

Dominator Chromatic Number of Interval Graphs

A. Sudhakaraiyah¹ and V. Raghava Lakshmi²

¹Department of Mathematics, S. V. University, Tirupati, Andhra Pradesh-517502, India

²Department of S&H (Mathematics), Vijaya Institute of Technology for Women, Vijayawada, Andhra Pradesh, India - 521108

Abstract: The present paper concentrates on the theory of dominator coloring in graphs and focuses on resolving the dominator chromatic number of interval graphs. Some categorized interval graphs are selected in this process of study. To facilitate the study and to establish the results, emphasis is given to the analogy between the nature and coherence of the intervals, which in turn played an essential role in determining the dominator chromatic number of the interval graphs. An interval graph is a circular- arc graph that can be represented with a set of arcs that do not cover the entire circle. A dominator coloring of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices are assigned with the same color and every vertex dominates all vertices of at least one color class, where a color class is the set of all vertices, having the same color. The minimum number of colors required for a dominator coloring of G is called the dominator chromatic number of G and is denoted by $\chi_d(G)$.

Mathematics Subject Classification: 05C15

Keywords: Chromatic number, dominator chromatic number, interval graphs

I. Introduction

Unless otherwise specified, throughout the paper the graph $G(V, E)$ is a finite and connected graph with the vertex set $V(G)$ and edge set $E(G)$ and standard definitions of graph theory as found in [1] are followed. Graphs presented in this article are all finite and connected Interval Graphs.

Interval graphs are a special class of circular- arc graphs that can be represented with a set of arcs that do not cover the entire circle. The extensive study of interval graphs has been done for several decades by both mathematicians and computer scientists. Let $I = \{I_1, I_2, I_3, \dots, I_k\}$ be an interval family, where each I_i is an interval on the real line and $I_i = [a_i, b_i]$, for $i = 1, 2, 3, \dots, k$. Here a_i is called the left end point and b_i is called the right end point. Without loss of generality, one can assume that, all end points of the intervals are distinct numbers between 1 and $2k$. The intervals are named in the increasing order of their left end points. The graph $G(V, E)$ is an interval graph if there is one-to-one correspondence between the vertex set V and the interval family I . Two vertices of G are joined by an edge if and only if their corresponding intervals in I intersect. That is if $I_i = [a_i, b_i]$ and $I_j = [a_j, b_j]$, then I_i and I_j will intersect if $a_i < b_j$ or $a_j < b_i$. Interval graphs are rich in combinatorial structures and have found applications in several disciplines such as traffic control, ecology, biology, computer sciences and particularly useful in cyclic scheduling and computers storage allocation problems etc. Having a representation of graph with intervals or arcs can be helpful in combinatorial problems of the graph, such as isomorphism testing and finding maximum independent set and cliques of graphs.

In section II, dominator chromatic number of some special classes of interval graphs is discussed. The dominator chromatic number of the graph G is the minimum number of colors required to properly color the graph, so that every vertex of the graph dominates an entire color class of the graph. Dominator chromatic number of the graph is denoted by $\chi_d(G)$.

The concept of dominator chromatic number was introduced by Gera et al. [2]. Dominating Set and Coloring have a number of applications and this has led to the algorithmic study of numerous variants of these problems. A number of basic combinatorial and algorithmic results on DC have been obtained in [2], [3], [4] and [5]. A systematic study of dominator coloring problem from the perspective of algorithms and complexity was initiated in [6].

II. Dominator chromatic numbers of Interval graphs

Theorem 2.1: Let $I = \{I_1, I_2, I_3, \dots, I_k\}$ be the interval family corresponding to an interval graph G . For any three consecutive intervals I_i, I_j and I_k , if I_j does not dominate any other interval except I_i and I_k , then

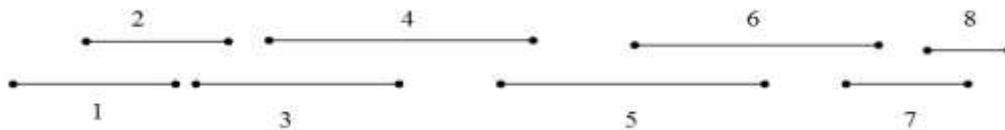
$$\chi_d(G) = 1 + \lceil k/3 \rceil; \text{ for } k=2, 3, 4, 5, 7 \\ = 2 + \lceil k/3 \rceil; \text{ otherwise}$$

Proof: Let G be the interval graph, whose interval family $I = \{I_1, I_2, I_3, \dots, I_k\}$ satisfies the condition mentioned in the theorem. By the hypothesis, the interval I_j intersects the interval I_2 .

the interval I_2 intersects the intervals I_1 and I_3 ;
 the interval I_3 intersects the intervals I_2 and I_4 ;
 the interval I_4 intersects the intervals I_3 and I_5 ;
;
;
 the interval I_{K-2} intersects the intervals I_{K-3} and I_{K-1} ;
 the interval I_{K-1} intersects the intervals I_{K-2} and I_K ;
 the interval I_k intersects the interval I_{K-1} .

