, y, t) + \sum_{m=1}^{\infty} \mathcal{U}_m(x, y, t) p^m \quad (4)$$

where the m th-order deformation component is defined by

$$\mathcal{U}_m(x, y, t) = \frac{1}{(m)!} \frac{\partial^m \varphi(x, y, t; p)}{\partial p^m} \Big|_{p=0} \quad (5)$$

If the series converges at $p = 1$, the solution of the original problem is

$$\mathcal{U}(x, y, t) = \sum_{m=0}^{\infty} \mathcal{U}_m(x, y, t) \quad (6)$$

Thus, HAM represents the required solution as a series of successively determined components.

2.3 Higher-Order Deformation Equation

Differentiating the zeroth-order deformation equation m times with respect to p , setting $p = 0$, and dividing by $m!$ gives

$$\mathcal{L}[\mathcal{U}_m - \chi_m \mathcal{U}_{m-1}] = \hbar\mathcal{H}(x, y, t)\mathcal{R}_m(\vec{\mathcal{U}}_{m-1}) \quad (7)$$

where the m th-order homotopy residual is

$$\mathcal{R}_m(\vec{\mathcal{U}}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} \mathcal{N}[\varphi(x, y, t; p)] \Big|_{p=0} \quad (8)$$

and

$$\chi_m = \begin{cases} 0, & m = 1, \\ 1, & m > 1. \end{cases} \quad (9)$$

Applying the inverse operator $\mathcal{L}^{-1}$ gives the recursive relation

$$\mathcal{U}_m = \chi_m \mathcal{U}_{m-1} + \mathcal{L}^{-1}[\hbar\mathcal{H}(x, y, t)\mathcal{R}_m(\vec{\mathcal{U}}_{m-1})] \quad (10)$$

Therefore, the original nonlinear problem is reduced to a sequence of simpler linear deformation equations. In many applications $\mathcal{H}(x, y, t) = 1$ is used, while $\hbar$ is selected to obtain a suitable convergence region.

For practical computations, the infinite series is truncated after M terms:

$$S_M(x, y, t) = \sum_{m=0}^M u_m(x, y, t) \quad (11)$$

where S_M is the M th-order HAM approximation.

3 Convergence of the HAM Series

The reliability of the HAM solution depends on the convergence of its series. Let the HAM components belong to a Banach space A . A simple sufficient condition for convergence is that, for sufficiently large m , ([8];[9];[10]).

$$\|u_{m+1}\| \leq r \|u_m\|, \quad 0 \leq r < 1 \quad (12)$$

This means that successive HAM terms become smaller. The series therefore behaves like a convergent geometric series; generally, a smaller value of r indicates faster convergence.

The truncation error after M terms can be bounded by

$$\|u - S_M\| \leq \frac{\|u_{m+1}\|}{1 - r} \quad (13)$$

Thus, as the higher-order terms decrease, the truncated HAM approximation approaches the converged solution.

If instead, for sufficiently large m ,

$$\|u_{m+1}\| \geq r \|u_m\|, \quad r > 1 \quad (14)$$

the terms cannot approach zero and the series diverges. When the ratio approaches one, the ratio test is inconclusive. Therefore, the convergence-control parameter $\hbar$ is adjusted in practice to obtain a suitable convergent region with $r < 1$.

3.1 Error Analysis

When an exact solution is available, the accuracy of the M th-order HAM approximation can be measured by the absolute error ([8];[9];[10]).

$$E_M = |u_{\text{exact}} - S_M| \quad (15)$$

The relative error is

$$E_M^{(r)} = \frac{|u_{\text{exact}} - S_M|}{|u_{\text{exact}}|}, \quad u_{\text{exact}} \neq 0 \quad (16)$$

The rate at which the numerical error decreases may also be examined through the successive error ratio

$$R_E = \frac{E_{M+1}}{E_M}, \quad E_M \neq 0 \quad (17)$$

A decrease in the absolute and relative errors as M increases provides numerical evidence that the HAM approximation is approaching the exact solution. Similarly, a successive error ratio below unity indicates that the error is decreasing from one approximation order to the next. These numerical indicators support the convergence study but do not, by themselves, replace the mathematical convergence condition.

In summary, HAM constructs the solution through a sequence of deformation equations, while the auxiliary parameter $\hbar$ provides a direct mechanism for controlling convergence. The truncated series, together with convergence and error measures, provides a practical framework for obtaining and validating semi-analytical solutions of nonlinear partial differential equations.

III. Result

4 Test Examples:

In this section, we apply the Homotopy Analysis Method (HAM) to obtain analytical solutions for selected two- and three-dimensional partial differential equations, with particular emphasis on the wave equation.

Example 4.1: We begin with the two-dimensional wave equation, expressed as:

$$\frac{\partial^2 \mathcal{U}}{\partial t^2} = -\frac{1}{2} \left(\frac{\partial^2 \mathcal{U}}{\partial x^2} + \frac{\partial^2 \mathcal{U}}{\partial y^2} \right), \quad (18)$$

Accompanied by the initial conditions:

$$\mathcal{U}(x, y, 0) = e^{x+y} \text{ and } \mathcal{U}_t(x, y, 0) = 0, \quad (19)$$

Dirichlet Boundary conditions :

$$\begin{aligned} \mathcal{U}(0, y, t) &= e^y \cos t, & \mathcal{U}(1, y, t) &= e^{1+y} \cos t, & 0 \leq y \leq 1, t \geq 0 \\ \mathcal{U}(x, 0, t) &= e^x \cos t, & \mathcal{U}(x, 1, t) &= e^{x+1} \cos t, & 0 \leq x \leq 1, t \geq 0 \end{aligned}$$

The exact solution corresponding to this problem is given by ([2];[16]):

$$\mathcal{U}(x, y, t) = e^{x+y} \cos t, \quad (20)$$

Choose the linear operator using the Homotopy analysis technique as follows:

$$\mathcal{L}[\varphi(x, y, t; p)] = \frac{\partial^2 \varphi(x, y, t; p)}{\partial t^2},$$

In section (**Error! Reference source not found.**) , the inverse linear operator used within the framework of the HAM is constructed for given problem.

