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Operator Regularization in Evaluating Feynman Diagrams in QED in (3+1) Dimensional Space-Time

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Abstract: *Operator regularization is one of the excellent prescriptions for studying gauge theories. Among the many regularization prescriptions Dimensional regularization and Pre-regularization are the best methods for evaluating loop diagrams perturbatively. On the other hand Operator regularization can also be said one of the best methods for studying gauge theories because of its two-fold use. With this prescription one can adopt path-integral method with the combination of background field quantization and Schwinger expansion to find the result of the required problem without considering any Feynman diagrams. Also from this prescription one can consider Feynman diagrams and evaluating these diagrams using the Operator regularization prescription. In this paper we have shown how one can use both the options of Operator regularization method to evaluate Feynman diagrams in QED in (3+1) dimensional space-time.*

Keywords: *Operator regularization, Dimensional regularization, Feynman diagrams in QED, Path-integral method, Background field quantization and Generating functional.*

I. Introduction

Feynman diagrammatic technique is a standard way of studying of gauge theories in a perturbative way. The problem is when one tries to evaluate the loop diagrams arising from the theory in consideration using Feynman integrals. Most of the time divergencies arise in these integrals. So one has to use some regularization method to overcome this problem.

Evaluating loop integrals using different regularization procedures give results which are very often dependent on the regulating parameters which are not expected. However in some cases choosing the appropriate value of the parameter one can get the exact result. In this paper three basic diagrams in QED in (3+1) dimensional space-time is studied by the method of Operator regularization [1] in two different approaches and Dimensional regularization [2]. We have shown here how the results in these three approaches are same.

Operator regularization is a little bit different than that of other regularization methods. Because all the methods mentioned here are perturbative method. That is one has to draw all possible Feynman graphs and following any regularization method one can find the transition amplitude of the required problem. But Operator regularization method can be used in a two-fold way. That is without considering Feynman graphs [1] which depend only on path integral method and also considering Feynman graphs [4]. In the first case one has to use back-ground field quantization in the Lagrangian then the operators and inverse operators have to regulated following a given prescription [1, 3]. After some simplification Schwinger expansion [5] has to be taken. From the expansion one can choose appropriate terms for the problem in consideration. It means that we do not have to consider Feynman graphs. In this method we do not have to face any divergencies at any stage of calculation. However, after quantizing with operator regularization, there is a way to consider Feynman graphs. Then following any regularization method one can find transition amplitudes of the problem. In this case this is a combination of Operator regularization and other regularization methods.

In this paper at first we will show how the second option of Operator regularization can be used in evaluating loop diagrams. That means from the path integral form of the Operator regularization how one can choose the part of the prescription which can be applied to find the amplitude of the basic Feynman graphs and then we will show how these problems can be obtained from the first option that is from path integral form of the method and the results will be compared later on.

II. Operator Regularization Prescription

Operator regularization is an alternative way of computing quantum correction in quantum field theory in context of background-field quantization, which was given by D.G.C. McKeon et.al. [1, 3]. In this method the Feynman diagrams of the usual perturbation series can be avoided because this method depends on path integrals. But at one stage there is an option to consider Feynman diagrams. That is from this prescription one can choose either path integral method or Feynman diagrammatic approach. In this approach we regulate operators and inverse operators rather than the initial Lagrangian. To one-loop order this scheme reduces to a perturbative expansion of the well-known ξ - function regularization [6-9] associated with the superdeterminant of an operator.

This prescription is given in ref. [3] but for completeness let us briefly describe it here. The background-field method in context of path-integral quantization [10-12] is the starting point of this procedure. We consider the general case where a field $\varphi_i(x)$ which may be either fermionic or bosonic. Let us consider a field $\varphi_i(x)$ is quantized into its background classical part $f_i(x)$ and quantum part $q_i(x)$.That is

$$\varphi_i(x) = f_i(x) + q_i(x) \tag{2.1}$$

The general form of the Lagrangian $L\,(f_i + q_i)$ that we will consider is

$$L\,(f_i, q_i) = \frac{1}{2} q_i M_{ij}(f_j) q_j + \frac{1}{3!} a_{ijk}(f_j) q_i q_j q_k + \frac{1}{4!} b_{ijkl} q_i q_j q_k q_l \tag{2.2}$$

The generating functional for Green's functions in the theory in the presence of a source function $J_i(x)$ is given by

$$Z[f_i, J_j] = \int dq_k \exp\left(\int dx\, [\, L\,(f_i + q_i) + J_i q_i]\right) \tag{2.3}$$

For simplicity let us deal only with one-loop effects. Then from the generating function we have to restrict our attention only to those terms in Eq. (2.2) that are bilinear in q_i.We thus consider only

$$L^{(2)} = \frac{1}{2} q_i M_{ij}(f_j) q_j \tag{2.4}$$

Upon substituting Eq. (2.4) into Eq. (2.3) we arrive at the one-loop generating functional

$$Z[f_i, 0] = \int dq_k \exp\left(\int dx \left[\frac{1}{2} q_i M_{ij}(f_i) q_j\right]\right) \tag{2.5}$$

Evaluation of the functional integral in Eq. (2.5) involves a determinant which we call as superdeterminant of M_{ij}, as q_i may be either fermionic or bosonic [13-16].

That means from equation (2.5) the one-loop generating functional for Green's functions is

$$Z_1[f_i, 0] = s\det^{-1/2}[M_{ij}(f_i)] \tag{2.6}$$

Equation (2.6) tells us that we have to regularize the superdeterminant of the operator M_{ij} and it's inverse.

The superdeterminant of an operator Ω can be written as

$$\det\Omega = \exp(tr \ln\Omega) \tag{2.7}$$

Let us regularize $\ln\Omega$ in the following way:

$$\ln\Omega = -\lim_{\varepsilon\to 0} \frac{d^n}{d\varepsilon^n}\left(\frac{\varepsilon^{n-1}}{n!}\Omega^{-\varepsilon}\right), \text{ (n = 1, 2, 3, ...)} \tag{2.8}$$

In facing no divergences we can always choose n to be greater than or equal to the number of "loop momentum integrals" or in other words order in $\hbar$.

Hence,

$$\det\Omega = \exp\left[tr\left\{ -\lim_{\varepsilon\to 0} \frac{d^n}{d\varepsilon^n}\left(\frac{\varepsilon^{n-1}}{n!}\Omega^{-\varepsilon}\right)\right\}\right] \tag{2.9a}$$

and

$$\begin{aligned}\Omega^{-m} &= \frac{(-1)^{m-1}}{(m-1)!}\frac{d^m}{d\Omega^m}(\ln\Omega) \\ &= -\lim_{\varepsilon\to 0}\frac{d^n}{d\varepsilon^n}\left(\frac{\varepsilon^{n-1}}{n!}\frac{\Gamma(\varepsilon+m)}{\Gamma(m)\Gamma(\varepsilon)}\Omega^{-\varepsilon-m}\right)\end{aligned} \tag{2.9b}$$

If we now rewrite $\Omega^{-\varepsilon}$ as

$$\Omega^{-\varepsilon} = \frac{1}{\Gamma(\varepsilon)}\int_0^\infty dt\; t^{\varepsilon-1}\exp(-\Omega t) \tag{2.10}$$

in eq. (2.9) we arrive at the result

$$\det \Omega = \exp[-\xi'(0)] \tag{2.11a}$$

where we have defined the ξ - function

$$\xi(\varepsilon) = \frac{1}{\Gamma(\varepsilon)}\int_0^\infty dt t^{\varepsilon-1} tr \exp(-\Omega t) \tag{2.11b}$$

This is the usual ξ - function regularization of the determinant of an operator.

Equations (2.8) and (2.9) are the main steps of the Operator regularization which is used in (2.6) to evaluate the Green's function of any problem. From this point we can divide the prescription in two fold way. That means if we use Schwinger expansion for the operator like

$$\det \Omega = \exp\left\{-\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}\left[\frac{1}{\Gamma(\varepsilon)}\int_0^\infty dt t^{\varepsilon-1} tr\left(e^{-\Omega_0 t} - t e^{\Omega_0 t}\Omega_I + \frac{t^2}{2}\int_0^1 du e^{-(1-u)\Omega_0 t}\ \Omega_I e^{-u\Omega_0 t}\Omega_I \right.\right.\right.$$
$$\left.\left.\left. -\frac{t^3}{3}\int_0^1 du u \int_0^1 dv e^{-(1-u)\Omega_0 t}\Omega_I e^{-u(1-v)\Omega_0 t}\Omega_I e^{-uv\Omega_0 t}\Omega_I + \ldots\ldots\right)\right]\right\} \tag{2.12}$$

where, $\Omega = \Omega_0 + \Omega_I$ with Ω_0 is independent of the background field f_i and Ω_I is at least linear in f_i. Then following the steps described in ref.[3] we can find the result of the problems in consideration.

