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Fractional Order Finite Difference Scheme For Soil Moisture Diffusion Equation And Its Applications

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Abstract: *The aim of this paper is to develop the implicit finite difference scheme for space fractional soil moisture diffusion equation (SFSMDE) with initial and boundary conditions. We prove that the scheme is unconditionally stable and convergent. Also, as an application of this scheme numerical solution for space fractional soil moisture diffusion equation is obtained by Mathematica software.*
Keywords: *Space fractional, Soil moisture diffusion equation, Finite difference scheme, Fractional derivatives, Mathematica.*

I. Introduction

Fractional order partial differential equations have recently found new applications in hydrology, science , engineering and finance [2, 5, 7, 8, 9]. A physical/mathematical approach to anomalous diffusion is based on generalized diffusion equation containing derivatives of fractional order in space or time or space-time. Recently fractional diffusion equation have been studied by many authors and developed a fractional order finite difference schemes for fractional diffusion equations. Therefore, in this connection we develop space-fractional implicit finite difference scheme for soil moisture diffusion equation of fractional order.

The solution of the linear partial differential equation of flow was first proposed by Casagrande through the use of the graphical flowent method [4]. This method is based on the assumptions that water flows region must be defined in terms of head or non-head flow. The flowent solutions proposed by Casagrande were for simple unconfined flow cases without flux boundary conditions. First experimental study on the movement of water in the soil was done by Henry Darcy (1856). Edgar Buckinghm (1907) described the water flow in unsaturated porous media modifying the equation of Darcy. Richard's (1931) combined the equations of Darcy and Buckingham with the equation of continuity to establish an overall relationship. Klute (1972) described several methods for estimating the hydraulic conductivity and diffusivity for unsaturated soils [1, 3, 4, 6]. To understand such phenomenon, soil scientists have made
some models for the flow of water into soil. Furthermore, many researchers developed different types of equations that models the water flow into soil. We consider the general diffusion equation of unsaturated flow of soil moisture as follows

$$\frac{\partial}{\partial x}\left(D\frac{\partial\theta}{\partial x}\right)+\frac{\partial}{\partial y}\left(D\frac{\partial\theta}{\partial y}\right)+\frac{\partial}{\partial z}\left(D\frac{\partial\theta}{\partial z}\right)+\frac{\partial K}{\partial z}=\frac{\partial\theta}{\partial t} \tag{1.1}$$

where,

θ (x, y, z, t) = the volumetric soil moisture content,

D = the diffusivity of soil moisture,

$D = D(\theta)$ is a function of moisture content and

K = the capillary or hydraulic conductivity of soil moisture.

If for equation (1.1), the flow takes place in the Z direction, as for infiltration of water into the soil, then the equation (1.1) becomes one-dimensional flow equation, which is given below

$$\frac{\partial}{\partial z}\left(D\frac{\partial\theta}{\partial z}\right)+\frac{\partial K}{\partial z}=\frac{\partial\theta}{\partial t} \tag{1.2}$$

where

$$D = K\frac{\partial h_t}{\partial\theta}$$

∂h_t = the tension head and

K = the capillary conductivity.

If the flow is considered in x direction (taken horizontal) then equation (1.1) becomes

$$\frac{\partial}{\partial x}\left(D\frac{\partial \theta}{\partial x}\right)=\frac{\partial \theta}{\partial t} \tag{1.3}$$

Now we assume that D is a constant then the one-dimensional diffusion equation is

$$\frac{\partial \theta}{\partial t}=D\frac{\partial^2 \theta}{\partial x^2} \tag{1.4}$$

which is exactly the diffusion heat flow equation [4] and it is well studied by Richard's [2] for water flow instead of heat flow. The model problem for the moisture flow in horizontal tube is given by

$$\frac{\partial \theta}{\partial t}=D\frac{\partial^2 \theta}{\partial x^2},\ \mathrm{t}>0,\mathrm{x}\geq 0 \tag{1.5}$$