Define the nonlinear operator associated with the problem as:

$$\mathcal{N}[\varphi(x, y, t; p)] = \frac{\partial^2 \varphi}{\partial t^2} + \frac{1}{2} \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right), \quad (21)$$

Here, the operator $\mathfrak{R}_m(\vec{\mathcal{U}}_{m-1})$ is defined as:

$$\mathfrak{R}_m(\vec{\mathcal{U}}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} \mathcal{N} \left[\sum_{i=0}^{m-1} p^i \mathcal{U}_i + \frac{1}{2} \left(\frac{\partial^2}{\partial x^2} \sum_{i=0}^{m-1} p^i \mathcal{U}_i + \frac{\partial^2}{\partial y^2} \sum_{i=0}^{m-1} p^i \mathcal{U}_i \right) \right]_{p=0}, \quad (22)$$

Now, we present the solution of the m^{th} order deformation equation, as given in section (**Error! Reference source not found.**). We begin the initial approximation $\mathcal{U}_0(x, y, t) = e^{x+y}$ and proceed by applying the iterative formula. The subsequent components are determined in a similar manner as follows:

The intermediate function $\mathfrak{R}_m$ are evaluated as:

$$\begin{aligned}\mathfrak{R}_1(\vec{\mathcal{U}}_0(x, y, t)) &= e^{x+y}, \\ \mathfrak{R}_2(\vec{\mathcal{U}}_1(x, y, t)) &= \left(-1 - \frac{t^2}{2!}\right) e^{x+y}, \\ \mathfrak{R}_3(\vec{\mathcal{U}}_2(x, y, t)) &= \left(\frac{t^2}{2!} + \frac{t^4}{4!}\right) e^{x+y}, \\ &\vdots \\ &\vdots\end{aligned}$$

This leads to the following approximate components of the solution:

$$\begin{aligned}\mathcal{U}_1(x, y, t) &= -\frac{t^2}{2!} e^{x+y}, \\ \mathcal{U}_2(x, y, t) &= \frac{t^4}{4!} e^{x+y}, \\ \mathcal{U}_3(x, y, t) &= -\frac{t^6}{6!} e^{x+y}, \\ &\vdots \\ &\vdots\end{aligned}$$

We construct the solution as a series by using the values of $\mathcal{U}_0$, $\mathcal{U}_1$, and $\mathcal{U}_2 \dots$, as follows.

$$\mathcal{U}(x, y, t) = \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \dots\right) e^{x+y} \quad (23)$$

This is the Taylor series expansion of $\cos t$. The final result is obtained as a similar of exact solution, as shown in Eq. **Error! Reference source not found..**

$$\mathcal{U}(x, y, t) = e^{x+y} \cdot \cos t \quad (24)$$

Which is indeed the exact solution.

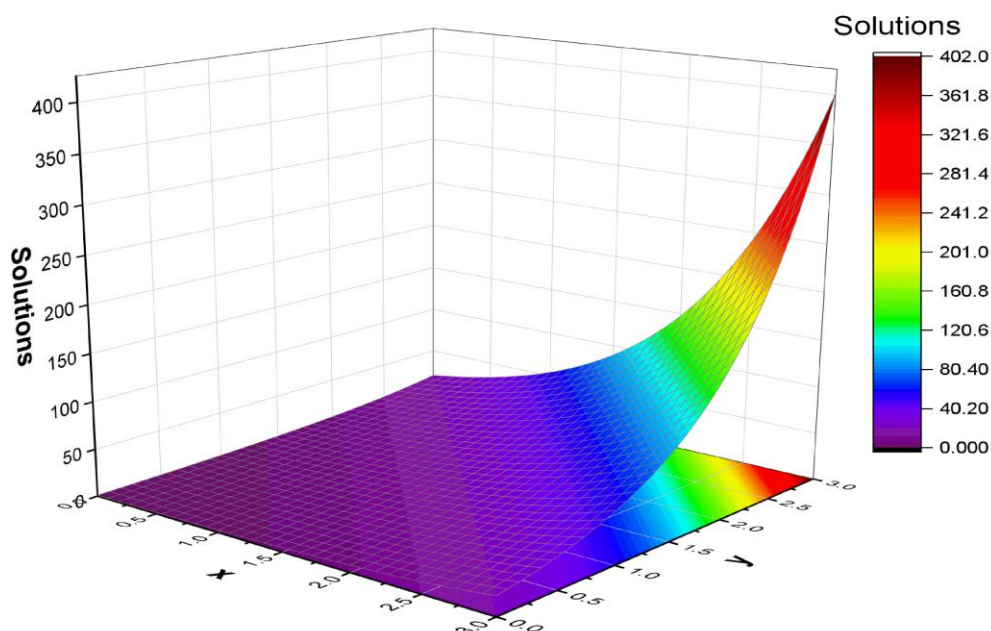


Figure Error! No text of specified style in document..1: Physical behaviour on the surface with projection of solutions “U” of example 4.1 at $0 \leq x, y \leq 3$ and $t = 0.1$.

Table Error! No text of specified style in document..1: A comparative analysis between the exact and HAM approximate solutions for example 4.1, is performed at $x = 0.5$, $y = 0.5$ and $t = 0.1$. The accuracy of HAM approximation is evaluate using absolute error, relative error, percentage error, convergence ratio and computation time to assess the accuracy of the approximation. In this analysis, one to eight terms of the homotopy series are used, and all computations are performed in MATLAB.