Also these equations can be used in evaluating Feynman loop-diagrams. For one-loop take n =1, for two-loops take n = 2 and so on.

Following (2.9b) we can write the general prescription of Operator regularization for the Feynman diagrams as follows [4]:

$$\Omega^{-m} = \lim_{\varepsilon\to 0}\frac{d^n}{d\varepsilon^n}\left[\left(1+\alpha_1\varepsilon+\alpha_2\varepsilon^2+\ldots+\alpha_n\varepsilon^n\right)\frac{\varepsilon^n}{n!}\Omega^{-\varepsilon-m}\right] \tag{2.13}$$

where the α_n s are arbitrary. For one-loop diagrams it is enough to use n = 1.When m = 2 and n = 1, then eq. (2.13) taken the form

$$\Omega^{-2} = \lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}\left[\varepsilon(1+\alpha\varepsilon)\Omega^{-\varepsilon-2}\right] \tag{2.14}$$

Now applying this to the three divergent one loop Feynman diagrams in QED.

2.1 One Loop Correction to the Fermion Line in (3+1) Dimensions

Starting with the Feynman diagram for the one loop correction to the fermions line which is represented by $\left(\sum(p)\right)$:

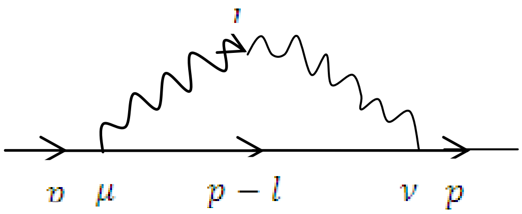

Fig.-1: One loop Feynman diagram for external fermion lines.

Using the Feynman rules one can write $\left(\sum(p)\right)$ as,

$$\left(\sum(p)\right) = -ie^2\int\frac{d^4 l}{(2\pi)^4}\gamma_\mu\frac{(p-l-m)}{\left[(p-l)^2-m^2\right]^2}\gamma_\mu$$

Using the Feynman identity, we can write

$$\left(\sum(p)\right) = -ie^2\int_0^1 dx\int\frac{d^4 l}{(2\pi)^4}\frac{\gamma_\mu(p-l-m)\gamma_\mu}{\left[(p-l)^2 x - m^2 x + l^2(1-x)\right]^2}$$

Shifting the variable of integration as $l' = l - px$ and simplifying we get

$$\left(\sum(p)\right) = -ie^2 \int_0^1 dx \int \frac{d^4 l'}{(2\pi)^4} \frac{\gamma_\mu [\not p(1-x) - m - \not l'] \gamma_\mu}{\left[l'^2 - m^2 x + p^2 x(1-x)\right]^2}$$

The term linear in l' integrates to zero because of symmetric integration, so

$$\left(\sum(p)\right) = -ie^2 \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \frac{\gamma_\mu [\not p(1-x) - m] \gamma_\mu}{\left[l^2 - m^2 x + p^2 x(1-x)\right]^2} \tag{2.1.1}$$

Which is taken as the common starting point for both Dimensional and Operator regularization.

Now proceeding with operator regularization, following the rule cited in Eqs. (2.11) and (2.12), the above result becomes,

$$\left(\sum(p)\right) = -ie^2 \int_0^1 dx \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \int \frac{d^4 l}{(2\pi)^4} \frac{\varepsilon(1+\alpha\varepsilon) \gamma_\mu [\not p(1-x) - m] \gamma_\mu}{\left[l^2 - m^2 x + p^2 x(1-x)\right]^{\varepsilon+2}} \tag{2.1.2}$$

Using the standard integral

$$\int \frac{d^{2w} l}{(2\pi)^{2w}} \frac{1}{\left(l^2 + M^2\right)^A} = \frac{1}{(4\pi)^w \Gamma(A)} \frac{\Gamma(A-w)}{\left(M^2\right)^{A-w}} \tag{2.1.3}$$

we get,

$$\left(\sum(p)\right) = -ie^2 \frac{1}{(4\pi)^2} \int_0^1 dx\, \gamma_\mu [\not p(1-x) - m] \gamma_\mu \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left[\frac{\varepsilon(1+\alpha\varepsilon)}{\Gamma(\varepsilon+2)} \cdot \frac{\Gamma(\varepsilon)}{\left(-m^2 x + p^2 x(1-x)\right)^\varepsilon} \right] \tag{2.1.4}$$

Here,

$$\lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left[\frac{\varepsilon(1+\alpha\varepsilon)}{\Gamma(\varepsilon+2)} \cdot \frac{\Gamma(\varepsilon)}{\left(-m^2 x + p^2 x(1-x)\right)^\varepsilon} \right]$$

$$= \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left[\frac{\varepsilon(1+\alpha\varepsilon)}{(\varepsilon+1) u^\varepsilon} \right], \text{ Taking } u = -m^2 x + p^2 x(1-x)$$

$$= \lim_{\varepsilon \to 0} \left[\frac{(1+\varepsilon) u^\varepsilon \alpha - (1+\alpha\varepsilon)\left\{u^\varepsilon + (1+\varepsilon) u^\varepsilon \ln u\right\}}{\left\{(1+\varepsilon) u^\varepsilon\right\}^2} \right]$$

$$= \alpha - 1 - \ln u \tag{2.1.5}$$

Therefore Eqn. (2.1.5) becomes,

$$\left(\sum(p)\right) = -ie^2 \frac{1}{(4\pi)^2} \int_0^1 dx\, \gamma_\mu [\not p(1-x) - m] \gamma_\mu \left[\alpha - 1 - \ln\left(-m^2 x + p^2 x(1-x)\right)\right]$$

The advantage of this method is that here we can use 4-dimensional γ -algebra. We do not have to go from 4 to n as in dimensional regularization.

Doing the γ -algebra we arrive at

$$\left(\sum(p)\right) = -2ie^2 \frac{1}{(4\pi)^2} \int_0^1 dx\, [\not p(1-x) + 2m] \left[\alpha - 1 - \ln\left(-m^2 x + p^2 x(1-x)\right)\right] \tag{2.1.6}$$

Let us now separate the finite part and divergent part from (2.1.6) as follows:

$$\left(\sum(p)\right) = -2ie^2 \frac{1}{(4\pi)^2} \int_0^1 dx\, [\not p(1-x) + 2m] \left[\alpha - 1 - \ln\left(-m^2 x + p^2 x(1-x)\right) - \ln\mu^2 + \ln\mu^2\right]$$

$$= -2ie^2 \frac{1}{(4\pi)^2} \int_0^1 dx\, [\not p(1-x) + 2m] \left[\left(\alpha + \ln\mu^2\right) - 1 - \ln\frac{-m^2 x + p^2 x(1-x)}{\mu^2}\right]$$

$$= -2ie^2 \frac{1}{(4\pi)^2} \int_0^1 dx\, [\not p(1-x) + 2m] \left[\alpha - 1 - \ln\frac{-m^2 x + p^2 x(1-x)}{\mu^2}\right]$$

where μ^2 taken from the arbitrary α .

Thus the finite part of $\left(\sum(p)\right)$ is,

$$-2i\,\frac{e^2}{(4\pi)^2}\int_0^1 dx\,\left[\not{p}(1-x)+2m\right]\left[\ln\frac{-m^2x+p^2x(1-x)}{\mu^2}\right] \tag{2.1.7}$$

and the divergent part is,

$$-2i\,\frac{e^2}{(4\pi)^2}\int_0^1 dx\,\left[\not{p}(1-x)+2m\right]\,(\alpha-1)$$

$$=-i\,\frac{e^2}{(4\pi)^2}\,(\alpha-1)\,\left[\not{p}+4m\right] \tag{2.1.8}$$

Comparing this to the result from Dimensional regularization [17-18], the finite part is

$$\frac{e^2}{(4\pi)^2}\int_0^1 dx\,\left[\not{p}(1-x)+2m\right]\ln\left[\frac{p^2x(1-x)-m^2x}{4\pi\mu^2}\right] \tag{2.1.9}$$

and the divergent part is

$$\frac{e^2}{(4\pi)^2}\left(\frac{2}{\varepsilon}-\gamma-1\right)\left[\not{p}+4m\right]-\frac{e^2}{(4\pi)^2}\;2m \tag{2.1.10}$$

where $\gamma\approx 0.5772$ is the Euler- Mascheroni constant.

There is a constant difference between these two methods that stems from dimensionally continuing the gamma matrices, but the resulting over all phase should not effect the physics.