We solve the particular model problem of moisture flow into a horizontal tube, we need to impose proper initial and boundary conditions. For that with an initial uniform moisture percentage of *θ* is θ_0 (θ_0 is constant) and for which at time *t* = 0, become initial condition and which is mathematically expressed as follows

$$\theta(x,t)=\theta_0, \quad t=0, x\geq 0 \tag{1.6}$$

For left boundary condition, there is applied a source of water placed at *x* = 0 so as to maintain at all times after t = 0 is θ_L, and which is mathematically expressed as follows

$$\theta(x,t)=\theta_L, \quad x=0, t\geq 0 \tag{1.7}$$

For the right boundary condition, there is applied a source of water placed at semi infinite plane so as to maintain at all times after *t* = 0 is θ_R, this is mathematically expressed as follows

$$\theta(x,t)=\theta_R, \quad x\to\infty,\ t\geq 0 \tag{1.8}$$

Therefore, we have the model IBVP for soil moisture flow which is given as follows

$$\frac{\partial \theta}{\partial t}=D\frac{\partial^2 \theta}{\partial x^2},\ \mathrm{t}>0,\mathrm{x}\geq 0 \tag{1.9}$$

Subject to the initial and boundary conditions

$$\theta(x,t)=\theta_0, \quad t=0, x\geq 0 \tag{1.10}$$

$$\theta(x,t)=\theta_L, \quad x=0,\ t\geq 0, \quad \theta_x(x,t)=\theta_R, \quad x\to\infty,\ t\geq 0 \tag{1.11}$$

Where *θ(x, t)* is volumetric water content and D is the diffusivity constant of soil moisture. In the next section, we develop the fractional order implicit finite difference scheme (SFIFDS) for space fractional soil moisture diffusion equation. The plan of the paper is as follows: In section 2 , the fractional order implicit finite difference scheme is develop for space fractional soil moisture diffusion equation. The section 3, is devoted for stability of the scheme and the question of convergence is proved in section 4. Numerical solution of space fractional soil moisture diffusion equation is obtained using Mathematica software in the last section.

II. Finite Difference Scheme

We consider the space fractional soil moisture diffusion equation (SFSMDE) with initial and boundary conditions as follows

$$\frac{\partial \theta(x,t)}{\partial t}=D\frac{\partial^\alpha \theta(x,t)}{\partial x^\alpha},\ t>0, x_L\leq x\leq x_R, 1<\alpha\leq 2 \tag{2.1}$$

initial condition: $\theta(x,0)=\theta_0,$ (2.2)

boundary conditions: $(0,t)=\theta_L, :\theta_x(x,t)=\theta_R,\ x\to\infty, t\geq 0$ (2.3)

where D is the diffusivity constant. We descretise the spatial *α*-order fractional derivative using the *Grünwald* finite difference formula at all time levels. The standard *Grünwald* estimate generally yields unstable finite difference equation regardless of whatever result in finite difference method is an explicit or an implicit system for related discussion [2]. Therefore we use a right shifted *Grünwald* formula to estimate the spatial *α*-order fractional derivative

$$\frac{\partial^\alpha \theta(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(-\alpha)} \lim_{N\to\infty} \frac{1}{h^\alpha} \sum_{k=0}^{N} \frac{\alpha(k-\alpha)}{\Gamma(k+1)} \theta(x-(k-1)h,t)$$

where N is the positive integer, $h = \frac{(x_R - x_L)}{N}$ and $\Gamma(.)$ is the gamma function. For the implicit numerical approximation scheme, we define $t_n = n\tau$ be the integration time $0 \le t_n \le T$ and $\Delta x = h > 0$ to be the grid size in x-direction, $h = \frac{(x_L - x_R)}{N}$ with $x_i = x_L + ih$ for i = 0,1, ..., N. Define $\theta_n^i = \theta(x_i, t_n)$ and let θ_n^i denote the numerical approximation to the exact solution $\theta(x_i, t_n)$. We also define the normalized *Grünwald* weights by

$$g_{\alpha,k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}, \mathrm{k} = 0, 1, \ldots$$