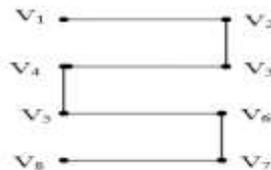
Let $v_1, v_2, v_3, \dots, v_k$ be the vertices corresponding to the intervals $I_1, I_2, I_3, \dots, I_k$ respectively. It follows that,
 $\text{nbrd } [v_i]=\{ v_{i-1}, v_i, v_{i+1} \}$ for $i=2,3,\dots,k-1$; $\text{nbrd } [v_1]=\{v_1, v_2\}$ and $\text{nbrd } [v_k]=\{v_{k-1}, v_k\}$, where
 $\text{nbrd } [v_i]=\{\text{the set of all vertices adjacent to } v_i\} \cup \{v_i\}$. Implies, G is a path graph. Hence, by [2]
 $\chi_d(G) = 1 + \lceil k/3 \rceil$; for $k=2, 3, 4, 5, 7$
 $= 2 + \lceil k/3 \rceil$; otherwise.

Illustration 2.1.1: Let the interval family $I = \{1, 2, 3, \dots, 8\}$ corresponding to the interval graph G be as follows:



Clearly the interval family satisfies the conditions mentioned in the theorem 2.1 for $k=8$. Therefore the dominator chromatic number of the graph G is $2 + \lceil 8/3 \rceil$ i.e., 4.

Verification: Let V_1, V_2, \dots, V_8 be the vertices of the interval graph G corresponding to the intervals 1, 2, 3, ..., 8 respectively. In this case, the graph G will be as follows:



Clearly the vertices form a path graph. Hence by [2]
 $\chi_d(G) = 2 + \lceil k/3 \rceil$; for $k = 8$
 $= 2 + \lceil 8/3 \rceil$
 $= 4$

Theorem 2.2: In the family of intervals $I = \{ I_1, I_2, \dots, I_{2k} \}$ corresponding to the Interval graph G , suppose that no other intersections are observed except the following:

- (i) I_i intersects no other interval except I_{i-1} which contains only the interval I_i
 for $i=2, 4, 6, \dots, 2k$ and
- (ii) I_i intersects no other non contained intervals except I_{i-2} and I_{i+2}
 for $i=3, 5, 7, \dots, 2(k-2)+1$

Then, $\chi_d(G) = k+1$ for $k \geq 3$, where $k \in \mathbb{N}$

Proof: Let the interval family $I = \{ I_1, I_2, \dots, I_{2k} \}$ corresponding to the interval family G satisfy the conditions mentioned in the theorem.

By first condition stated in the theorem

I_2 intersects no other interval except I_1 ;

I_4 intersects no other interval except I_3 ;

I_6 intersects no other interval except I_5 ;

.....;

.....;

I_{2k} intersects no other interval except I_{2k-1} ;

Every dominating set of G contains either I_i or I_{i-1} for $i=2, 4, 6, \dots, 2k$. As a result, the set of intervals $\{ I_1, I_3, \dots, I_{2k-1} \}$ and $\{ I_2, I_4, \dots, I_{2k} \}$ are minimal dominating sets of the graph G . It follows that, $\gamma(G) = K$.

By the first and second conditions stated in the theorem

I_1 intersects no other interval except I_2 and I_3 ;

I_3 intersects no other interval except I_1, I_4 and I_5 ;

I_5 intersects no other interval except I_3, I_6 and I_7 ;

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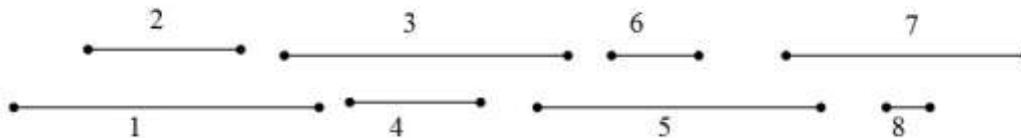
I_{2k-3} intersects no other interval except I_{2k-5}, I_{2k-2} and I_{2k-1} ;

I_{2k-1} intersects no other interval except I_{2k-3} , and I_{2k} ;

