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Best Approximation in Real Linear 2-Normed Spaces

R.Vijayaragavan
Applied Analysis Division, School of Advanced Sciences, VIT University, Vellore – 632 014, Tamilnadu, India.

Abstract: *This paper delineates existence, characterizations and strong unicity of best uniform approximations in real linear 2-normed spaces.*
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I. Introduction

The concepts of linear 2-normed spaces were initially introduced by Gahler [5] in 1964. Since then many researchers (see also [2,4]) have studied the geometric structure of 2-normed spaces and obtained various results. This paper mainly deals with existence, characterizations and unicity of best uniform approximation with respect to 2-norm. Section 2 provides some important definitions and results that are used in the sequel. Some main results of the set of best uniform approximation in the context of linear 2-normed spaces are established in Section 3.

II. Preliminaries

Definition 2.1. Let X be a linear space over real numbers with dimension greater than one and let $\| \cdot, \cdot \|$ be a real-valued function on $X \times X$ satisfying the following properties for all x, y, z in X.

(i) $\| x, y \| = 0$ if and only if x and y are linearly dependent,

(ii) $\| x, y \| = \| y, x \|$,

(iii) $\| \alpha x, y \| = |\alpha| \, \| x, y \|$, where α is a real number,

(iv) $\| x, y + z \| \le \| x, y \| + \| x, z \|$.

Then $\| \cdot, \cdot \|$ is called a 2-norm and the linear space X equipped with 2-norm is called a linear 2-normed space. It is clear that 2-norm is non-negative.

Example 2.2. Let $X = R^3$ with usual component wise vector additions and scalar multiplications. For $x = (a_1, b_1, c_1)$ and $y = (a_2, b_2, c_2)$ in X, define

$$\| x, y \| = \max\{|a_1 b_2 - a_2 b_1|, |b_1 c_2 - b_2 c_1|, |a_1 c_2 - a_2 c_1|\}.$$

Then clearly $\| \cdot, \cdot \|$ is a 2-norm on X.

Definition 2.3. Let G be a subset of a real linear 2-normed space X and $x \in X$. Then $g_0 \in G$ is said to be a best approximation to x from the elements of G if

$$\| x - g_0, z \| = \inf_{g \in G} \| x - g, z \| , \ z \in X \backslash V(x, G)$$

where V (x, G) is the subspace generated by x and G.

The set of all elements of best approximation to $x \in X$ from G with respect to the set z is denoted by $P_{G,Z}(x)$.

Definition 2.4. A linear 2-normed space $(X, \| \cdot, \cdot \|)$ is said to be strictly convex if

$$\| a + b, c \| = \| a, c \| + \| b, c \| , \ \| a, c \| = \| b, c \| = 1 \text{ and } c \notin V(a, b) \Rightarrow a = b.$$

or

A linear 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be strictly convex if and only if

$\|x,z\| = \|y,z\| = 1$, $x \neq y$ and $z \in X \backslash V(x, y) \Rightarrow \left\|\frac{1}{2}(x+y), z\right\| < 1$

Example 2.5. Let $X = R^3$ with 2-norm defined as follows: For $x = (a_1, b_1, c_1)$ and $y = (a_2, b_2, c_2)$ in X, let

$$\|x, y\| = \{(a_1 b_2 - a_2 b_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (a_1 c_2 - a_2 c_1)^2\}^{\frac{1}{2}}$$

Then the space $(X, \|.,.\|)$ is strictly convex linear 2-normed space.

Definition 2.6. For all functions $h \in C([a, b] \times [c, d])$

$\|h\|_\infty = \{\sup |h(t, t')| : t \in [a, b], t' \in [c, d]\}$.

The set of extreme points of a function $h \in C([a, b] \times [c, d])$ is defined by $E(h) = \{x \in [a, b], y \in [c, d] : |h(x, y)| = \|h\|_\infty\}$. Best approximation with respect to this norm is called best uniform approximation.

Definition 2.7. Let G be a subspace of $C([a, b] \times [c, d]) = \{f: [a, b] \times [c, d] \mapsto \mathbf{R}$

A function $g_0 \in G$ is called a strongly unique best uniform approximation of $f \in C([a, b] \times [c, d])$ if there exists a constant $k_f > 0$ such that for all $g \in G$,

$\|f - g\|_\infty \geq \|f - g_0\|_\infty + k_f \|g - g_0\|_\infty$.