n	$(U)_E$	$(U_n)_A$	Abs. Error	Rel. Error	% Error	Convergence Ratio	Computation Time (s)
1	2.704701742	2.718281828	1.36E-02	5.02E-03	5.02E-01		1.16E-05
2	2.704701742	2.704690419	1.13E-05	4.19E-06	4.19E-04	8.34E-04	8.02E-06
3	2.704701742	2.704701745	3.77E-09	1.40E-09	1.40E-07	3.33E-04	1.34E-05
4	2.704701742	2.704701742	6.74E-13	2.49E-13	2.49E-11	1.79E-04	7.82E-06
5	2.704701742	2.704701742	0.00E+00	0.00E+00	0.00E+00	0.00E+00	1.02E-05
6	2.704701742	2.704701742	0.00E+00	0.00E+00	0.00E+00		1.22E-05
7	2.704701742	2.704701742	0.00E+00	0.00E+00	0.00E+00		1.36E-05
8	2.704701742	2.704701742	0.00E+00	0.00E+00	0.00E+00		1.67E-05

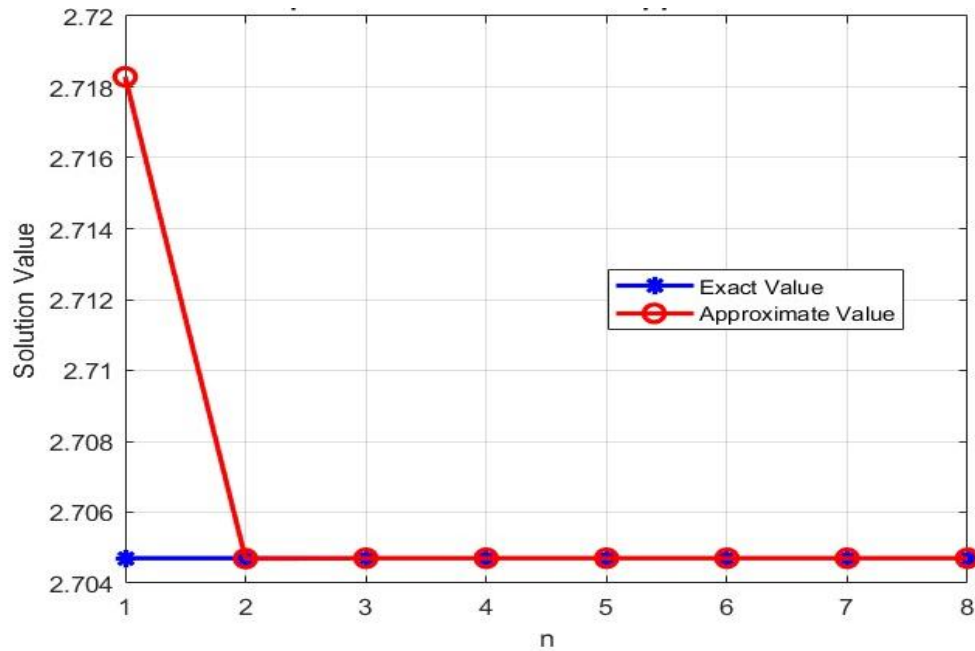


Figure Error! No text of specified style in document..2: Comparison between the exact and HAM approximate solutions “U” for example 4.1 at $t = 0.1$.

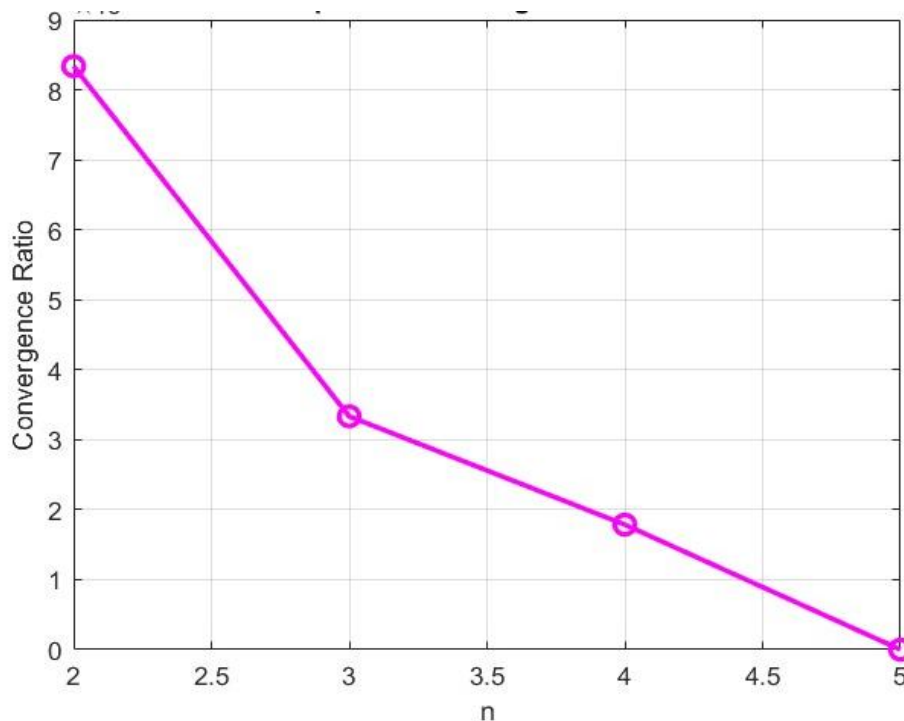


Figure Error! No text of specified style in document..3: Convergence ratio behavior of the HAM approximate solution “U” for example 4.1 at $t = 0.1$.

Example 4.2: We begin with the two-dimensional Wave equation, expressed as:

$$\frac{\partial^2 \mathcal{U}}{\partial t^2} = \frac{1}{2\pi^2} \left(\frac{\partial^2 \mathcal{U}}{\partial x^2} + \frac{\partial^2 \mathcal{U}}{\partial y^2} \right), \quad (25)$$

Accompanied by the initial conditions:

$$\mathcal{U}(x, y, 0) = \sin(\pi x) \cdot \sin(\pi y) \text{ and } \mathcal{U}_t(x, y, 0) = 0 \quad (26)$$

Dirichlet Boundary conditions :

$$\mathcal{U}(0, y, t) = 0, \quad \mathcal{U}(1, y, t) = 0, \quad 0 \leq y \leq 1, t \geq 0$$

$$\mathcal{U}(x, 0, t) = 0, \quad \mathcal{U}(x, 1, t) = 0, \quad 0 \leq x \leq 1, t \geq 0$$

The exact solution corresponding to this problem is given by ([2];[16]):

$$\mathcal{U}(x, y, t) = \sin(\pi x) \cdot \sin(\pi y) \cos t, \quad (27)$$

Choose the linear operator using the Homotopy analysis technique as follows:

$$\mathcal{L}[\varphi(x, y, t; p)] = \frac{\partial^2 \varphi(x, y, t; p)}{\partial t^2},$$

In section (**Error! Bookmark not defined.**), the inverse linear operator used within the framework of the HAM is constructed for given problem.