Partition the whole set of intervals I into two subsets A and B , where $A = \{ I_1, I_3, I_5, \dots, I_{2k-1} \}$ and $B = \{ I_2, I_4, \dots, I_{2k} \}$. Assign colors $C_1, C_2, C_3, \dots, C_k$ to the intervals $I_1, I_3, I_5, \dots, I_{2k-1}$ respectively of the set A . The set of intervals $\{ I_2, I_4, \dots, I_{2k} \}$ is independent set of intervals. The same color, say C_{k+1} that is different from the set of colors $C_1, C_2, C_3, \dots, C_k$ can be assigned to every interval in the set of intervals B . The coloring is a proper coloring. Every interval in the independent set of intervals B dominates at least one color class in the set of color classes $C_1, C_2, C_3, \dots, C_k$ as the graph G is a connected graph. Every color class corresponding to the intervals of set A consists of only one interval. As a result, every interval of A dominates its own color class. It follows that the coloring of the graph is a dominator coloring of the graph. Implies, $\chi_d(G) \leq k+1$.

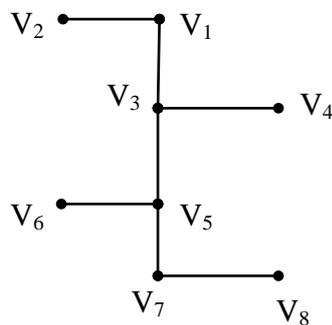
If not, let one of the intervals in the set A , say I_{2m-1} , $1 \leq m \leq k$ be assigned with a used color C_p , where $1 \leq p \leq k$ instead of the color C_m . The coloring is not a dominating coloring. In this case, the interval I_{2m} can neither dominate the color class C_{k+1} , as it contains all the intervals of the set B which are not dominated by the interval I_{2m} nor the color class C_p as the color class C_p contains the intervals I_{2p-1} and I_{2m-1} , among which, I_{2p-1} is not dominated by I_{2m} . That is at least k different colors are needed to color the k intervals of the set A and another new color for the intervals in the set B . Implies, $\chi_d(G) \geq k+1$. Hence, $\chi_d(G) = k+1$.

Illustration 2.2.1: Let the interval family $I = \{1, 2, 3, \dots, 8\}$ corresponding to the interval graph G be as follows:



Clearly the interval family satisfies the conditions mentioned in the theorem 2.2 for $k=4$. Therefore the dominator chromatic number of the graph G is $k+1$ i.e., 5

Verification: Let V_1, V_2, \dots, V_8 be the vertices of the interval graph G corresponding to the intervals 1, 2, 3, ..., 8 respectively. In this case, the graph G will be as follows:



Clearly the vertices V_1, V_3, V_5 & V_7 form a path graph. At least these four vertices must receive their unique color and one more color is needed for the rest of the vertices. Hence, $\chi_d(G) = 4+1=5$.

Theorem 2.3 : In the family of intervals $I = \{ I_1, I_2, \dots, I_{3k} \}$ corresponding to the Interval graph G , suppose that no other intersections are observed except the following:

- (i) The non-intersecting intervals I_i and I_{i-1} intersect no other interval except I_{i-2} which contains only the intervals I_i and I_{i-1} , for $i=3, 6, 9, \dots, 3k$ and
- (ii) I_i intersects no other non contained intervals except I_{i-3} and I_{i+3} ,
 for $i=4, 7, 11, \dots, 3(k-2)+1$

Then, $\chi_d(G) = k+1$ for $k \geq 3$, where $k \in \mathbb{N}$

Proof : Let the interval family $I = \{ I_1, I_2, \dots, I_{3k} \}$ corresponding to the interval family G satisfy the conditions mentioned in the theorem.

By first condition stated in the theorem

I_2 and I_3 intersect no other interval except I_1 ;

I_5 and I_6 intersect no other interval except I_4 ;

I_8 and I_9 intersect no other interval except I_7 ;

..... ;

.....;

I_{3k} and I_{3k-1} intersect no other interval except I_{3k-2} ;

Every dominating set of G contains I_i , for $i = 1, 4, 7, \dots, 3k-2$. As a result, the set of intervals $\{ I_1, I_4, \dots, I_{3k-2} \}$ is a minimum dominating set of the graph G . It follows that $\gamma(G) = K$.