Example 2.8. Consider the space $G = \text{span}\{g_1\}$ of $C([-1, 1] \times [-1, 1])$, where $g_1(t, t^*) = t \in [-1, 1]$ and $f \in (C[-1, 1] \times [-1, 1])$. Then $(0, 1)$ is the best approximation of $[1, 1]$.

Definition 2.9. Let G be a subset of $C([a, b] \times [c, d])$ and let $f \in C([a, b] \times [c, d])$ have a unique best uniform approximation $g_0 \in G$. Then the projection $P_G : C([a, b] \times [c, d]) \rightarrow POW(G)$ is called Lipchitz-continuous at f if there exists a const $k_f > 0$ such that for all $\bar{f} \in C([a, b] \times [c, d])$ and all $g_{\bar{f}} \in P_G(\bar{f})$, $\|g_f - g_{\bar{f}}\|_\infty \leq k_f \|f - \bar{f}\|_\infty$.

III. Main Results

Theorem 3.1. Let G be a finite-dimensional subspace of a real linear 2-normed space X. Then for every $x \in X$, there exists a best approximation from G. **Proof.** Let $x \in X$.

Then by the definition of the infimum there exists a sequence $\{g_n\} \in G$ such that

$$\|x - g_n, z\| \rightarrow \inf_{g \in G} \|x - g, z\|.$$

This implies that there exists a constant k>0 such that for all n,

$$\|g_n, z\| - \|x, z\| \leq \|x - g_n, z\| \leq \inf_{g \in G} \|x - g, z\| + k$$

$$\leq \|x, z\| + k.$$

Hence for all n, $\|g_n, z\| \leq 2\|x, z\| + k$.

$\Rightarrow \{g_n\}$ is bounded sequence. Then there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ converging to $g_0 \in G$.

$$\therefore \|x-g_0,z\| = \lim_{k\to\infty}\|x-g_{n_k}\| = \inf_{g\in G}\|x-g,z\|, z\in X\setminus V(x,G)$$

$\Rightarrow g_0 \in P_{G,Z}(x)$,which completes the proof.

Theorem 3.2. Let G be a finite –dimensional subspace of a strictly convex linear 2- normed space X. Then for every $x \in X$,there exists a unique best approximation from G.

Proof. Let $x \in X$ .Since G is a finite-dimensional, by Theorem 3.1 there exists an element $g_0 \in G$ Such that , $g_0 \in P_{G,Z}(x)$, $z \in X \setminus V(x,G)$.

Now we show that, $P_{G,Z}(x) = \{g_0\}$. For that first we prove that $P_{G,z}(x)$ is convex. Let $g_1, g_2 \in P_{G,Z}(x)$ and $0 \le \alpha \le 1$. Then,

$$\|x-(\alpha g_1 + (1-\alpha)g_2), z\| = \|\alpha(x-g_1)+(1-\alpha)(x-g_2), z\|$$
$$\le \alpha\|x-g_1,z\| + (1-\alpha)\|x-g_2,z\|$$
$$= \alpha\inf_{g\in G}\|x-g,z\| + (1-\alpha)\inf_{g\in G}\|x-g,z\|$$
$$= \inf_{g\in G}\|x-g,z\|$$
$$\le \|x-g,z\|, \text{ for all } g\in G.$$

Since $\alpha g_1 + (1-\alpha)g_2 \in G$, $\alpha g_1 + (1-\alpha)g_2 \in P_{G,Z}(x)$. We shall suppose that $g^* \in P_{G,Z}(x)$. Then $\frac{1}{2}(g_0 + g^*) \in P_{G,Z}(x)$, which implies that

$$\left\|\frac{1}{2}\{(x-g_0)+(x-g^*)\}, z\right\| = \left\|x-\frac{1}{2}(g_0-g^*), z\right\|$$
$$= \inf_{g\in G}\|x-g,z\|_1$$

Since $\|x-g_0,z\| = \|x-g^*,z\| = \inf_{g\in G}\|x-g,z\|$ and X is strictly convex, we obtain

$x-g_0 = x-g^* \Rightarrow g_0 = g^*$. This proves that $P_{G,Z}(x) = \{g_0\}$.

Theorem 3.3. Let G be a finite-dimensional subspace of a real linear 2-normed space X with the property that every function with domain $X \times X$ has a unique best approximation from G. Then for all $f_1, f_2 \in X\times X$,

$$\left|\inf_{g\in G}\|f_1-g,z\| - \inf_{g\in G}\|f_2-g,z\|\right| \le \|f_1-f_2,z\|, z\in X\setminus G$$ and $P_G : X \text{ x } X \to G$ is continuous.