Define the nonlinear operator associated with the problem as:

$$N[\varphi(x, y, t; p)] = \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2\pi^2} \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right), \quad (28)$$

Here, the operator $\mathfrak{R}_m(\vec{\mathcal{U}}_{m-1})$ is defined as:

$$\mathfrak{R}_m(\vec{\mathcal{U}}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} \left[\frac{\partial^2}{\partial t^2} \sum_{i=0}^{m-1} p^i \mathcal{U}_i - \frac{1}{2\pi^2} \left(\frac{\partial^2}{\partial x^2} \sum_{i=0}^{m-1} p^i \mathcal{U}_i + \frac{\partial^2}{\partial y^2} \sum_{i=0}^{m-1} p^i \mathcal{U}_i \right) \right]_{p=0}, \quad (29)$$

Now, we present the solution of the m^{th} order deformation equation, as given in section (**Error! Bookmark not defined.**) . We begin the initial approximation $\mathcal{U}_0(x, y, t) = \sin(\pi x) \cdot \sin(\pi y)$ and proceed by applying the iterative formula. The subsequent components are determined in a similar manner as follows:

The intermediate function $\mathfrak{R}_m$ are evaluated as:

$$\mathfrak{R}_1(\vec{\mathcal{U}}_0(x, y, t)) = \sin(\pi x) \cdot \sin(\pi y),$$

$$\mathfrak{R}_2(\vec{\mathcal{U}}_1(x, y, t)) = \left(-1 - \frac{t^2}{2!} \right) \cdot \sin(\pi x) \cdot \sin(\pi y),$$

$$\Re_3(\vec{U}_2(x, y, t)) = \left(\frac{t^2}{2!} + \frac{t^4}{4!}\right) \cdot \sin(\pi x) \cdot \sin(\pi y),$$

$$\vdots$$

$$\vdots$$

This leads to the following approximate components of the solution:

$$\mathcal{U}_1(x, y, t) = -\frac{t^2}{2!} \sin(\pi x) \cdot \sin(\pi y),$$

$$\mathcal{U}_2(x, y, t) = \frac{t^4}{4!} \sin(\pi x) \cdot \sin(\pi y),$$

$$\mathcal{U}_3(x, y, t) = -\frac{t^6}{6!} \sin(\pi x) \cdot \sin(\pi y),$$

$$\vdots$$

$$\vdots$$

We construct the solution as a series using the values of $\mathcal{U}_0$, $\mathcal{U}_1$, and $\mathcal{U}_2 \dots$, as follows.

$$\mathcal{U}(x, y, t) = \sin(\pi x) \cdot \sin(\pi y) \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots\right) \quad (30)$$

This is the Taylor series expansion of $\cos t$. The final result is obtained as a similar of exact solution, as shown in Eq. **Error! Reference source not found..**

$$\mathcal{U}(x, y, t) = \sin(\pi x) \cdot \sin(\pi y) \cdot \cos t. \quad (31)$$

Which is indeed the exact solution.

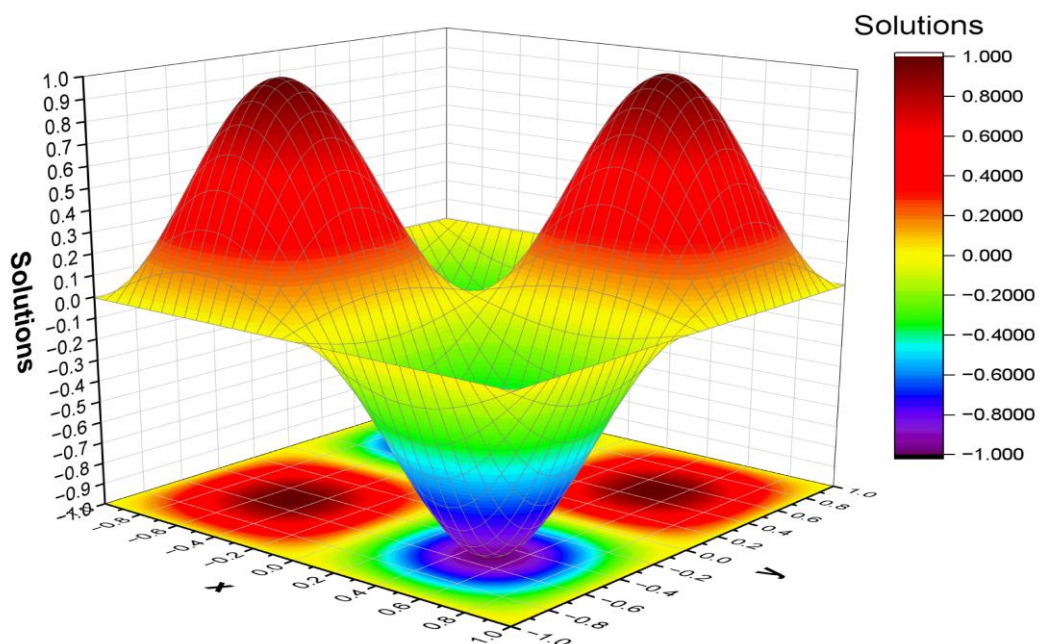


Figure Error! No text of specified style in document..4: Physical behaviour on the surface with projection of solutions “U” of example 4.2 at $-1 \leq x, y \leq 1$ and $t = 0.1$.