2.2 One Loop Correction to the Boson (Photon) Line in (3+1) Dimensions

Let us consider the Feynman diagram for the one loop correction to the photon line which is represented by $\Pi_{\mu\nu}(p)$:

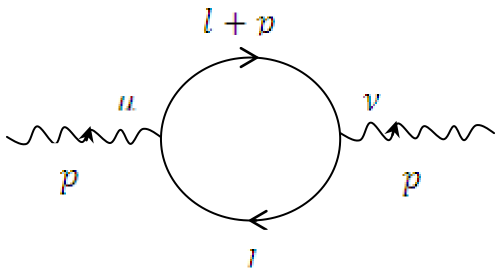

Fig.-2: One loop Feynman diagram for external boson lines.

The QED one loop correction to the photon line in 4-dimensions is

$$\Pi_{\mu\nu}(p)=e^2\int\frac{d^4l}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{\gamma_\mu(\not{l}+\not{p}-m)\gamma_\nu(\not{l}-m)}{\left[(l+p)^2-m^2\right]\left(l^2-m^2\right)}\right]$$

Combining the denominator using the Feynman identity and simplifying, we get

$$\Pi_{\mu\nu}(p)=e^2\int_0^1 dx\int\frac{d^4l}{(2\pi)^4}\,\frac{Tr\left[\gamma_\mu(\not{l}+\not{p}-m)\gamma_\nu(\not{l}-m)\right]}{\left[(l+p)^2x-m^2x+\left(l^2-m^2\right)(1-x)\right]^2} \tag{2.2.1}$$

Now putting $l'=l+px$ in Eq. (2.2.1), then we get,

$$\Pi_{\mu\nu}(p)=e^2\int_0^1 dx\int\frac{d^4l'}{(2\pi)^4}\,\frac{Tr\left[\gamma_\mu(\not{l}'-\not{p}x+\not{p}-m)\gamma_\nu(\not{l}'-\not{p}x-m)\right]}{\left[(l'+p(1-x))^2x-m^2x+\left((l'-px)^2-m^2\right)(1-x)\right]^2}$$

$$=\frac{N_{\mu\nu}}{D_{\mu\nu}}$$

Here

$$D_{\mu\nu}=\left[\{l'+p(1-x)\}^2x-m^2x\;+\left\{(l'-px)^2-m^2\right\}(1-x)\right]^2$$

$$=\left[l'^2 x+2l'p(1-x)x+p^2(1-x)^2 x-m^2 x+l'^2(1-x)-2l'p(1-x)+p^2x^2(1-x)-m^2+m^2x\right]^2$$
$$=\left[l'^2-m^2+p^2x(1-x)\right]^2$$

and

$$N_{\mu\nu}= Tr\left[\gamma_\mu\{l'+p(1-x)-m\}\gamma_\nu(l'-px-m)\right]$$
$$=Tr\left[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\{l'+p(1-x)\}_\alpha\ (l'-px)_\beta+m^2\gamma_\mu\gamma_\nu\right]$$
$$=4\left[(\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\nu}\delta_{\alpha\beta}+\delta_{\mu\beta}\delta_{\alpha\nu})\{l'+p(1-x)\}_\alpha\ (l'-px)_\beta+m^2\delta_{\mu\nu}\right]$$
$$=4\left[2l'_\mu l'_\nu-2x(1-x)(p_\mu p_\nu-p^2\delta_{\mu\nu})-\delta_{\mu\nu}\{l'^2-m^2+p^2x(1-x)\}\right]$$

Hence equation (2.2.1) with $l'\to l$ becomes,

$$\Pi_{\mu\nu}(p)=4e^2\int_0^1 dx\int\frac{d^4l}{(2\pi)^4}\left[\frac{2l_\mu l_\nu}{\{l^2-m^2+p^2x(1-x)\}^2}-\frac{2x(1-x)(p_\mu p_\nu-p^2\delta_{\mu\nu})}{\{l^2-m^2+p^2x(1-x)\}^2}-\frac{\delta_{\mu\nu}}{\{l^2-m^2+p^2x(1-x)\}}\right] \tag{2.2.2}$$

If we apply the following integrals in the first and third terms in the integrand of equation (2.2.2),

I. $$\int d^dl\frac{l_\mu l_\nu}{(l^2+2lq-m^2)^\alpha}=\frac{i\pi^{d/2}}{\alpha}\frac{1}{(-q^2-m^2)^{\alpha-d/2}}\ \text{x}\ \left[q_\mu q_\nu\Gamma\alpha-d/2+\frac{1}{2}g_{\mu\nu}(-q^2-m^2)\Gamma(\alpha-1-d/2)\right]$$

II. $$\int d^dl\frac{1}{(l^2+2lq-m^2)^\alpha}=(-1)^\alpha i\pi^{d/2}\frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)(-q^2-m^2)^{\alpha-d/2}}$$

We arrive at,

$$\Pi_{\mu\nu}(p)=-8e^2\left(p_\mu p_\nu-p^2\delta_{\mu\nu}\right)\int_0^1 dx\int\frac{d^4l}{(2\pi)^4}\left[\frac{x(1-x)}{\{l^2-m^2+p^2x(1-x)\}^2}\right] \tag{2.2.3}$$

Which is again taken as the common starting point for both Dimensional and Operator regularization for one loop correction to the photon lines.

Using the operator regularization rule which describe in section-2 in above eq., we obtain,

$$\Pi_{\mu\nu}(p)=8e^2\left(p_\mu p_\nu-p^2\delta_{\mu\nu}\right)\int_0^1 dx\lim_{\varepsilon\to0}\frac{d}{d\varepsilon}\int\frac{d^4l}{(2\pi)^4}\frac{\varepsilon(1+\alpha\varepsilon)x(1-x)}{\left[l^2-m^2+p^2x(1-x)\right]^{\varepsilon+2}} \tag{2.2.4}$$

Due to the momentum integral (2.1.3), from eq. (2.2.4) we get, $A=\varepsilon+2$, $w=2$, $M^2=-m^2+p^2x(1-x)$, then Eq. (2.2.4) becomes,

$$\Pi_{\mu\nu}(p)=-8e^2\frac{1}{(4\pi)^2}\left(p_\mu p_\nu-p^2\delta_{\mu\nu}\right)\int_0^1 dx\ x(1-x)\lim_{\varepsilon\to0}\frac{d}{d\varepsilon}\left[\frac{\varepsilon(1+\alpha\varepsilon)}{\Gamma(\varepsilon+2)}\cdot\frac{\Gamma(\varepsilon)}{(-m^2+p^2x(1-x))^\varepsilon}\right] \tag{2.2.5}$$

From Eq. (2.1.5) we get,

$$\lim_{\varepsilon\to0}\frac{d}{d\varepsilon}\left[\frac{\varepsilon(1+\alpha\varepsilon)}{\Gamma(\varepsilon+2)}\cdot\frac{\Gamma(\varepsilon)}{(-m^2+p^2x(1-x))^\varepsilon}\right]=\alpha-1-\ln u,\quad \text{where } u=-m^2+p^2x(1-x).$$

Thus equation (2.1.4) becomes,

$$\Pi_{\mu\nu}(p)=8e^2\frac{1}{(4\pi)^2}\left(p_\mu p_\nu-p^2\delta_{\mu\nu}\right)\int_0^1 dx\ x(1-x)\left[\alpha-1-\ln(-m^2+p^2x(1-x))\right]$$
$$=8e^2\frac{1}{(4\pi)^2}\left(p_\mu p_\nu-p^2\delta_{\mu\nu}\right)\int_0^1 dx\ x(1-x)\left[\alpha-1-\ln\frac{-m^2+p^2x(1-x)}{4\pi\mu^2}\right]$$

where μ^2 taken from the arbitrary α .

Therefore the finite part of $\Pi_{\mu\nu}(p)$ is,

$$8e^2 \frac{1}{(4\pi)^2}\left(p_\mu p_\nu - p^2\delta_{\mu\nu}\right)\int_0^1 dx\; x(1-x)\left[\ln\frac{p^2x(1-x)-m^2}{4\pi\mu^2}\right] \tag{2.2.6}$$

and the divergent part is,

$$-8e^2 \frac{1}{(4\pi)^2}\left(p_\mu p_\nu - p^2\delta_{\mu\nu}\right)\int_0^1 dx\; x(1-x)\,(\alpha-1)$$

$$=-\frac{4}{3}\frac{e^2}{(4\pi)^2}\left(p_\mu p_\nu - p^2\delta_{\mu\nu}\right)(\alpha-1) \tag{2.2.7}$$

Comparing this against the result of Dimensional regularization [17-18], the finite part is

$$\frac{-8ie^2\left(p_\mu p_\nu - \delta_{\mu\nu}p^2\right)}{(4\pi)^2}\int_0^1 dx\; x(1-x)\left[\ln\left(\frac{p^2x(1-x)-m^2}{2\pi\mu^2}\right)\right] \tag{2.2.8}$$

and the divergent part of $\Pi_{\mu\nu}(p)$ is,

$$\frac{-8ie^2\left(p_\mu p_\nu - \delta_{\mu\nu}p^2\right)}{(4\pi)^2}\int_0^1 dx\; x(1-x)\left(\frac{2}{\varepsilon}-\gamma\right)$$

$$=\frac{-4ie^2\left(p_\mu p_\nu - \delta_{\mu\nu}p^2\right)}{3(4\pi)^2}\left(\frac{2}{\varepsilon}-\gamma\right) \tag{2.2.9}$$

We see that both results are the same in form.