Proof. Suppose that P_G is not continuous. Then there exists an element $f \in X\times X$ and a sequence $\{f_n\} \in X\times X$ such that $\{f_n \times g_n\}$.

$P_{G,z}(f_n)$ does not converge to $P_G(f)$. Since G is finite dimensional, there ex- ists a subsequence $\{f_{nk}\}$ of $\{f_n\}$ such that $P_{G,Z}(f_{nk}) \to g_0 \in G$, $g_0 \neq P_G(f)$ and we shall show that the mapping

$f \to \inf_{g\in G}\|f-g,z\|$ is continuous $(f \in X\times X)$

Let $f_1, f_2 \in X\times X$. Then there exists a $g_2 \in G$ such that

$$\|f_2-g_2,z\| = \inf\|f_2-g,z\|$$

$\Rightarrow \inf_{g\in G} \| f_1 - g, z\| \leq \| f_1 - g_2, z\|$

$\leq \| f_1 - f_2, z\| + \| f_2 - g_2, z\|$

$= \| f_1 - f_2, z\| + \inf_{g\in G} \| f_2 - g, z\|$

$\Rightarrow \inf \| f_1 - g, z\| - \inf \| f_2 - g, z\| \leq \| f_1 - f_2, z\|$, $z \in X \setminus G$.

$g\in G$

This proves that

$g\in G$

$$\left| \inf_{g\in G} \| f_1 - g, z\| - \inf_{g\in G} \| f_2 - g, z\| \right| \leq \| f_1 - f_2, z\|, z \in X \setminus G$$

By continuity it follows that

$$\| f_{nk} - P_G(f_{nk}), z\| = \inf \| f_{nk} - g, z\|$$

$$\rightarrow \inf_{g\in G} \lim_{k\to\infty} \| f_{n_k} - g, z\|$$

$$= \inf_{g\in G} \| f - g, z\|$$

and $\lim \| f_{n_k} - P_G(f_{n_k}), z\| \rightarrow \| f - g_0, z\|$

$$\Rightarrow \| \lim(f_{n_k} - P_G(f_{n_k}), z\| = \| f - P_G(f), z\|$$

$\Rightarrow g_0$ and $P_G(f)$ are two distinct best approximation of f and G which contradicts the uniqueness of best approximation.

Therefore $P_G : X \times X \rightarrow G$ is continuous.

A 2 –functional is a real valued mapping with domain $A \times C$ with A and C are linear manifolds of a 2- normed space X.

A linear 2- functionals is 2- functional such that

(i) $F(a+c, b+d) = F(a,b) + F(a,d) + F(c,b) + F(c,d)$

(ii) $F(\alpha a, \beta b) = \alpha\beta F(a,b)$.

F is called a bounded 2-functional if there is a real constant $k \geq 0$ such that $|F(a,b)| \leq k \|a,b\|$ for all a,b in the domain of F and

$$\|F\| = \inf \{k : |f(a,b)| \leq k \|a,b\|, (a,b \in D(F)\}$$

$$= \sup\{ f(a,b) : \|a,b\| = 1, (a,b) \in D(F)\}$$

$$= \sup\left\{ \frac{|f(a,b)|}{\|a,b\|} : \|a,b\| \neq 0, (a,b) \in D(F) \right\}$$

Theorem 3.4. Let G be a subspace of $C([a,b]\times[a,b])$, $f \in C([a,b]\times[a,b])$ and $g_0 \in G$. Then the following statements are equivalent:
(i) The function g_0 is a best uniform approximation of f from G. (ii) For every function $g \in G$,

$$\min_{t,t^* \in E(f-g_0)} f(t,t^*) - g_0(t,t^*))\,(g(t,t^*)) \le 0$$

Proof. (ii) $\Rightarrow$ (i). Suppose that (ii) holds and let $g \in G$. Then by (ii) there exist the points $t,t' \in E(f-g_0)$ such that

$(f(t,t^*) - g_0(t,t^*))(g(t,t^*) - g_0(t,t^*)) \le 0$. Then we have

$$\| f - g \|_\infty \ge |f(t,t^*) - g(t,t^*)|$$
$$= |f(t,t^*) - g_0(t,t^*) + g_0(t,t^*) - g(t,t^*)|$$
$$= |f(t,t^*) - g_0(t,t^*)| + |g(t,t^*) - g_0(t,t^*)|$$
$$\ge \| f(t,t^*) - g_0(t,t^*) \|_\infty$$

which shows that (i) holds.