By the first and second conditions stated in the theorem

I_1 intersects no other interval except I_2, I_3 and I_4 ;

I_4 intersects no other interval except I_1, I_5, I_6 and I_7 ;

I_7 intersects no other interval except I_4, I_8, I_9 and I_{10} ;

..... ;

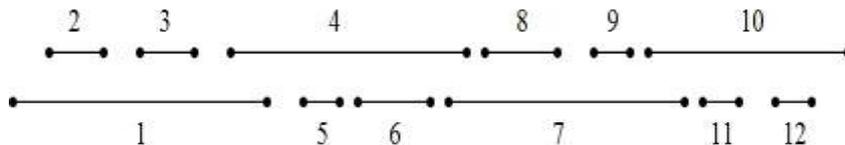
.....;

I_{3k-5} intersects no other interval except $I_{3k-8}, I_{3k-4}, I_{3k-3}$ and I_{3k-2} ;

I_{3k-2} intersects no other interval except I_{3k-5}, I_{3k-1} and I_{3k} ;

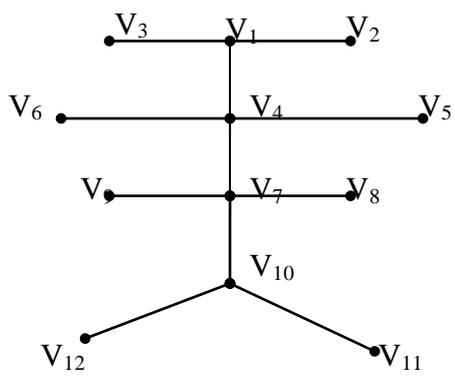
Partition the whole set of intervals I into two subsets A and B , where $A = \{ I_1, I_4, I_7, \dots, I_{3k-2} \}$ and $B = \{ I_2, I_3, I_5, I_6, I_8, I_9, \dots, I_{3k-4}, I_{3k-3}, I_{3k-1}, I_{3k} \}$. Assign colors $C_1, C_2, C_3, \dots, C_k$ to the intervals in the set A . The set B is an independent set of intervals. The same color, say C_{k+1} that is different from the set of colors $C_1, C_2, C_3, \dots, C_k$ can be assigned to every interval in the set of intervals B . The coloring is a proper coloring. Every interval in the independent set of intervals B is dominated by at least one color class from the set of color classes $C_1, C_2, C_3, \dots, C_k$ as the graph G is a connected graph. Every color class corresponding to the intervals of set A consists of only one interval. As a result, every interval of A dominates its own color class. It follows that the coloring of the graph is a dominator coloring of the graph. Implies, $\chi_d(G) \leq k+1$ and as in theorem 2.2 it can be proved that the coloring is a minimum dominator coloring. As a result, $\chi_d(G) = k+1$

2. 3. 1 Illustration: Let the interval family $I = \{1, 2, 3, \dots, 12\}$ corresponding to the interval graph G be as follows:



Clearly the interval family satisfies the conditions mentioned in the theorem 2.3 for $k=4$. Therefore the dominator chromatic number of the graph G is $k+1$ i.e., 5.

Verification: Let V_1, V_2, \dots, V_{12} be the vertices of the interval graph G corresponding to the intervals 1, 2, 3, ..., 12 respectively. In this case, the graph G will be as follows:



Clearly the vertices V_1, V_4, V_7 & V_{10} form a path graph. At least these four vertices must receive their unique color and one more color is needed for the rest of the vertices. Hence, it follows that $\chi_d(G) = 4+1=5$.

2.4 Generalized Theorem: In the family of intervals $I = \{ I_1, I_2, \dots, I_{mk} \}$ corresponding to the Interval graph G , suppose that no other intersections are observed except the following:

- (i) The non-intersecting intervals $I_i, I_{i-1}, I_{i-2}, \dots, I_{i-(m-2)}$ intersect no other interval except $I_{i-(m-1)}$ which contains only the intervals $I_i, I_{i-1}, I_{i-2}, \dots, I_{i-(m-2)}$ for $i = m, 2m, 3m, \dots, km$ and
- (ii) I_i intersects no other non contained intervals except I_{i-m} and I_{i+m}
for $i = m+1, 2m+1, 3m+1, \dots, m(k-2) + 1$

Then, $\chi_d(G) = k+1$ for $k \geq 3$, where $m, k \in \mathbb{N}$

Proof : Let the interval family $I = \{ I_1, I_2, \dots, I_{mk} \}$ corresponding to the interval family G satisfy the conditions mentioned in the theorem.