Table Error! No text of specified style in document..2: A comparative analysis between the exact and HAM approximate solutions for example 4.2, is performed at $x = 0.5$, $y = 0.5$ and $t = 0.1$. The accuracy of HAM approximation is evaluate using absolute error, relative error, percentage error, convergence ratio and computation time to assess the accuracy of the approximation. In this analysis, one to eight terms of the homotopy series are used, and all computations are performed in MATLAB.

n	$(U)_E$	$(U_n)_A$	Abs. Error	Rel. Error	% Error	Convergence Ratio	Computation Time (s)
1	0.995004165	1.000000000	5.00E-03	5.02E-03	5.02E-01		6.24E-06
2	0.995004165	0.995000000	4.17E-06	4.19E-06	4.19E-04	8.34E-04	8.07E-06
3	0.995004165	0.995004167	1.39E-09	1.40E-09	1.40E-07	3.33E-04	7.26E-06
4	0.995004165	0.995004165	2.48E-13	2.49E-13	2.49E-11	1.79E-04	8.32E-06
5	0.995004165	0.995004165	1.11E-16	1.12E-16	1.12E-14	4.48E-04	1.02E-05
6	0.995004165	0.995004165	1.11E-16	1.12E-16	1.12E-14	1.00E+00	1.04E-05
7	0.995004165	0.995004165	1.11E-16	1.12E-16	1.12E-14	1.00E+00	1.29E-05
8	0.995004165	0.995004165	1.11E-16	1.12E-16	1.12E-14	1.00E+00	1.49E-05

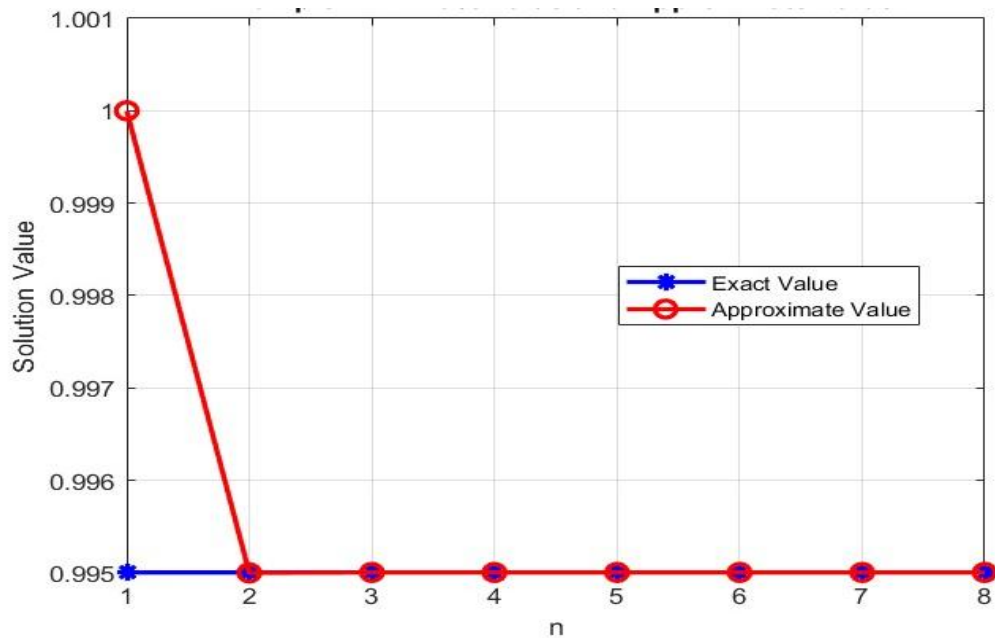


Figure Error! No text of specified style in document..5: Comparison between the exact and HAM approximate solutions “U” for example 4.2 at $t = 0.1$.

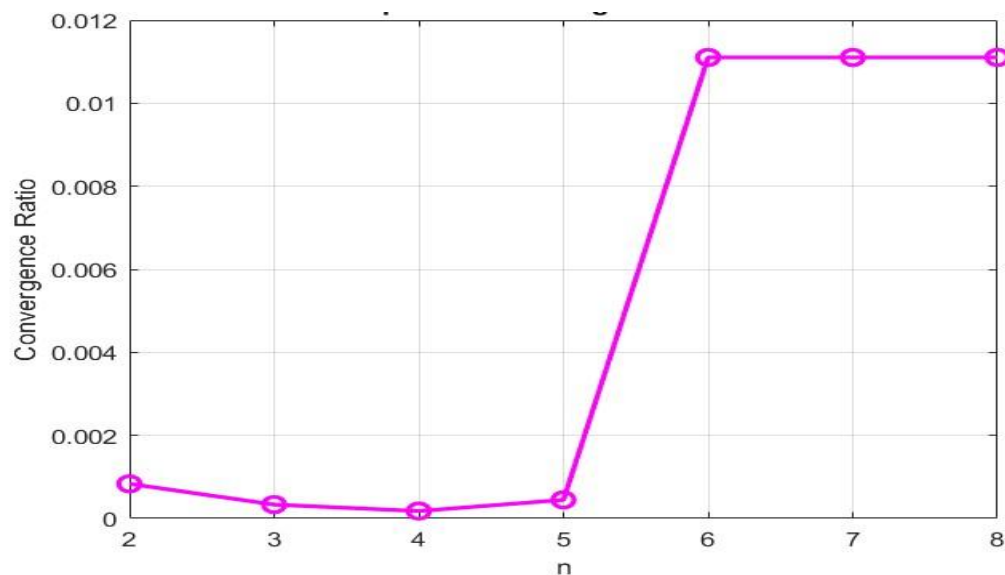


Figure Error! No text of specified style in document..6: Convergence ratio behavior of the HAM approximate solution “U” for example 4.2 at $t = 0.1$.