2.3 One Loop Correction to the Vertex in (3+1) Dimensions

Let us now consider the Feynman diagram for the one loop correction to the vertex which is represented by $\Gamma_\rho(p,q)$.

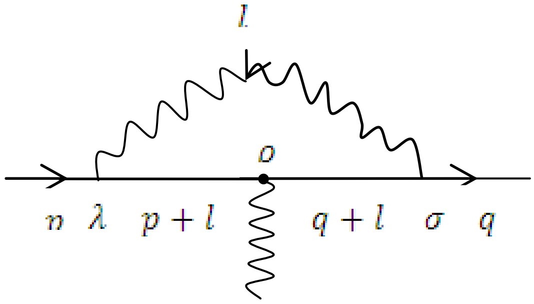

Fig.-3: One loop Feynman diagram for vertex function**.**

The QED one loop correction to the vertex in 4-dimensions is

$$\Gamma_\rho(p,q) = \int\frac{d^4l}{(2\pi)^4}\left[(-ie\gamma_\lambda)\frac{i}{(\not p+\not l+m)}(-ie\gamma_\rho)\frac{i}{\not q+\not l+m}(-ie\gamma_\sigma)\frac{\delta_{\tau\sigma}}{l^2}\right]$$

$$=-ie^3\int\frac{d^4l}{(2\pi)^4}\;\frac{\left[\gamma_\lambda(\not p+\not l-m)\gamma_\rho(\not q+\not l-m)\gamma_\lambda\right]}{l^2\left[(p+l)^2-m^2\right]\left[(q+l)^2-m^2\right]} \tag{2.3.1}$$

Now we introduce the 3-parameter Feynman formula for combining the denominator,

$\frac{1}{abc} = 2\int_0^1 dx\int_0^{1-x} dy\frac{1}{\left[a(1-x-y)+bx+cy\right]^3}$ in the eqn. (2.3.1), we obtain,

$$\Gamma_\rho(p,q) = -2ie^3\int_0^1 dx\int_0^{1-x} dy\int\frac{d^4l}{(2\pi)^4}\;\frac{\left[\gamma_\lambda(\not p+\not l-m)\gamma_\rho(\not q+\not l-m)\gamma_\lambda\right]}{\left[l^2(1-x-y)+\left\{(p+l)^2-m^2\right\}x+\left\{(q+l)^2-m^2\right\}y\right]^3}$$

If we change the variables $l \to l + px + qy$ and simplify the denominator and numerator, we obtain,

$$\Gamma_\rho(p,q) = -2ie^3 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^4 l}{(2\pi)^4} \frac{\left[\gamma_\lambda (l + p(1+x) + qy - m)\gamma_\rho (l + q(1+y) + px - m)\gamma_\lambda\right]}{\left[l^2 - m^2(x+y) + p^2 x(1-x) + q^2 y(1-y) - 2p.qxy\right]^3}$$

This integral contains convergent and divergent pieces. The part of the numerator quadratic in l is divergent, the rest convergent, so separating the divergent piece $\Gamma^{(1)}{}_\rho(p,q)$ and convergent piece $\Gamma^{(2)}{}_\rho(p,q)$, i.e.

$$\Gamma_\rho(p,q) = \Gamma^{(1)}{}_\rho(p,q) + \Gamma^{(2)}{}_\rho(p,q) .$$

Thus the divergent piece is,

$$\Gamma^{(1)}{}_\rho(p,q) = -2ie^3 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^4 l}{(2\pi)^4} \frac{\gamma_\sigma l \gamma_\rho l \gamma_\sigma}{\left(l^2 - M^2\right)^3} \tag{2.3.2}$$

where, $M^2 \equiv m^2(x+y) - p^2 x(1-x) - q^2 y(1-y) + 2p.qxy$.

Which is taken as the common starting point for both Dimensional and Operator regularization for one-loop correction to the vertex.

Again, using the operator regularization rule which describe in section-2 in above eq., we obtain,

$$\Gamma^{(1)}{}_\rho(p,q) = -2ie^3 \int_0^1 dx \int_0^{1-x} dy \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \int \frac{d^4 l}{(2\pi)^4} \varepsilon(1+\alpha\varepsilon) \frac{\gamma_\sigma l \gamma_\rho l \gamma_\sigma}{\left(l^2 - M^2\right)^{\varepsilon+3}} \tag{2.3.3}$$

Now performing the momentum integral, we get

$$\Gamma^{(1)}{}_\rho(p,q) = -ie \frac{e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left[\frac{\varepsilon(1+\alpha\varepsilon)}{\Gamma(\varepsilon+3)} . \frac{\Gamma(\varepsilon)}{\left(M^2\right)^\varepsilon} \right] \gamma_\sigma \gamma_\lambda \gamma_\rho \gamma_\tau \gamma_\sigma \tag{2.3.4}$$

Here,

$$\lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left[\frac{\varepsilon(1+\alpha\varepsilon)}{\Gamma(\varepsilon+3)} . \frac{\Gamma(\varepsilon)}{(u)^\varepsilon} \right]$$, where u = M^2.

$$= \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left[\frac{(1+\alpha\varepsilon)}{\left(\varepsilon^2 + 3\varepsilon + 2\right)u^\varepsilon} \right]$$

$$= \lim_{\varepsilon \to 0} \left[\frac{\left(\varepsilon^2 + 3\varepsilon + 2\right)u^\varepsilon \alpha - (1+\alpha\varepsilon)\{(2\varepsilon+3)u^\varepsilon\}}{[\left(\varepsilon^2 + 3\varepsilon + 2\right)u^\varepsilon]^2} + \frac{\left(\varepsilon^2 + 3\varepsilon + 2\right)u^\varepsilon \ln u}{[\left(\varepsilon^2 + 3\varepsilon + 2\right)u^\varepsilon]^2} \right]$$

$$= \frac{2\alpha - 3 - 2\ln u}{4}$$

Again performing the γ -algebra in 4-dimentions, eq. (2.3.4) reduces to,

$$\Gamma^{(1)}{}_\rho(p,q) = -ie \frac{e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \frac{1}{4}(-3 + 2\alpha - 2\ln u)\, 4\gamma_\rho$$

Hence the finite part of $\Gamma^{(1)}{}_\rho(p,q)$ is,

$$2ie\, \gamma_\rho \frac{e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \ln\left(\frac{M^2}{\mu^2}\right) \tag{2.3.5}$$

where, $M^2 \equiv m^2(x+y) - p^2 x(1-x) - q^2 y(1-y) + 2p.qxy$

and the divergent part is

$$-ie\, \gamma_\rho \frac{e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy\, (-3 + 2\alpha)$$

$$=-ie\,\gamma_\rho\,\frac{e^2}{(4\pi)^2}\left(-\frac{3}{2}+\alpha\right) \qquad (2.3.6)$$

Comparing this result with the Dimensional regularization [17-18] result, the finite part is

$$\frac{-8e^3\gamma_\rho}{(4\pi)^2}\int_0^1 dx\int_0^{1-x} dy\left[\ln\left(\frac{M^2}{4\pi\mu^2}\right)\right] \qquad (2.3.7)$$

$$\text{where, } M^2 \equiv m^2(x+y)-p^2x(1-x)-q^2y(1-y)+2p.qxy$$

and the divergent part is,

$$\frac{-8e^3\gamma_\rho}{(4\pi)^2}\int_0^1 dx\int_0^{1-x} dy\;\left(\frac{2}{\varepsilon}-\gamma-1\right)$$

$$=\frac{-4e^3\gamma_\rho}{(4\pi)^2}\left(\frac{2}{\varepsilon}-\gamma-1\right) \qquad (2.3.8)$$

Which agree in form, recalling that Operator regularization goes further than Dimensional regularization in so much as that it actually removes the divergences.

III. Path Integral Form of Operator Regularization for One Loop Generating Functional in QED

Let us consider the QED Lagrangian,

$$L=-\frac{1}{4}\left(\partial_\mu\Omega_\mu-\partial_\nu\Omega_\nu\right)^2+\overline{\Psi}\left(-i\gamma^\mu\partial_\mu-e\gamma^\mu\Omega_\mu-m\right)\Psi-\frac{1}{2\alpha}\left(\partial_\mu\Omega^\mu\right)^2 \qquad (3.1)$$

Let us expand this Lagrangian taking background field quantization of the fields. Let the background field expansion of gauge field Ω_μ and fermionic field ψ are respectively,

$$\Omega_\mu=V_\mu+Q_\mu$$
$$\psi=\eta+q$$

where V_μ and η are the classical fields and Q_μ and q are the quantum fields.

