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Analysis on a Common Fixed Point Theorem

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***Abstract*:** *The aim of this paper is to prove a common fixed point theorem which generalizes the result of Brian Fisher [1] and etal. by weaker conditions. The conditions of continuity, compatibility and completeness of a metric space are replaced by weaker conditions such as reciprocally continuous and compatible, weakly compatible, and the associated sequence.*
Keywords: *Fixed point, self maps, reciprocally continuous, compatible maps, weakly compatible mappings.*

I. Introduction

Two self maps S and T are said to be commutative if ST = TS. The concept of the commutativity has been generalized in several ways. For this Gerald Jungck [2] initiated the concept of compatibility.

1.1 Compatible Mappings.

Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n,TSx_n)=0$,whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = Tx_n = t$ for some $t\in X$.

It can be easily verified that when the two mappings are commuting then they are compatible but not conversely.

In 1998, Jungck and Rhoades [4] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but not conversely.

1.2. Weakly Compatible.

A pair of maps A and S is called weakly compatible pair if they commute at coincidence points.

Brian Fisher and others [1] proved the following common Fixed Point theorem for four self maps of a complete metric space.

Theorem 1.3 Suppose A, B, S and T are four self maps of metric space(X,d)such that

1.3.1 (X,d) is a complete metric space

1.3.2 $A(x)\subseteq T(x)$, $B(x)\subseteq S(x)$

1.3.3 The pairs (A,S) and,(B,T) are compatible

1.3.4 $d(Ax,By)^2 < c_1 \max\{d(Sx,Ax)^2,d(Ty,By)^2,d(Sx,Ty)^2\} + c_2 \max\{d(Sx,Ax), d(Sx,By), d(Ax,Ty),d(By,Ty)\}+c_3\{d(Sx,By),d(Ty,Ax)\}$

Where $c_1, c_2, c_3 \geq 0$, $c_1+2c_2<1$ and $c_1+c_3> 1$,then A,B,S and T have a unique common fixed point $z\in X$.

1.4 Associated Sequence.

Suppose A, B, S and T are self maps of a metric space (X, d) satisfying the condition (1.3.2), Then for any $x_0\in X$,$Ax_0\in A(X)$ and hence, $Ax_0\in T(X)$ so that there is a $x_1\in X$ with $Ax_0= Tx_1$. Now $Bx_1\in B(X)$ and hence there is $x_2\in X$ with $Bx_1=Sx_2$. Repeating this process to each $x_0\in X$, we get a sequence $<x_n>$ in X such that $Ax_{2n}= Tx_{2n+1}$ and $Bx_{2n+1} = Sx_{2n+2}$ for $n \geq 0$. We shall call this sequence as an associated sequence of x_0 relative to the Four self maps A, B,S and T.

Now we prove a lemma which plays an important role in proving our theorem.

1.5 Lemma. Suppose A, B, S and T are four self maps of a metric space (X, d) satisfying the conditions (1.3.2) and (1.3.4) of Theorem(1.3) and Further if (1.3.1) (X, d) is a complete metric space then for any $x_0\in X$ and for any of its associated sequence $<x_n>$ relative to Four self maps, the sequence $Ax_0, Bx_1, Ax_2, Bx_3, \ldots\ldots\ldots, Ax_{2n}, Bx_{2n+1}, \ldots\ldots\ldots$, converges to some point $z\in X$.

Proof: For simplicity let us take $d_n=d(y_n,y_{n+1})$ for n=0,1,2,………

We have
$d^2_{2n+1}=[d(y_{2n+1},y_{2n+2})]^2=[d(Ax_{2n},Bx_{2n+1})]^2$

$\le c_1\max\{[d(Sx_{2n},Ax_{2n})]^2, d(Tx_{2n+1},Bx_{2n+1})]^2,[d(Sx_{2n},Tx_{2n+1})]^2\}$
$+c_2\max\{d(Sx_{2n},Ax_{2n}), d(Sx_{2n},Bx_{2n+1}), d(Ax_{2n},Tx_{2n+1})d(Bx_{2n+1},Tx_{2n+1})\}$
$+c_3\max\{d(Sx_{2n}, Bx_{2n+1}), d(Tx_{2n+1},Ax_{2n})\}$

$\le c_1\max\{d^2_{2n},d^2_{2n+1}\}+c_2\{d_{2n}\, d(y_{2n},y_{2n-2})\}$

$\le c_1\max\{d^2_{2n},d^2_{2n+1}\}+c_2[d^2_{2n}+d_{2n}d_{2n-1}]$
$\le c_1\max\{d^2_{2n},d^2_{2n+1}\}+c_2[\frac{3}{2}d^2_{2n}+\frac{1}{2}d^2_{2n-1}]$ …………………………….(1.5.1)