For $D_x^\alpha = \theta(x_i, t_{k+1})$, we adopt the right shifted *Grünwald* formula at all time levels for approximating the second order space derivative by implicit type numerical approximation to equation (2.1), we get

$$\frac{\theta_i^{n+1} - \theta_i^n}{\tau} = D\delta_{\alpha,x}\theta_i^{n+1}$$

where the above fractional partial differential operator is defined as

$$\delta_{\alpha,x}\theta_i^n = \frac{1}{h^\alpha} \sum_{k=0}^{i+1} g_{\alpha,k}\theta_{i-k+1}^n$$

which is an $O(h^\alpha)$ approximation to the α-order fractional derivative. Therefore, the fractional approximated equation is

$$\frac{\theta_i^{n+1} - \theta_i^n}{\tau} = \frac{D}{h^\alpha} \sum_{k=0}^{i+1} g_{\alpha,k}\theta_{i-k+1}^{n+1}$$

After simplification, we get

$$\theta_i^{n+1} - r\sum_{k=0}^{i+1} g_{\alpha,k}\theta_{i-k+1}^{n+1} = \theta_i^n,\ i = 1, \ldots, N,\ n = 0,1,2,\ldots$$

where $r = \frac{D\tau}{h^\alpha}$.

The initial condition is approximated as $\theta_i^0 = \theta_0$, i = 1, 2, ... , N.

The left boundary condition is approximated as $\theta_0^n = \theta_L$, n =0, 1, 2, ... , N.

Now using central difference the right boundary condition is approximated as follows

$$\frac{\theta_{N+1}^n - \theta_{N-1}^n}{2h} = 0,\ n = 0,1,2,\ldots,N$$

Therefore, the fractional approximated IBVP is

$$\theta_i^{n+1} - r\sum_{k=0}^{i+1} g_{\alpha,k}\theta_{i-k+1}^{n+1} = \theta_i^n,\ i = 1, \ldots, N,\ n = 0,1,2,\ldots \tag{2.4}$$

initial condition: $\theta_i^0 = \theta_0$, i = 1, 2, ... , N. (2.5)

boundary conditions: $\theta_0^n = \theta_L$ *and* $\theta_{N+1}^n = \theta_{N-1}^n$ (2.6)

where $r = \frac{D\tau}{h^\alpha}$.

Therefore, the fractional approximated IBVP (2.4) − (2.6) can be written in the following matrix equation form

$$AU^{n+1} = U^n \tag{2.7}$$

where $U^n = (\theta_0^n, \theta_1^n, \theta_2^n, ..., \theta_N^n)^T$ and $A = (a_{ij})$ is a square matrix of coefficient of order N. For i = 0, 1, 2, ...,N, j = 0, 1, 2, ...,N the coefficients are

$$a_{ij} = \begin{cases} 0, when\ j \geq i+2 \\ -rg_0, when\ j = i+1 \\ 1 - rg_{1,} when\ j = i = 1,2,4,... \\ -rg_k, otherwise\ k = 2,3,4,..., N. \end{cases} \tag{2.8}$$

While $a_{N,N-1} = -r(g_0 + g_2)$

The above system of algebraic equations is solved by using Mathematica software in section 5.

III. Stability

This section is devoted for the stability of the fractional implicit finite difference scheme (2.4)−(2.6) for the space fractional soil moisture diffusion equation (SFSMDE) (1.9) − (1.11).

Lemma 3.1: If $\lambda_j(A), j = 1,2,...., N$ represents eigen value of matrix A then we prove the following results:

(i) $|\lambda_j(A)| > 1, j = 1,2,...., N$

(ii) $\|A^{-1}\|_2 \leq 1$

Proof: The Gerschgorin theorem states that each eigenvalue λ of a square matrix A is in at least one of the following disk

$$|\lambda - a_{jj}| \leq \sum_{l=1,\ l\neq j}^{N} a_{lj},\ j = 1,2,...., M \tag{3.1}$$

Therefore, each eigenvalue λ of matrix A satisfies at least one of the following inequalities:

$$|\lambda| \leq |\lambda - a_{jj}| \leq \sum_{l=1,\ l\neq j}^{N} |a_{lj}| \leq \sum_{l=1}^{N} |a_{lj}| \tag{3.2}$$

$$|\lambda| \geq |a_{jj}| - |\lambda - a_{jj}| \geq |a_{jj}| - \sum_{l=1,\ l\neq j}^{N} |a_{lj}| \tag{3.3}$$