Therefore Eq. (3.1) becomes,

$$L=-\frac{1}{4}\left(\partial_\mu V_\mu+\partial_\mu Q_\mu-\partial_\nu V_\nu-\partial_\nu Q_\nu\right)^2+\left(\overline{\eta}+\overline{q}\right)\left(-i\gamma^\mu\partial_\mu-e\gamma^\mu\left(V_\mu+Q_\mu\right)-m\right)(\eta+q)-\frac{1}{2\alpha}\left(\partial_\mu V_\mu+\partial_\mu Q_\mu\right)^2$$

$$=\frac{1}{4}\left[\left(\partial_\mu Q_\mu-\partial_\nu Q_\nu\right)+\left(\partial_\mu V_\mu-\partial_\nu V_\nu\right)\right]^2+\overline{\eta}\left(-i\gamma^\mu\partial_\mu-m\right)\eta+\overline{\eta}\left(-i\gamma^\mu\partial_\mu-m\right)q$$

$$+\overline{q}\left(-i\gamma^\mu\partial_\mu-m\right)\eta+e\left(\overline{\eta}\gamma^\mu V_\mu\eta+\overline{\eta}\gamma^\mu V_\mu q+\overline{\eta}\gamma^\mu Q_\mu\eta+\overline{\eta}\gamma^\mu Q_\mu q+\overline{q}\gamma^\mu V_\mu\eta\right.$$

$$\left.+\overline{q}\gamma^\mu V_\mu q+\overline{q}\gamma^\mu Q_\mu\eta+\overline{q}\gamma^\mu Q_\mu q\right)-\frac{1}{2\alpha}\left(\partial_\mu V_\mu+\partial_\mu Q_\mu\right)^2 \qquad (3.2)$$

To find the one-loop 1PI generating functional we need to consider only the terms in the Lagrangian bilinear in the quantum fields; then from Eq b .(3.2) we obtain

$$L^{(2)}=\overline{q}\gamma^\mu\left(-i\partial_\mu-eV_\mu\right)q-\left[\frac{1}{4}\left(\partial_\mu Q_\mu-\partial_\nu Q_\nu\right)^2+\frac{1}{2\alpha}\left(\partial_\mu Q_\mu\right)^2\right]-e\overline{\eta}\gamma^\mu Q_\mu q-e\overline{q}\gamma^\mu Q_\mu\eta$$

$$=\overline{q}\not{D}q-\frac{1}{2}Q_\mu\left[p^2\delta_{\mu\nu}-\left(1-\frac{1}{\alpha}\right)p_\mu p_\nu\right]Q_\nu-e\overline{\eta}\not{Q}q-e\overline{q}\not{Q}\eta \qquad (3.3)$$

$$\text{where, } \not{D}\equiv\gamma^\mu\left(-i\partial_\mu-eV_\mu\right)=\not{p}-e\not{V}.$$

The formalism of Section-**2** cannot be directly applied to the bilinear Lagrangian $L^{(2)}$ of Eq.(3.3) as q and $\overline{q}$ are independent quantum fields in the associated path integral. However, it is possible to rewrite $L^{(2)}$ in the form of Eq. (2.4) by the following device. We introduce the notation

$\theta = \begin{bmatrix} q \\ \bar{q}^T \end{bmatrix}$, $\theta^T = \lfloor q^T, \bar{q} \rfloor$ and we identify the quantum field $\begin{bmatrix} q_1 \\ q_2 \\ . \\ . \\ q_n \end{bmatrix}$ of the original formalism of the background field method with $\begin{bmatrix} Q_\mu \\ \theta \end{bmatrix}$ i.e. $\begin{bmatrix} Q_\mu \\ q \\ \bar{q}^T \end{bmatrix}$.

Thus the Lagrangian (3.3) can be written as,

$$L^{(2)} = \frac{1}{2}\begin{bmatrix} Q_\mu & q^T & \bar{q} \end{bmatrix} \begin{bmatrix} p^2\delta_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right)p_\mu p_\nu & -e\bar{\eta}\gamma_\mu & e\eta^T\gamma_\mu^T \\ e\gamma_\nu^T\bar{\eta}^T & 0 & -\not{D}^T \\ -e\gamma_\nu\eta & \not{D} & 0 \end{bmatrix} \begin{bmatrix} Q_\nu \\ q \\ \bar{q}^T \end{bmatrix}$$

$$= \frac{1}{2} h_i^T M_{ij} h_j \text{, where, } h_j = \begin{bmatrix} Q_\nu \\ q \\ \bar{q}^T \end{bmatrix} \tag{3.4}$$

Evaluation of the path integral (2.5) leads at once to the one-loop generating functional

$$Z_1 = \int dq_k \exp\left(\int dx \left[\frac{1}{2} h_i M_{ij}(f_i) h_j\right]\right) \tag{3.5}$$

Evaluation of the functional integral in Eq. (3.5) involves the "superdeterminant'' [13-16] of M_{ij} as h_i may be either fermionic or bosonic.

Now we can proceed in two ways: either (a) complete the square in the fermionic variables $\bar{f}$ and f or (b) complete the square in the bosonic variable b [1, 3].

Following ref. [1] let us complete the square in fermionic variables in the argument of the exponential on the right-hand side of Eq. (3.5), then we get

$$I = \int d\bar{f} df db \exp\left[\frac{1}{2} b^T M_{bb} b - b^T M_{bf} M_{\bar{f}f}^{-1} M_{\bar{f}b} b + \left(\bar{f} + b M_{bf} M_{\bar{f}f}^{-1}\right) M_{\bar{f}f}\left(f + M_{\bar{f}f}^{-1} M_{\bar{f}b} b\right)\right] \tag{3.6}$$

The change of variables,

$$f' = f + M_{\bar{f}f}^{-1} M_{\bar{f}b} b \text{ and } \bar{f}' = \bar{f} + b M_{bf} M_{\bar{f}f}^{-1} \text{, then we get,}$$

$$I = \int d\bar{f}' df' db \exp\left[\frac{1}{2} b^T \left(M_{bb} - 2M_{bf} M_{\bar{f}f}^{-1} M_{\bar{f}b}\right) b + \bar{f}' M_{\bar{f}f} f'\right] \tag{3.7}$$

Now using the standard Gaussian integrals

$$\int db \exp\left(\frac{1}{2} b^T A b\right) = \det{}^{-1/2} A$$

$$\text{and } \int d\bar{f}' df' \exp\left(\bar{f}' B f'\right) = \det B$$

in Eq. (3.7), we obtain,

$$I = \frac{\det M_{\bar{f}f}}{\det^{1/2}\left(M_{bb} - 2M_{bf} M_{\bar{f}f}^{-1} M_{\bar{f}b}\right)}$$

Thus one-loop generating functional for Green's functions Z_1 in Eq. (3.5) becomes,

$$Z_1 = \frac{\det \not{D}}{\det^{1/2}\left[p^2\delta_{\mu\nu} - \left(1-\frac{1}{\alpha}\right)p_\mu p_\nu - e^2\bar{\eta}\gamma_\mu \not{D}^{-1}\gamma_\nu\eta - e^2\bar{\eta}\gamma_\nu \not{D}^{-1}\gamma_\mu\eta\right]} = \frac{\det A}{\det B} \text{ (say)} \tag{3.8}$$

Here we see that Z_1 is the ratio of determinant of operators. Each of the determinants occurring in Eq. (3.8) requires regularization and a corresponding ξ-function. The numerator and denominator separately contribute to Green's functions with only external boson lines and with both external fermions lines and vertex function in massless QED respectively.