By first condition stated in the theorem

$I_2, I_3, I_4, \dots, I_m$ intersect no other interval except I_1 ;

$I_{m+2}, I_{m+3}, I_{m+4}, I_{m+5}, \dots, I_{2m}$ intersect no other interval except I_{m+1} ;

$I_{2m+2}, I_{2m+3}, I_{2m+4}, I_{2m+5}, \dots, I_{3m}$ intersect no other interval except I_{2m+1} ;

..... ;

..... ;

$I_{(k-1)m+2}, I_{(k-1)m+3}, I_{(k-1)m+4}, \dots, I_{km}$ intersect no other interval except $I_{(k-1)m+1}$

Every dominating set of G contains I_i for $i = 1, m+1, 2m+1, 3m+1, \dots, (k-1)m+1$

As a result the set of intervals $\{ I_1, I_{m+1}, I_{2m+1}, \dots, I_{(k-1)m+1} \}$ is a minimum dominating set of the graph G . It follows that $\gamma(G) = K$.

By the first and second conditions stated in the theorem

I_1 intersects no other interval except $I_2, I_3, \dots, I_m, I_{m+1}$;

I_{m+1} intersects no other interval except $I_1, I_{m+2}, I_{m+3}, \dots, I_{2m}, I_{2m+1}$

I_{2m+1} intersects no other interval except $I_{m+1}, I_{2m+2}, I_{2m+3}, \dots, I_{3m}, I_{3m+1}$

..... ;

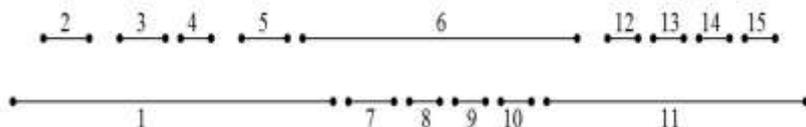
..... ;

$I_{(k-2)m+1}$ intersects no other interval except $I_{(k-3)m+1}, I_{(k-2)m+2}, I_{(k-2)m+3}, \dots, I_{(k-1)m+1}$

$I_{(k-1)m+1}$ intersects no other interval except $I_{(k-2)m+1}, I_{(k-1)m+2}, I_{(k-1)m+3}, \dots, I_{km}$

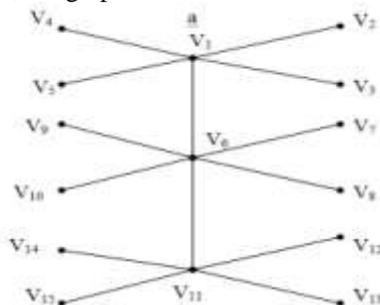
Partition the whole set of intervals I into two subsets A and B , where $A = \{ I_1, I_{m+1}, I_{2m+1}, \dots, I_{(k-1)m+1} \}$ and $B = \{ I_2, I_3, \dots, I_m, I_{m+2}, I_{m+3}, \dots, I_{2m}, \dots, I_{(k-1)m+2}, I_{(k-1)m+3}, \dots, I_{km} \}$. Assign colors $C_1, C_2, C_3, \dots, C_k$ to the intervals in the set A . The set B is an independent set of intervals. The same color, say C_{k+1} that is different from the set of colors $C_1, C_2, C_3, \dots, C_k$ can be assigned to every interval in the set of intervals B . The coloring is a proper coloring. Every interval in the independent set of intervals B is dominated by at least one color class from the set of color classes $C_1, C_2, C_3, \dots, C_k$ as the graph G is a connected graph. Every color class corresponding to the intervals of set A consists of only one interval. As a result, every interval of A dominates its own color class. It follows that the coloring of the graph is a dominator coloring of the graph. Implies, $\chi_d(G) \leq k+1$ and as in theorem 2. 2 it can be proved that the coloring is a minimum dominator coloring. As a result, $\chi_d(G) = k+1$

2.4.1 Illustration: Let the interval family $I = \{1, 2, 3, \dots, 15\}$ corresponding to the interval graph G be as follows:



Clearly the interval family satisfies the conditions mentioned in the theorem 2.4 for $k=3$ and $m=5$. Therefore the dominator chromatic number of the graph G is $k+1$ i.e., 4

Verification: Let V_1, V_2, \dots, V_{15} be the vertices of the interval graph G corresponding to the intervals 1, 2, 3, ..., 15 respectively. In this case, the graph G will be as follows:



Clearly the vertices $V_1, V_6,$ & V_{11} form a path graph. At least these three vertices must receive their unique color and one more color is needed for the rest of the vertices. Hence, $\chi_d(G) = 3+1=4$.

III. Conclusion

Resolving the dominator chromatic number of some special classes of interval graphs has been the main focus of the paper. Especially, the nature of the arcs played a major role in determining the dominator chromatic number of the interval graphs with amazing ease. Some categorized graphs have been chosen in the process of exploration. In future, efforts will be put to identify the interval graphs with domination chromatic number equal to either domination number or chromatic number.

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