To prove (i), we use equation (3.3) to matrix A, then each eigenvalue λ of matrix A satisfies the following inequality.

$$\begin{aligned} |\lambda_1(A)| &\geq |1 - rg_1| - |-rg_0| \\ &\geq 1 - rg_1 - rg_0 \\ &\geq 1 - r(g_0 + g_1) > 1, (\text{since } r(g_0 + g_1) < 0) \end{aligned}$$

$$|\lambda_1(A)| > 1$$

$$\begin{aligned} |\lambda_2(A)| &\geq |1 - rg_1| - |-rg_2 - rg_0| \\ &\geq 1 - r(g_1) - (rg_2 + rg_0) \\ &\geq 1 - r(g_0 + g_1 + g_2) > 1, (\text{since } r(g_0 + g_1 + g_2) < 0) \end{aligned}$$

. . .

$$\begin{aligned} |\lambda_N(A)| &\geq |1 - rg_1| - |-rg_N - rg_{N-1} - ... - r(g_0 + g_2) - rg_1| \\ &\geq 1 - r(g_N + g_{N-1} + ... + g_1 + g_0) > 1 \end{aligned}$$

$$|\lambda_N(A)| > 1$$

Therefore, this proves $\left|\lambda_j(A)\right| > 1,\ j = 1,2,\ldots,N$

To prove (ii), we have $\|A\|_2 = \max\limits_{1\le j\le N} \left|\lambda_J(A)\right| > 1$

$$\left\|A^{-1}\right\|_2 \le \frac{1}{\left|\lambda_j(A)\right|} \le 1$$

Theorem 3.1 *The solution of the fractional approximated IBVP* (2.4)−(2.6) *is unconditionally stable.*
Proof: To prove that the above scheme is unconditionally stable.

We must show that $\left\|U^n\right\|_2 \le \left\|U^0\right\|_2$ *for* $n \ge 1$

From the equation (2.7), we have

$$AU^n = U^{n-1},\ n = 1,2,\ldots \tag{3.4}$$

Clearly, matrix A is invertible. Now for n = 1, 2, ... from equation (3.4), we get

$$AU^1 = U^0$$
$$U^1 = A^{-1}U^0$$
$$AU^2 = U^1$$
$$U^2 = A^{-1}U^1$$
$$= A^{-1}(A^{-1}U^0)$$
$$U^2 = (A^{-1})^2 U^0$$
$$\ldots$$
$$U^n = (A^{-1})^n U^0,\ n \ge 1 \tag{3.5}$$

From equation (3.5), we get

$$\left\|U^n\right\|_2 \le \left\|A^{-1}\right\|_2^n \left\|U^0\right\|_2,\ (By\ lemma\ 3.1, \left\|A^{-1}\right\|_2 \le 1)$$
$$\left\|U^n\right\|_2 \le \left\|U^0\right\|_2$$

This shows that the finite difference scheme for fractional equation is unconditionally stable. Hence the proof is completed.

IV. Convergence

In this section we discuss the convergence of the finite difference scheme. Consider the another vector

$$\vec{U}^n = \left[\theta(x_0,t_n),\ldots,\theta(x_i,t_n),\ldots,\theta(x_N,t_n)\right]^T$$

which represents the exact solution at time level t_n, whose size is N. The finite difference scheme (3.4) will become

$$A\vec{U}^n = \vec{U}^{n-1} + \tau^n,\ n = 1,2,\ldots \tag{4.1}$$

where τ^n is the vector of the truncation errors at level t_n.