3.1 One-Loop Generating Functional and Loop Corrections for External Boson Lines

To find the loop corrections or to write the generating functional for external boson lines one has to make a close look at the numerator of eq. (3.8) and on the other hand for external fermion lines one has to take care of the denominator of eq. (3.8). So for bosonic case we have to regulate the $\det A$ through the use of ξ-function in Eq. (2.11a) yielding

$$\det A = Z_1^A = \exp\left[-\underset{\varepsilon\to 0}{Lit}\, \xi'^A(\varepsilon)\right], \tag{3.1.1}$$

$$\text{where, } \xi(\varepsilon) = \frac{1}{\Gamma(\varepsilon)}\int_0^\infty dt\, t^{\varepsilon-1} tr \exp(-\Omega t) \quad \text{with } \Omega = \not{p} - e\not{V}\ . \tag{3.1.2}$$

As we mentioned in section-**II**, after regularization we have to consider Schwinger expansion, to this view let us now identify the operator Ω_0 and Ω_I with $\not{p}$ and $-e\not{V}$, respectively, then by Eq. (2.10), Eq. (3.1.1) can be written as,

$$Z_1^A = \text{epx}\left[-\underset{\varepsilon\to 0}{Lim}\frac{d}{d\varepsilon}\left\{\frac{1}{\Gamma(\varepsilon)}\int_0^\infty dt t^{\varepsilon-1} tr\left(e^{-\not{p}t} - te^{-\not{p}t} + (-e\not{V})\frac{t^2}{2}\int_0^1 du e^{-(1-u)\not{p}t}(-e\not{V})e^{-u\not{p}t}(-e\not{V})\right.\right.\right.$$
$$\left.\left.\left. -\frac{t^3}{3}\int_0^1 du u\int_0^1 dv e^{-(1-u)\not{p}t}\,(-e\not{V})e^{-u(1-v)\not{p}t}\,(-e\not{V})e^{-uv\not{p}t}(-e\not{V}) + \ldots\ldots\right)\right\}\right] \tag{3.1.3}$$

To one-loop order this series plays the same role as Feynman rules in the usual perturbation theory. Here we want to evaluate the one-loop correction to the two-point function for external photon in QED; we restrict our attention to the term bilinear in V_μ on the right-hand side of Eq. (3.1.3). This leaves us with

$$Z_{1VV}^A = \exp\left[-\underset{\varepsilon\to 0}{Lim}\frac{d}{d\varepsilon}\left\{\frac{e^2}{\Gamma(\varepsilon)}\int_0^\infty dt\frac{t^{\varepsilon+1}}{2}\ tr\left(\int_0^1 du e^{-(1-u)\not{p}t}\not{V}e^{-u\not{p}t}\not{V}\right)\right\}\right] \tag{3.1.4}$$

Now let us complete the functional trace

$$\text{T} = tr\left(\int_0^1 du e^{-(1-u)\not{p}t}\not{V}e^{-u\not{p}t}\not{V}\right) \tag{3.1.5}$$

Schwinger has pointed out that such traces are most easily evaluated in momentum space. We introduce a complete orthonormal set of states $|\not{p}\rangle$ that are eigenstates of the operator p_μ, where, in n dimensions,

$$\langle x|\not{p}\rangle = e^{ip.x}\Big/(2\pi)^{n/2} \tag{3.1.6a}$$

and $$\langle \not{p}|f|\not{q}\rangle = f(\not{p}-\not{q})\Big/(2\pi)^{n/2} \tag{3.1.6b}$$

On the right-hand side of Eq. (3.1.6b), $f(\not{p}-\not{q})$ is the Fourier transform of $f(x)$:

$$f(\not{p}-\not{q}) = \int\frac{d^n x}{(2\pi)^{n/2}} f(x)e^{-ix.(\not{p}-\not{q})} \tag{3.1.7}$$

Equation (3.1.5) takes the form,

$$T=\int d^4p\,d^4q\,d^4r\,d^4s\ \langle p|e^{-(1-u)pt}|q\rangle\ \langle q|V|r\rangle\ \langle r|e^{-upt}|s\rangle\ \langle s|V|r\rangle \tag{3.1.8}$$

Upon inserting the complete set $1=\int d^4p|p\rangle\langle p|$ at the appropriate places, and using (3.1.6), we rewrite Eq. (3.1.8) as,

$$\begin{aligned}T &= \int d^4p\,d^4q\,d^4r\,d^4s\ \frac{V(p-q)}{(2\pi)^2}e^{-(1-u)pt}\cdot\frac{e^{ir.q}}{(2\pi)^2}\delta(r-q)\cdot\frac{V(r-s)}{(2\pi)^2}e^{-urt}\cdot\frac{e^{is.p}}{(2\pi)^2}\delta(s-p)\\ &= \int\frac{d^4p\,d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^2}\ e^{-(1-u)pt}V(p-q)V(q-s)\ e^{-uqt}e^{is.p}\delta(s-p)\\ &= \int\frac{d^4p\,d^4q}{(2\pi)^4}\ e^{-(1-u)pt-uqt}V(p-q)V(q-p)\end{aligned} \tag{3.1.9}$$

After shifting the variable of integration $p\to p+q$, Eq. (3.1.9) becomes,

$$T=\int\frac{d^4p\,d^4q}{(2\pi)^4}\ e^{-[q+(1-u)p]t}V(p)V(-p) \tag{3.1.10}$$

Upon substituting Eq. (3.1.10) into Eq. (3.1.4), we find that

$$Z_{1VV}^{A}=\exp\left[-\lim_{\varepsilon\to 0}\xi_{VV}'^{A}(\varepsilon)\right] \tag{3.1.11a}$$

where, $$\xi_{VV}^{A}(\varepsilon)=\frac{e^2}{2\Gamma(\varepsilon)}\int_0^\infty dt\ t^{\varepsilon+1}\int_0^1 du\ \int d^4p\,V(p)V(-p)\ \int\frac{d^4q}{(2\pi)^4}e^{-[q+(1-u)p]t} \tag{3.1.11b}$$

We use Eq. (2.10) to integrate over t, then (3.1.11b) becomes,

$$\begin{aligned}\xi_{VV}^{A}(\varepsilon) &= \frac{e^2}{2\Gamma(\varepsilon)}\int d^4p\,V(p)V(-p)\int_0^1 du\int\frac{d^4q}{(2\pi)^4}\frac{\Gamma(\varepsilon+2)}{[q+(1-u)p]^{\varepsilon+2}}\\ &= \frac{e^2\Gamma(\varepsilon+2)}{2\Gamma(\varepsilon)}\int d^4p\,V(p)V(-p)\int_0^1 du\int\frac{d^4q}{(2\pi)^4}\frac{[q-(1-u)p]^{\varepsilon+2}}{\left[q^2-(1-u)^2p^2\right]^{\varepsilon+2}}\end{aligned} \tag{3.1.12}$$

Now the last integral I_1 (say) of Eq. (3.1.12) can be calculated as follows:

$$\begin{aligned}I_1 &= \int\frac{d^4q}{(2\pi)^4}\frac{[q-(1-u)p]^{\varepsilon+2}}{\left[q^2-(1-u)^2p^2\right]^{\varepsilon+2}}\\ &= \int\frac{d^4q}{(2\pi)^4}\frac{\left[q^{s+2}-(\varepsilon+2)(q)^{\varepsilon+1}(1-u)p+\ldots\ldots+(1-u)^{\varepsilon+2}(p)^{\varepsilon+2}\right]}{\left[q^2-(1-u)^2p^2\right]^{\varepsilon+2}}\end{aligned}$$

Differentiating eq. (3.1.11b) with respect to ε and taking $\varepsilon\to 0$, we see that the product terms in ε will vanish. Hence in the numerator of I_1 only the first and last term will contribute.

$$\therefore\ I_1=\int\frac{d^4q}{(2\pi)^4}\left[\frac{\left(q^2\right)^{\frac{\varepsilon+2}{2}}}{\left(q^2-(1-u)^2p^2\right)^{\varepsilon+2}}+\frac{(1-u)^{\varepsilon+2}(p)^{\varepsilon+2}}{\left(q^2-(1-u)^2p^2\right)^{\varepsilon+2}}\right] \tag{3.1.13}$$

To evaluate this integral let us consider the standard integral,

$$\int\frac{d^nq}{(2\pi)^n}\frac{\left(q^2\right)^r}{\left(q^2+c^2\right)^m}=\frac{1}{\left(16\pi^2\right)^{n/4}}\left(c^2\right)^{(n/2)+r-m}\frac{\Gamma\left(r+n/2\right)\Gamma\left(m-r-n/2\right)}{\Gamma\left(n/2\right)\Gamma(m)} \tag{3.1.14}$$

Using eq. (3.1.14) in (3.1.13) we get,

$$I_1=\frac{1}{\left(16\pi^2\right)^{4/4}}\left[\left\{-(1-u)^2p^2\right\}^{\left(2+\frac{\varepsilon+2}{2}-(\varepsilon+2)\right)}\frac{\Gamma\left(\frac{\varepsilon+2}{2}+2\right)\Gamma\left(\varepsilon+2-\frac{\varepsilon+2}{2}-2\right)}{\Gamma(2)\Gamma(\varepsilon+2)}\right.$$

$$\left.+(1-u)^{\varepsilon+2}(p)^{\varepsilon+2}\left\{-(1-u)^2p^2\right\}^{(2+0-\varepsilon-2)}\cdot\frac{\Gamma(2)\Gamma(\varepsilon+2-2)}{\Gamma(2)\Gamma(\varepsilon+2)}\right]$$

$$=\frac{1}{\left(16\pi^2\right)}\frac{1}{\Gamma(\varepsilon+2)}\left[(-1)^{1-\frac{\varepsilon}{2}}(1-u)^{2-\varepsilon}(p)^{2-\varepsilon}\Gamma\left(\varepsilon/2+3\right)\Gamma\left(\frac{\varepsilon}{2}-1\right)+(-1)^{-\varepsilon}(1-u)^{2-\varepsilon}(p)^{\varepsilon+2}(p)^{-2\varepsilon}\Gamma(\varepsilon)\right]$$