Example 4.3: We begin with the three-dimensional Wave equation, expressed as:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{3} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad (32)$$

Accompanied by the initial conditions:

$$\mathcal{U}(x, y, z, 0) = \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z), \text{ and } \mathcal{U}_t(x, y, z, 0) = 0, \quad (33)$$

Dirichlet Boundary conditions :

$$\begin{aligned} \mathcal{U}(0, y, z, t) &= 0, & \mathcal{U}(1, y, z, t) &= 0, & 0 \leq y, z \leq 1, t \geq 0 \\ \mathcal{U}(x, 0, z, t) &= 0, & \mathcal{U}(x, 1, z, t) &= 0, & 0 \leq x, z \leq 1, t \geq 0 \\ \mathcal{U}(x, y, 0, t) &= 0, & \mathcal{U}(x, y, 1, t) &= 0, & 0 \leq x, y \leq 1, t \geq 0 \end{aligned}$$

The exact solution corresponding to this problem is given by ([2]; [16]):

$$\mathcal{U}(x, y, z, t) = \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z) \cdot \cos(\pi t), \quad (34)$$

Choose the linear operator using the Homotopy analysis technique as follows:

$$\mathcal{L}[\varphi(x, y, z, t; p)] = \frac{\partial \varphi(x, y, z, t; p)}{\partial t},$$

In section (**Error! Bookmark not defined.**), the inverse linear operator used within the framework of the HAM is constructed for the given problem.

Define the nonlinear operator associated with the problem as:

$$\mathcal{N}[\varphi(x, y, z, t; p)] = \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{3} \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \right), \quad (35)$$

Here, the operator $\mathfrak{R}_m(\vec{\mathcal{U}}_{m-1})$ is defined as:

$$\begin{aligned} \mathfrak{R}_m(\vec{\mathcal{U}}_{m-1}) &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} \left[\frac{\partial^2}{\partial t^2} \sum_{i=0}^{m-1} p^i \mathcal{U}_i \right. \\ &\quad \left. - \frac{1}{3} \left(\frac{\partial^2}{\partial x^2} \sum_{i=0}^{m-1} p^i \mathcal{U}_i + \frac{\partial^2}{\partial y^2} \sum_{i=0}^{m-1} p^i \mathcal{U}_i + \frac{\partial^2}{\partial z^2} \sum_{i=0}^{m-1} p^i \mathcal{U}_i \right) \right]_{p=0}, \end{aligned} \quad (36)$$

Now, we present the solution of the m^{th} order deformation equation, as given in section (**Error! Bookmark not defined.**). We begin the initial approximation $\mathcal{U}_0(x, y, z, t) = \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z)$, and proceed by applying the iterative formula. The subsequent components are determined in a similar manner as follows;

The intermediate function $\mathfrak{R}_m$ are evaluated as:

$$\begin{aligned} \mathfrak{R}_1(\vec{\mathcal{U}}_0(x, y, z, t)) &= \pi^2 \cdot \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z), \\ \mathfrak{R}_2(\vec{\mathcal{U}}_1(x, y, z, t)) &= \left(-\pi^2 - \frac{\pi^4 t^2}{2!} \right) \cdot \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z), \end{aligned}$$

$$\begin{aligned}\Re_3(\vec{\mathcal{U}}_2(x, y, z, t)) &= \left(\frac{\pi^4 t^2}{2!} + \frac{\pi^6 t^4}{4!} \right) \cdot \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z), \\ &\vdots \\ &\vdots\end{aligned}$$

This leads to the following approximate components of the solution:

$$\mathcal{U}_1(x, y, z, t) = -\frac{\pi^2 t^2}{2!} \cdot \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z),$$

$$\mathcal{U}_2(x, y, z, t) = \frac{\pi^4 t^4}{4!} \cdot \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z),$$

$$\mathcal{U}_3(x, y, z, t) = -\frac{\pi^6 t^6}{6!} \cdot \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z),$$

$\vdots$
 $\vdots$

We construct the solution as a series by using the values of $\mathcal{U}_0$, $\mathcal{U}_1$ and $\mathcal{U}_2 \dots$, as follows.

$$\mathcal{U}(x, y, z, t) = \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z) \cdot \left(1 - \frac{(\pi t)^2}{2!} + \frac{(\pi t)^4}{4!} - \frac{(\pi t)^6}{6!} + \dots \right), \quad (37)$$

This is the Taylor series expansion of $\cos \pi t$. The final result is obtained as a similar of exact solution, as shown in Eq. **Error! Reference source not found.**

$$\mathcal{U}(x, y, z, t) = \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z) \cdot \cos(\pi t). \quad (38)$$

Which is indeed the exact solution.

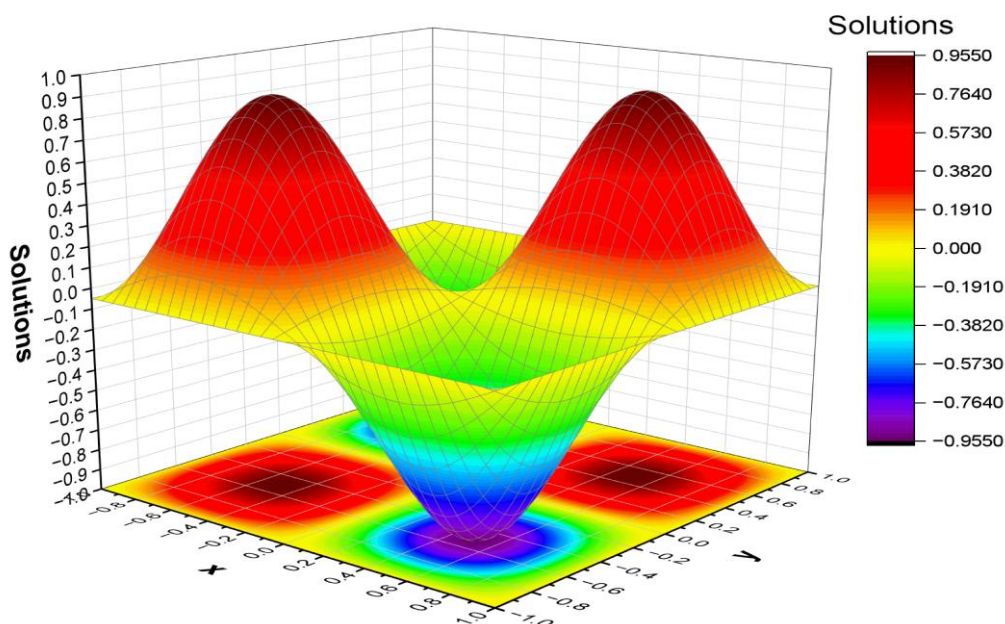


Figure Error! No text of specified style in document..7: Physical behaviour on the surface with projection of solutions “ U ” for example 4.3 at $-1 \leq x, y \leq 1$, $t = 0.1$ and $z = 0.5$.