If $d_{2n+1}>d_{2n}$, inequality (1.5.1) implies $d^2_{2n+1}\le\frac{2c2}{2-2\,c1-c2}d^2_{2n}$ a contradiction, since $\frac{3\,c2}{2-2\,c1-c2}<1$. Thus $d_{2n+1}\le d_{2n}$ and inequality (1.5.1) implies that $d_{2n+1}=d(y_{2n+1},y_{2n-2})\le h\, d(y_{2n},y_{2n+1})=h^2 d_{2n}$ Where $h^2=\frac{2c1+3c2}{2-c2}<1$.
Similarly,
$d_{2n}{}^2=[d(y_{2n},y_{2n+1}]^2=[d(Ax_{2n},Bx_{2n-1})]^2\le c_1\max\{d^2_{2n-1},d^2_{2n}\}+c_2(\frac{3}{2}d^2_{2n-1}+\frac{1}{2}d^2_{2n})$ and it follows above that
$d_{2n}=d(y_{2n},y_{2n+1})\le h\, d(y_{2n-1},y_{2n})=d_{2n-1}$

Consequently, $d(y_{n+1},y_n)\le h\, d(y_n,y_{n-1})$, For n=1, 2, 3……since h<1, this implies that $\{y_n\}$ is a cauchy sequence in X.

Hence the Lemma.

The converse of the lemma is not true.

That is, suppose A, B, S and T are self maps of a metric space (X, d) satisfying the conditions (1.3.2) and (1.3.4), even for each associated sequence $<x_n>$ of x_0, the associated sequence converges, the metric space (X,d) need not be complete. For this we provide an example.

1.6 Example. Let X= (-1, 1) with $d(x,y)=|x-y|$

$$Ax=Bx=\begin{cases}\frac{1}{5} & if\ -1<x<\frac{1}{6}\\ \frac{1}{6} & if\ \frac{1}{6}\le x<1\end{cases}\quad Sx=\begin{cases}\frac{1}{5} & if\ -1<x<\frac{1}{6}\\ \frac{6x+5}{36} & if\ \frac{1}{6}\le x<1\end{cases}\quad Tx=\begin{cases}\frac{1}{5} & if\ -1<x<\frac{1}{6}\\ \frac{1}{3}-x & if\ \frac{1}{6}\le x<1\end{cases}$$

Then $A(X)=B(X)=\left\{\frac{1}{5},\frac{1}{6}\right\}$ while $S(X)==\left\{\frac{1}{5}\cup\left[\frac{1}{6},\frac{11}{36}\right)\right\}$, $T(X)=\left\{\frac{1}{5}\cup\left[\frac{1}{6},\frac{-2}{3}\right)\right\}$ so that $A(X)\subset T(X)$ and $B(X)\subset S(X)$ proving the condition (1.3.2)of Theorem (1.3). Clearly (X, d) is not a complete metric space. It is easy to prove that the associated sequence Ax_0, Bx_1, Ax_2, Bx_3,.., Ax_{2n}, Bx_{2n+1}.., converges to $\frac{1}{5}$ if $-1<x<\frac{1}{6}$; and converges to $\frac{1}{6}$, if $\frac{1}{6}\le x<1$.

II. Main Result

Theorem 2. Suppose A, B, S and T are four self maps of metric space(X,d)such that

2.1 $A(x)\subseteq T(x)$, $B(x)\subseteq S(x)$,
2.2 The pair (A,S) is reciprocally continuous and compatible, and the pair (B,T) is weakly compatible.
2.3 $d(Ax,By)^2 < c_1\max\{d(Sx,Ax)^2,d(Ty,By)^2,d(Sx,Ty)^2\}+c_2\max\{d(Sx,Ax), d(Sx,By), d(Ax,Ty),d(By,Ty)\}+c_3$
$\{d(Sx,By),d(Ty,Ax)\}$ where $c_1,c_2,c_3,\ge 0$, $c_1+2c_2<1$ and $c_1+c_3>1$

Further if

2.4 The sequence Ax_0, Bx_1, Ax_2, Bx_3….. Ax_2n, Bx_{2n+1}…. converges to $z \in X$ then A ,B,S and T have a unique common fixed point $z \in X$.