Thus Eq. (3.1.12) becomes,

$$\xi_{VV}^{\Omega}(\varepsilon)=\frac{e^2\Gamma(\varepsilon+2)}{2\Gamma(\varepsilon)}\frac{1}{\left(16\pi^2\right)}\frac{1}{\Gamma(\varepsilon+2)}\int d^4pV(p)V(-p)\int_0^1du\left[(-1)^{1-\frac{\varepsilon}{2}}(1-u)^{2-\varepsilon}(p)^{2-\varepsilon}\Gamma\left(\varepsilon/2+3\right)\Gamma\left(\frac{\varepsilon}{2}-1\right)\right.$$

$$\left.+(-1)^{-\varepsilon}(1-u)^{2-\varepsilon}(p)^{\varepsilon+2}(p)^{-2\varepsilon}\Gamma(\varepsilon)\right]$$

$$=\frac{e^2}{32\pi^2}\frac{1}{\Gamma(\varepsilon)}\int d^4pV(p)V(-p)\left[(-1)^{1-\frac{\varepsilon}{2}}\frac{1}{3-\varepsilon}(p)^{2-\varepsilon}\Gamma\left(\varepsilon/2+3\right)\Gamma\left(\frac{\varepsilon}{2}-1\right)\right.$$

$$\left.+(-1)^{-\varepsilon}\frac{1}{3-s}(p)^{\varepsilon+2}(p)^{-2\varepsilon}\Gamma(\varepsilon)\right]$$

$$=\frac{e^2}{32\pi^2}\frac{1}{\Gamma(\varepsilon)}\int d^4pV(p)V(-p)$$

$$\left[(-1)^{2-\frac{\varepsilon}{2}}\frac{1}{\varepsilon-3}(p)^{2-\varepsilon}\left(\frac{\varepsilon}{2}+2\right)\left(\frac{\varepsilon}{2}+1\right)\left(\frac{\varepsilon}{2}\right)\Gamma\left(\varepsilon/2\right)\frac{\Gamma\left(\frac{\varepsilon}{2}-1\right)\left(\frac{\varepsilon}{2}-1\right)}{\left(\frac{\varepsilon}{2}-1\right)}+(-1)^{-\varepsilon+1}\frac{1}{\varepsilon-3}(p)^{\varepsilon+2}(p)^{-2\varepsilon}\Gamma(\varepsilon)\right]$$

$$=\frac{e^2}{32\pi^2}\frac{1}{\Gamma(\varepsilon)}\int d^4pV(p)V(-p)\left[(-1)^{2-\frac{\varepsilon}{2}}\frac{2}{(\varepsilon-3)(\varepsilon-2)}(p)^{2-\varepsilon}\left(\frac{\varepsilon+4}{2}\right)\left(\frac{\varepsilon+2}{2}\right)\left(\frac{\varepsilon}{2}\right)\left[\Gamma\left(\frac{\varepsilon}{2}\right)\right]^2\right.$$

$$\left.+(-1)^{-\varepsilon+1}\frac{1}{\varepsilon-3}(p)^{\varepsilon+2}(p)^{-2\varepsilon}\Gamma(\varepsilon)\right]$$

$$=\frac{e^2}{32\pi^2}\int d^4pV(p)V(-p)\left[(-1)^{2-\frac{\varepsilon}{2}}\frac{\left(\varepsilon^4+6\varepsilon^3+8\varepsilon^2\right)}{16\left(\varepsilon^2-5\varepsilon+6\right)}(p)^{2-\varepsilon}+(-1)^{1-\varepsilon}(\varepsilon-3)^{-1}(p)^{\varepsilon+2}(p)^{-2\varepsilon}\right] \qquad (3.1.15)$$

where we have used $\left[\Gamma\left(\frac{\varepsilon}{2}\right)\right]^2=\left(\frac{\varepsilon}{4}\right)\Gamma(\varepsilon)$.

Now differentiating Eq. (3.1.15) w. r. to ε, we get

$$\xi_{VV}'^{\Omega}(\varepsilon)=\frac{e^2}{32\pi^2}\int d^4pV(p)V(-p)+\frac{1}{16}(-1)^{2-\frac{\varepsilon}{2}}.\left\{\frac{\left(\varepsilon^2-5\varepsilon+6\right)\left(4\varepsilon^3+18\varepsilon^2+16\varepsilon\right)}{\left(\varepsilon^2-5\varepsilon+6\right)^2}\right.$$

$$\left.-\frac{\left(\varepsilon^4+6\varepsilon^3+8\varepsilon^2\right)(2\varepsilon-5)}{\left(\varepsilon^2-5\varepsilon+6\right)^2}\right\}(p)^{2-\varepsilon}+\frac{1}{16}(-1)^{2-\frac{\varepsilon}{2}}.\left(\frac{\varepsilon^4+6\varepsilon^3+8\varepsilon^2}{\varepsilon^2-5\varepsilon+6}\right)(p)^{2-\varepsilon}\ln(p)(-1)$$

$$+(-1)^{1-\varepsilon}\ln(-1)(-1)(\varepsilon-3)^{-1}(p)^{\varepsilon+2}(p)^{-2\varepsilon}+(-1)^{2-\varepsilon}(\varepsilon-3)^{-2}(p)^{\varepsilon+2}(p)^{-2\varepsilon}$$

$$\left.+(-1)^{1-\varepsilon}(\varepsilon-3)^{-1}(p)^{\varepsilon+2}\ln(p)(p)^{-2\varepsilon}+(-1)^{1-\varepsilon}(\varepsilon-3)^{-1}(p)^{\varepsilon+2}(p)^{-2\varepsilon}\ln(p)(-2)\right]$$

$$\therefore\ \xi'^{A}_{VV}(0) = \underset{\varepsilon\to 0}{Lim}\,\xi'^{A}_{VV}(\varepsilon)$$

$$= \frac{e^2}{32\pi^2}\int d^4p V(p)V(-p)\left[-\frac{1}{3}p^2\ln(-1)+\frac{1}{9}p^2+\frac{1}{3}p^2\ln(p)-\frac{2}{3}p^2\ln(p)\right]$$

$$= \frac{e^2}{96\pi^2}\int d^4p V(p)V(-p)p^2\left[\frac{1}{3}-\ln(-1)-\frac{1}{2}\ln p^2\right] \quad (3.1.16)$$

Substituting of Eq. (3.1.16) into Eq. (3.1.11a) yields our final expression for Z^{A}_{1VV} as,

$$Z^{A}_{1VV} = \exp\left[-\frac{e^2}{6(4\pi)^2}\int d^4p V(p)V(-p)p^2\left(\frac{1}{3}-\ln(-1)-\frac{1}{2}\ln p^2\right)\right] \quad (3.1.17)$$

This contributes to the to the one–loop generating functional for external bosons (photon) lines.

To find one-loop correction for external boson lines from above generating functional, we have to take logarithm on Eq. (3.1.17) and then functional differentiation of the expansion with respect to momentum p.

Thus the one-loop correction for the external boson lines is,

$$-\frac{e^2}{6(4\pi)^2}p^2\left(\frac{1}{3}-\ln(-1)-\frac{1}{2}\ln p^2\right)$$

$$= \frac{e^2}{96\pi^2}p^2\left(\frac{1}{2}\ln p^2-\ln(-1)-\frac{1}{3}\right) \quad (3.1.18)$$

where,

the finite part is $\frac{e^2}{96\pi^2}p^2\left(\frac{1}{2}\ln p^2-\frac{1}{3}\right)$ (3.1.19)

and the divergent part is $-\frac{e^2}{96\pi^2}p^2\ln(-1)$ (3.1.20)

The result in (3.1.18) is of the same form as we obtained by the diagrammatic form of Operator regularization and Dimensional regularization methods in section-**II**. In this section we have shown and explained how one can choose the appropriate terms from the Schwinger expansion for the problem in hand.