Table Error! No text of specified style in document..3: A comparative analysis between the exact and HAM approximate solutions for example 4.3, is performed at $x = 0.5$, $y = 0.5$, $z = 0.5$ and $t = 0.1$. The accuracy of HAM approximation is evaluate using absolute error, relative error, percentage error, convergence ratio and computation time to assess the accuracy of the approximation. In this analysis, one to eight terms of the homotopy series are used, and all computations are performed in MATLAB.

n	$(U)_E$	$(U_n)_A$	Abs. Error	Rel. Error	% Error	Convergence Ratio	Computation Time (s)
1	0.951056516	1.000000000	4.89E-02	5.15E-02	5.15E+00		5.50E-06
2	0.951056516	0.950651978	4.05E-04	4.25E-04	4.25E-02	8.27E-03	5.99E-06
3	0.951056516	0.951057849	1.33E-06	1.40E-06	1.40E-04	3.29E-03	5.33E-06
4	0.951056516	0.951056514	2.35E-09	2.47E-09	2.47E-07	1.76E-03	8.91E-06
5	0.951056516	0.951056516	2.58E-12	2.71E-12	2.71E-10	1.10E-03	1.04E-05
6	0.951056516	0.951056516	1.89E-15	1.98E-15	1.98E-13	7.32E-04	1.04E-05
7	0.951056516	0.951056516	0.00E+00	0.00E+00	0.00E+00	0.00E+00	1.22E-05
8	0.951056516	0.951056516	0.00E+00	0.00E+00	0.00E+00		1.67E-05

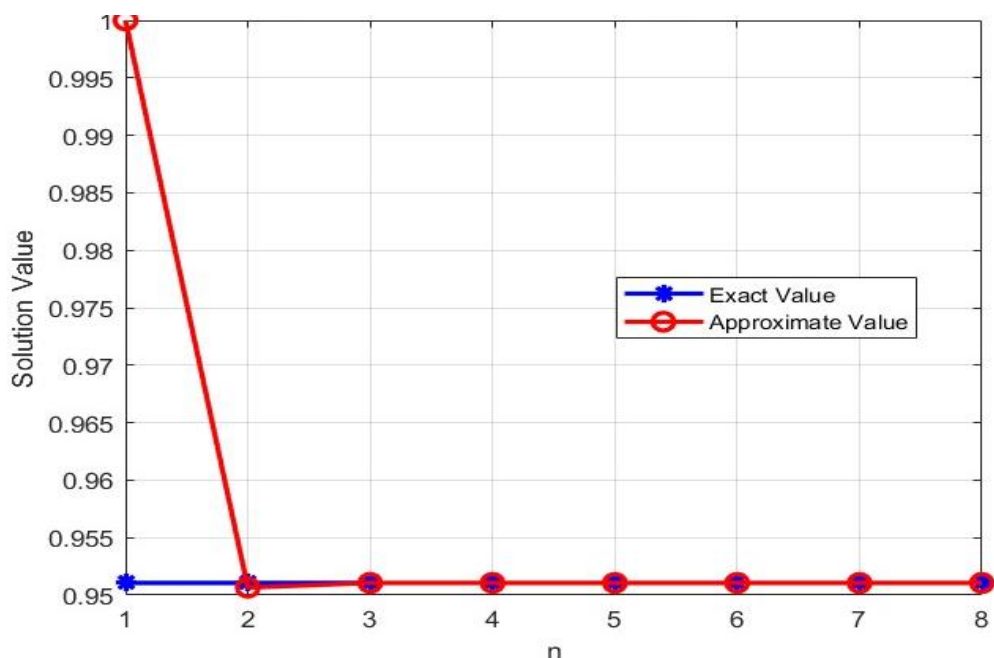


Figure Error! No text of specified style in document..8: Comparison between the exact and HAM approximate solutions “U” for example 4.3 at $t = 0.1$.

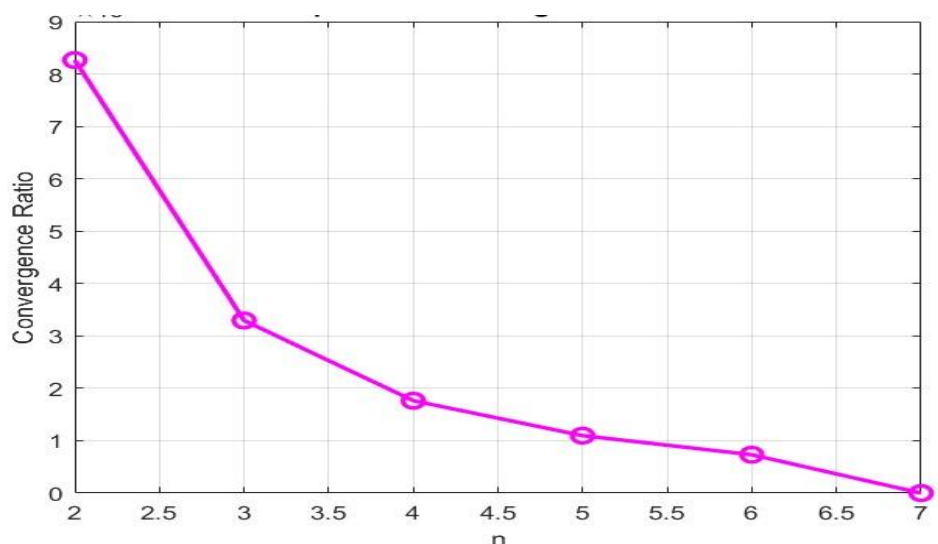


Figure Error! No text of specified style in document..9: Convergence ratio behavior of the HAM approximate solution “U” for example 4.3 at $t = 0.1$.