Theorem 4.1 *The fractional order finite difference scheme* (2.4) − (2.6) *for SFSMDE is convergent.*
Proof: If we subtract (3.4) from (4.1), we get

$$A(\vec{U}^n - U^n) = (\vec{U}^{n-1} - U^{n-1}) + \tau^n \tag{4.2}$$

Consider the error vector $E^n = \vec{U}^n - U^n$, from equation (4.2), we get

$$AE^n = E^{n-1} + \tau^n \tag{4.3}$$

In equation (4.3), putting n = 1, 2,…, we get

$$AE^1 = E^0 + \tau^1$$
$$E^1 = A^{-1}E^0 + A^{-1}\tau^1$$
$$AE^2 = E^1 + \tau^2$$
$$\begin{aligned} E^2 &= A^{-1}E^1 + A^{-1}\tau^2 \\ &= A^{-1}[A^{-1}E^0 + A^{-1}\tau^1] + A^{-1}\tau^2 \\ &= (A^{-1})E^0 + A^{-1}[A^{-1}\tau^1 + \tau^2] \end{aligned}$$
$$\vdots$$

$$E^n = (A^{-1})^n E^0 + A^{-1}\sum_{k=0}^{n-1}(A^{-1})^k \tau^{n-k} \qquad (4.4)$$

We take $U^0 = \vec{U}^0$, then $E^0 = 0$ is a zero vector, then from (4.4), we get

$$\left\|E^n\right\|_2 \le \left\|A^{-1}\right\|_2 \left(\sum_{k=0}^{n-1}\left\|A^{-1}\right\|_2^k\right). \max_{1\le M\le n}\left\|\tau^M\right\|_2 \qquad (4.5)$$

Since by Lemma (3.1), $\left\|A^{-1}\right\|_2 \le 1$ and $\lim_{(h,\tau)\to(0,0)}\left\|\tau^M\right\|_2 = 0,\ (1 \le M \le n)$

Therefore, from equation (4.5), we get

$$\left\|E^n\right\|_2 \to 0 \ as \ (h,\tau) \to (0,0)$$

The proof is completed.

V. Numerical Solutions

In this section, we obtain the approximated solution of space fractional soil moisture diffusion equation with initial and boundary conditions. To obtain the numerical solution of the space fractional soil moisture diffusion equation (SFSMDE) by the finite difference scheme, it is important to use some analytical model. Therefore, we present an example to demonstrate that SFSMDE can be applied to simulate behavior of a fractional diffusion equation by using Mathematica Software. We consider the following, dimensionless one-dimensional space fractional soil moisture diffusion equation with suitable initial and boundary boundary conditions

$$\frac{\partial\theta(x,t)}{\partial t} = D\frac{\partial^\alpha \theta(x,t)}{\partial x^\alpha},\ 0 < x < 1,\ 1 < \alpha \le 2, t > 0$$

$$\text{initial condition}: \theta\,(\mathrm{x},0) = 0,\ 0 \le x \le 1$$

$$\text{boundary conditions}: \theta(0,\mathrm{t}) = 1,$$

$$\theta_x(x,t) = 0,\ as\ x \to \infty,\ t > 0$$

with the diffusion coefficient $D = 1$.

The numerical solution obtained at $t = 0.05$ by considering the parameters $\tau = 0.005$, $h = 0.1$, $\alpha = 1.7$, 1.8 and 1.9, which is simulated in the following figure.

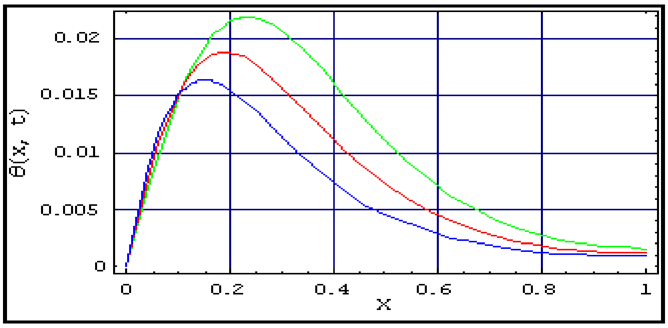

*Fig.*5.1 : The soil moisture diffusion profile with $t = 0.05$, $h = 0.1$, $\alpha = 1.7$(*blue*) $\alpha = 1.8$(*red*) *and* $\alpha = 1.9$(*green*)

VI. Conclusions.

(i) We develop fractional order finite difference scheme for space fractional soil moisture diffusion equation.
(ii) The numerical example is analyzed to show that the numerical results are in good agreement with theoretical analysis.
(iii) The fractional order implicit finite difference scheme is numerically stable.

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