Proof: From condition IV, Ax_{2n}, Bx_{2n+1} converges to z as $n \to \infty$.
Since the pair (A, S) is reciprocally continuous means ASx_{2n} converges to Az and SAx_{2n} converges to Sz as $n \to \infty$.
Also since the pair (A,S) is compatible, we get $\lim_{n\to\infty} d\left(ASx_{2n}, SAx_{2n}\right) = 0$ or d(Az,Sz)=0 or Az=Sz .
Now $d(Az,z)^2 = d(Az,Bx_{2n+1})^2 \le c_1 \max\{[d(Sz,Az)^2, d(Tx_{2n+1},Bx_{2n+1})^2, d(Sz,,Tx_{2n+1})^2]\} + c_2 \max\{d(Sz,Az), d(Sz,Bx_{2n+1}), d(Az,Tx_{2n+1}), d(Bx_{2n+1},Tx_{2n})\} + c_3 \{d(Sz,Bx_{2n+1}), d(Tx_{2n+1},Az)\}$

Letting $n \to \infty$,we get $d(Az,z)^2 \le c_1\ d(Az,z)^2 + c_3 d(Az,z)^2 = (c_1+c_3)\ d(Az,z)^2$
This gives $d(Az,z)^2[1-(c_1+c_3)] \le 0$.
Since $c_1+c_3 < 1$, we get $d[(Az,z)]^2=0$ or Az=z. Therefore z=Az=Sz.

Also Since $A(x) \subseteq T(x)$ $\exists\ u \in x$ such that z=Az=Tu .
We prove Bu= Tu.
Consider $d[(z.Bu)]^2 = [d(Az,Bu)]^2 \le c_1 \max\{[d(Sz,Az)^2, d(Tu,Bu)^2, d(Sz,Tu)^2]\}$
$+c_2 \max\{d(Sz,Az), d(Sz,Bu), d(Az,Tu), d(Bu,Tz)\} + c_3 \{d(Sz,Bu), d(Tu,Az)\}$
$= c_1\ d(z,Bu)^2 + c_3 d(z,Bu)^2$
$d(z,Bu)^2 \le c_1 + c_3 d(z,Bu)^2$
$d(z,Bu)^2[1-(c_1+c_3)] \le 0$ since $c_1+c_3<1$,we get $d(z,Bu)^2=0$ or Bu=z.
Therefore z=Bu=Tu.
Since the pair (B,T) is weakly compatible and z= Bu=Tu, we get d(BBu, TTu)=0 or Bz=Tz.

Now consider $d(z,Bz)^2 = d(Az,Bz)^2 \le c_1 \max\{[d(Sz,Az)^2, d(Tz,Bz)^2, d(Sz,Tz)^2]\} + c_2 \max\{d(Sz,Az), d(Sz,Bz), d(Az,Tz), d(Bz,Tz)\} + c_3 \{d(Sz,Bz), d(Tz,Az)\} = c_1 d(z,Bz)^2 + c_3 d(z,Bz)^2$.
This gives
$d(z,Bz)^2 \le (c_1+c_3)\ d(z,Bz)^2$
$d(z,Bz)^2[1-(c_1+c_3)] \le 0$, since $c_1+c_3<1$, we get $d(z,Bz)^2=0$ or z=Bz .
Therefore z=Bz=Tz
Since z=Az=Bz=Sz=Tz, z is a common fixed point of A, B, S and T.
The uniqueness of common fixed point can be easily proved.

Now, we discuss our earlier example in the following two remarks to justify our result.

Remark 2.5: From the example given earlier, clearly the pair (A,S) is reciprocally continuous, since if $x_n = \left(\frac{1}{6} + \frac{1}{6^n}\right)$ for $n \ge 1$, then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{6}$ and $\lim_{n\to\infty} ASx_n = \frac{1}{6}$ =A(t) also $\lim_{n\to\infty} SAx_n = \frac{1}{6}$ =S(t).But none of A and S are continuous. Also, since $\lim_{n\to\infty} d(ASx_n, SAx_n)=0$, the pair (A,S) is compatible. Moreover the pair (B,T) is weakly compatible as they commute at coincident points $\frac{1}{5}$ and $\frac{1}{6}$. The contractive condition holds for the values of $c_1, c_2, c_3, \ge 0$, $c_1+2c_2<1$ and $c_1+c_3>1$. Further $\frac{1}{6}$ is the unique common fixed point of A, B, S and T.

Remark 2.6: Finally we conclude that from the earlier example, the mappings A,B,S and T are not continuous, the pair (A,S) is reciprocally continuous and compatible and (B,T) is weakly compatible. Also the associated sequence relative to the self maps A,B,S and T such that the sequence Ax_0, Bx_1, Ax_2, Bx_3, ………, Ax_{2n}, Bx_{2n+1}, ………, converges to the point $\frac{1}{6} \in X$, but the metric space X is not complete. Moreover, $\frac{1}{6}$ is the unique common fixed point of A, B, S and T. Hence, Theorem (2) is a generalization of Theorem (1.3).

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