3.2 One-Loop Generating Functional and Loop Corrections for External Fermion Lines and Vertex Function

In this case we focus on the denominator in Eq. (3.8), so that let us regulate the $\det B$ through use of the ξ - function in Eq. (3.11a) yielding

$$\det B = Z^{B}_{1} = \exp\left[-\frac{1}{2}\underset{\varepsilon\to 0}{Lit}\,\xi'^{B}(\varepsilon)\right], \quad (3.2.1)$$

where,
$$\xi^{B}(\varepsilon)=\frac{1}{\Gamma(\varepsilon)}\int_0^\infty dt\ t^{\varepsilon-1}\ tr\exp\left\{-t\left[p^2\delta_{\mu\nu}-\left(1-\frac{1}{\alpha}\right)p_\mu p_\nu - e^2\bar{\eta}\gamma_\mu\frac{1}{p-eV}\gamma_\nu\eta - e^2\bar{\eta}\gamma_\nu\frac{1}{p-eV}\gamma_\mu\eta\right]\right\} \quad (3.2.2)$$

In Eq. (3.2.2) it is understood that the exponential is $tr[\exp(-tB)]$, where

$$B_{\mu\nu}\equiv p^2\delta_{\mu\nu}-\left(1-\frac{1}{\alpha}\right)p_\mu p_\nu - e^2\bar{\eta}\gamma_\mu\frac{1}{p-eV}\gamma_\nu\eta - e^2\bar{\eta}\gamma_\nu\frac{1}{p-eV}\gamma_\mu\eta$$
$$\equiv B_{0\mu\nu}+B_{I\mu\nu} \quad (3.2.3)$$

where, $B_{0\mu\nu}$ is independent of the background field η and $\bar{\eta}$, and $B_{I\mu\nu}$ is at least linear in η and $\bar{\eta}$.

Now as before to use Schwinger expansion in this case let us use the eq. (2.10) and then taking bilinear in η and $\overline{\eta}$ on the on the right-hand side of Eq. (3.2.1), we end up with

$$Z_1^B = \exp\left[-\frac{1}{2}\left\{\underset{s\to 0}{Lim}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty dt\ t^s\ tr\left\{\exp\left(p^2\delta_{\mu\nu}-\left(1-\frac{1}{\alpha}\right)p_\mu p_\nu\right)t\ \left(-2e^2\overline{\eta}\gamma_\mu D\!\!\!/^{-1}\gamma_\nu\eta\right)\right\}\right)\right\}\right] \quad (3.2.4)$$

The exponential factor in the trace of Eq. (3.2.4) can be simplified using the complete set of orthonormal projections operators:

$$T_{\mu\nu}(p) = \left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right) \quad (3.2.5a)$$

$$L_{\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2} \quad (3.2.5b)$$

These allows us to write $\left(e^{-tB_0}\right)_{\mu\nu}$ as

$$\exp\left(p^2\delta_{\mu\nu}-\left(1-\frac{1}{\alpha}\right)p_\mu p_\nu\right)t = \sum_{n=0}^{\infty}\frac{1}{n!}\left[p^2t\left(T_{\mu\nu}+\frac{1}{\alpha}L_{\mu\nu}\right)\right]^n$$
$$= e^{-tp^2}\,T_{\mu\nu} + e^{-tp^2/\alpha}\,L_{\mu\nu} \quad (3.2.6)$$

and let us expand $D\!\!\!/^{-1}$ in powers of the back-ground field in the ξ-function Eq. (3.2.4):

$$D\!\!\!/^{-1} = \frac{1}{p\!\!\!/ - eV\!\!\!/} = \frac{1}{p\!\!\!/}\left(1-eV\!\!\!/\frac{1}{p\!\!\!/}\right)^{-1}$$
$$=\left(\frac{1}{p\!\!\!/}+\frac{1}{p\!\!\!/}eV\!\!\!/\frac{1}{p\!\!\!/}+\frac{1}{p\!\!\!/}eV\!\!\!/\frac{1}{p\!\!\!/}eV\!\!\!/\frac{1}{p\!\!\!/}+\ldots.\ \ldots.\right)$$

It is interesting to note that at this stage this is straightforward to apply the perturbative expansion of Eq. (2.10) to this ξ-function and to select from the expansion those terms appropriate for any particular Greens function. This means that from the expansion we can choose appropriate terms that are associated with the related problems that we are interested in. Let us consider here the ξ-function for the fermion two-point function and the vertex function, we find

$$\xi^B(\varepsilon) \cong \frac{e^2}{\Gamma(\varepsilon)}\int_0^\infty dt\ t^\varepsilon\ tr\left[\overline{\eta}\gamma_\mu\left(\frac{1}{p\!\!\!/}+\frac{1}{p\!\!\!/}eV\!\!\!/\frac{1}{p\!\!\!/}\right)\gamma_\nu\,\eta\left(e^{-tp^2}\,T_{\mu\nu}+e^{-tp^2/\alpha}\,L_{\mu\nu}\right)\right] \quad (3.2.7)$$

Following ref. [19] in the approach-A, we compute from Eq. (3.2.7) the ξ-function in the limit of zero momentum transfer to the photon:

$$\xi^B(\varepsilon) = \frac{e^2}{8\pi^2}\int d^4p\left(\frac{1}{p^2}\right)\frac{\Gamma(1-\varepsilon)}{\Gamma(3-\varepsilon)}\left[2+\left(\alpha^{\varepsilon+1}-1\right)\frac{(2-\varepsilon)}{(1+\varepsilon)}\right]$$
$$\times\ \overline{\eta}(p)\left[p\!\!\!/+\frac{e}{4\pi^2}\left(V\!\!\!/(0)-2\varepsilon\frac{p\!\!\!/}{p^2}\,p.V(0)\right)\right]\eta(-p) \quad (3.2.8)$$

Therefore by Eq. (3.2.1) the contributions to the one-loop generating functional is

$$Z_1^B \cong \exp\left\{\frac{1}{2}\left[\frac{e^2}{8\pi^2}\int d^4p\left(\frac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2\right)\ \overline{\eta}(p)\left(p\!\!\!/+\frac{e}{4\pi^2}V\!\!\!/(0)\right)\eta(-p)\ -\alpha\frac{e^3}{16\pi^4}\right.\right.$$
$$\left.\left.\int d^4p\,\overline{\eta}(p)\,p\!\!\!/\,\eta(-p)\frac{p.V(0)}{p^2}\right]\right\} \quad (3.2.9)$$

This contributes to the one-loop generating functional for external fermion (electron) lines and vertex function in QED.

To obtain the one-loop correction for external fermion lines and vertex function, we have to take logarithm of Eq.(3.2.9) and then functional differentiation with respect to momentum p.

Hence from Eq. (3.2.9), we get

$$=\frac{e^2}{16\pi^2}\left(\frac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2\right)\overline{\eta}(p)\not{p}\,\eta(-p)+\frac{e^3}{64\pi^4}\left(\frac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2\right)\overline{\eta}(p)\ \not{V}(0)\eta(-p)-\alpha\frac{e^3}{32\pi^4}$$

$$\overline{\eta}(p)\gamma^{\mu}p_{\mu}\,\eta(-p)\frac{p^{\nu}V_{\nu}(0)}{p^2}$$

$$=\frac{e^2}{16\pi^2}\left(\frac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2\right)\overline{\eta}(p)\not{p}\,\eta(-p)+\frac{e^3}{64\pi^4}\left(\frac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2\right)\overline{\eta}(p)\ \not{V}(0)\eta(-p)$$

$$-\alpha\frac{e^3}{32\pi^4}\ \overline{\eta}(p)\gamma^{\mu}V_{\nu}(0)\eta(-p)\frac{p_{\mu}p^{\nu}}{p^2}$$

$$=\frac{e^2}{16\pi^2}\left(\frac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2\right)\overline{\eta}(p)\not{p}\,\eta(-p)+\frac{e^3}{64\pi^4}\left(\frac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2\right)\overline{\eta}(p)\ \not{V}(0)\eta(-p)$$

$$-\alpha\frac{e^3}{32\pi^4}\ \overline{\eta}(p)\gamma^{\nu}\delta_{\mu\nu}V_{\nu}(0)\eta(-p)\frac{p_{\mu}p^{\nu}}{p^2}$$

$$=\frac{e^2}{16\pi^2}\left(\frac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2\right)\overline{\eta}(p)\not{p}\,\eta(-p)+\frac{e^3}{64\pi^4}\left(\frac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2-2\alpha\right)\overline{\eta}(p)\ \not{V}(0)\eta(-p)$$

(3.2.10)

From the expansion (3.2.10) we can find the one-loop correction for the external fermion lines and one-loop vertex function by choosing the appropriate terms. This expression is of the same form as obtained by DR and OR methods with Feynman diagrams in section- **II**.

Thus the one-loop correction for the external fermion lines is,

$$\frac{e^2}{16\pi^2}\ \not{p}\left(\frac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2\right) \tag{3.2.11}$$

and the one-loop correction to the vertex function is,

$$\frac{e^3}{64\pi^4}\left(\frac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2-2\alpha\right) \tag{3.2.12}$$

The result in (3.2.11) and (3.2.12) is of the same form as we obtained by the diagrammatic form of Operator regularization and Dimensional regularization methods in section-**II**.

IV. Conclusion

In this paper we have evaluated basic QED loop diagrams in (3+1) dimensions with and without considering Feynman diagrams by the Operator regularization. We have compared the results of both procedure of Operator regularization and that of Dimensional regularization. We have seen that in both cases the result is of the same form with Dimensional regularization except a finite constant term difference. This will not affect the renormalization procedure.

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