IV. Conclusion

This study presented a **dimension-unified Homotopy Analysis Method (HAM) framework for two- and three-dimensional wave equations**. The same HAM procedure was successfully applied in both dimensions through suitable choices of the linear operator, nonlinear operator, initial approximation, and recursive deformation relations. In each example, the HAM solution was obtained as a rapidly convergent series and reproduced the corresponding exact solution.

For the first two-dimensional wave problem, the HAM series reduced to the Taylor expansion of $\cos t$ and yielded

$$\mathcal{U}(x, y, t) = e^{x+y} \cos t.$$

The numerical results showed a rapid reduction in absolute and relative errors. The second two-dimensional example similarly produced

$$\mathcal{U}(x, y, t) = \sin(\pi x) \sin(\pi y) \cos t,$$

with the absolute error decreasing from approximately 5.00×10^{-3} at the first approximation to about 10^{-16} at higher orders. The three-dimensional example further demonstrated that the same framework extends naturally to an additional spatial dimension and produced

$$\mathcal{U}(x, y, z, t) = \sin(\pi x) \cdot \sin(\pi y) \cdot \sin(\pi z) \cdot \cos(\pi t).$$

For this case, the absolute error decreased from approximately 4.89×10^{-2} at the first approximation to values close to machine precision at higher orders. Overall, the numerical comparisons, convergence behaviour, and graphical results show that the HAM approximations approach the exact solutions rapidly as the number of terms increases. The proposed dimension-unified formulation therefore provides a simple, systematic, and accurate analytical framework for treating both two- and three-dimensional wave equations within a common structure.

V. Future Scope

The present dimension-unified HAM framework can be extended to more complex multidimensional wave models. Future studies may consider nonlinear wave equations, variable coefficients, external forcing, damping effects, and coupled systems within the same analytical structure. The approach can also be applied to fractional-order wave equations to examine memory and nonlocal effects in complex media. Further work may focus on selecting the convergence-control parameter $\hbar$ more efficiently through $\hbar$ -curves, residual-error minimization, or optimization techniques. Hybrid formulations combining HAM with integral transforms, spectral methods, or numerical schemes may also improve computational efficiency for difficult problems. Comparative studies with other semi-analytical and numerical methods can provide deeper insight into accuracy and convergence. The framework may ultimately be applied to practical problems in acoustics, structural vibration, elasticity, electromagnetic wave propagation, and fluid-wave interaction.

VI. References

- [1]. Basener, W. F. (2006). *Topology and its applications*. John Wiley & Sons.
- [2]. Evans, L. C. (2010). *Partial differential equations* (2nd ed.). American Mathematical Society. [ams.org](https://doi.org/10.1017/CBO9780511666278)
- [3]. Hilton, P. (1953). *An introduction to homotopy theory*. Cambridge University Press. <https://doi.org/10.1017/CBO9780511666278>
- [4]. Liao, S. (2004). On the homotopy analysis method for nonlinear problems. *Applied Mathematics and Computation*, 147(2), 499–513. [https://doi.org/10.1016/S0096-3003\(02\)00790-7](https://doi.org/10.1016/S0096-3003(02)00790-7) [sciencedirect.com](https://doi.org/10.1016/S0096-3003(02)00790-7)
- [5]. Liao, S. (2012). *Homotopy analysis method in nonlinear differential equations*. Springer. <https://doi.org/10.1007/978-3-642-25132-0> Springer Nature Link
- [6]. Liao, S. J. (1992). *The proposed homotopy analysis technique for the solution of nonlinear problems* (Doctoral dissertation). Shanghai Jiao Tong University.
- [7]. Liao, S. J. (1999). An explicitly analytic solution of the Blasius equation. *International Journal of Non-Linear Mechanics*, 34(5), 759–778.
- [8]. Liao, S. J. (2003). *Beyond perturbation: Introduction to the homotopy analysis method*. Chapman & Hall/CRC. <https://doi.org/10.1201/9780203491164>
- [9]. Liao, S. J. (2004). On the homotopy analysis method for nonlinear problems. *Applied Mathematics and Computation*, 147(2), 499–513. [https://doi.org/10.1016/S0096-3003\(02\)00790-7](https://doi.org/10.1016/S0096-3003(02)00790-7)
- [10]. Liao, S. J. (2012). *Homotopy analysis method in nonlinear differential equations*. Springer.

- [11]. Sharma, S., & Singh, I. (2024a). Elzaki transform homotopy analysis techniques for solving fractional (2+1)-D and (3+1)-D nonlinear Schrödinger equations. *Communications on Applied Nonlinear Analysis*, 31(6s), 305–317. <https://doi.org/10.52783/cana.v31.1224>
- [12]. Sharma, S., & Singh, I. (2024b). Mathematical analysis of some (3+1)-D models via Rangaig transform. *Communications on Applied Nonlinear Analysis*, 32(3), 383–399. <https://doi.org/10.52783/cana.v32.1996>
- [13]. Singh, I., & Sharma, S. (2024). Efficient techniques for solving system of (2+1)-D & (3+1)-D PDEs using Rangaig transforms based HAM. *International Journal of Intelligent Systems and Applications in Engineering*, 12(23s), 2399–2423.
- [14]. Singh, I., & Sharma, S. (2025a). A review of the homotopy analysis method with various hybrid techniques. *International Journal of Mathematics and Statistics Invention*, 13(1), 1–15. <https://doi.org/10.35629/4767-13010115>
- [15]. Singh, I., & Sharma, S. (2025b). Rangaig transforms based HAM for solving some two-dimensional PDEs. *Journal of Mechanics of Continua and Mathematical Sciences*, 20(1), 115–136. <https://doi.org/10.26782/jmcms.2025.01.00008>
- [16]. Strauss, W. A. (2007). *Partial differential equations: An introduction* (2nd ed.). John Wiley & Sons.