(i) $\Rightarrow$ (ii). Suppose that (i) holds and assume that (ii) fails. Then there exists a function $g_1 \in G$ such that for all $t,t^* \in E(f-g_0)$, $(f(t,t^*) - g_0(t,t^*))g_1(t,t^*) > 0$. Since $E(f-g_0)$ is compact, there exists a real number $c > 0$ such that for all $t,t^* \in E(f-g_0)$

$$(f(t,t^*) - g_0(t,t^*))g_1(t,t^*) > c. \qquad (1)$$

Further, there exists an open neighborhood U of $E(f-g_0)$ such that for all

$t,t^* \in U$, and $c(f(t,t^*) - g_0(t,t^*))g_1(t,t^*) > 2$ (2) $-$

$$|f(t,t^*) - g_0(t,t^*)| \ge_2 \| f - g_0 \|_\infty \qquad (3)$$

Since $[a,b]\backslash U$ is compact, there exists a real number $d > 0$ such that for all $t,t^* \in [a,b]\backslash U$,

$$|f(t,t^*) - g_0(t,t^*)| < \| f - g_0 \|_\infty - d \qquad (4)$$

Now we shall assume that

$$\| g_1 \|_\infty \le \min\{d, \| f - g_0 \|\}. \qquad (5)$$

Let $g_2 = g_0 + g_1$. Then by (4) and (5) for all $t,t^* \in [a,b]\backslash U$,

$$|f(t,t^*) - g_2(t,t^*)| = |(f(t,t^*) - g_0(t,t^*)) - g_1(t,t^*)|$$
$$\le |f(t,t^*) - g_0(t,t^*)| + |g_1(t,t^*)|$$
$$\le \| f(t,t^*) - g_0(t,t^*) \|_\infty - d + \| g_1(t,t^*) \|$$
$$\le \| f(t,t^*) - g_0(t,t^*) \|_\infty.$$

For all $t \in U$, by (2), (3) and (5),

$|f(t,t^*) - g_2(t,t^*)| = |(f(t,t^*) - g_0(t,t^*)) - g_1(t,t^*)|$
$\leq |f(t,t^*) - g_0(t,t^*)| - |g_1(t,t^*)|$
$\leq \| f - g_0 \|_\infty.$

$\Rightarrow \| f - g_2 \|_\infty \leq \| f - g_0 \|_\infty$

$\Rightarrow g_0$ is not the best uniform approximation of f which is a contradiction. Hence the proof.

Theorem 3.5. Let G be a subset of $C([a, b] \times [c, d])$ and f has a strongly unique best uniform approximation from G, then

$P_G : C([a,b] \times [c,b])$ POW(G) is Lipschitz-continuous at f.

Proof. Let $f \in X = C([a,b] \times [c,b])$ have a strongly unique best uniform approximation $g_f \in G$. Then there exists $k_f > 0$ such that for all $g \in G$.

$\| f - g \|_\infty \geq \| f - g_f \|_\infty + k_f \| g - g_f \|_\infty$.

Then for all
$\tilde{f} \in X$ and for all $g_{\tilde{f}} \in P_G(\tilde{f})$.

We obtain $k_f \| g_f - g_{\tilde{f}} \|_\infty \leq \| f - g_{\tilde{f}} \|_\infty - \| f - g_f \|_\infty$
$\leq \| f - \tilde{f} \|_\infty + \| \tilde{f} - g_{\tilde{f}} \|_\infty - \| f - g_f \|_\infty$
$\leq \| f - \tilde{f} \|_\infty + \| f - \tilde{f} \|_\infty = 2 \| f - \tilde{f} \|_\infty. \Rightarrow L_f = \dfrac{2}{k_f}$ is the desired constant.

Theorem 3.6. Let G be a finite dimensional subspace of $X = C([a,b] \times [c,d])$,
$f \in X \setminus G$ and $g_0 \in G$. Then the following statements are equivalent:

(i) The function g_0 is a strongly unique best uniform approximation of f from G.
(ii) For every nontrivial function $g \in G$,
$\min_{x,y \in E(f-g_0)} (f(x,y) - g_0(x,y)) g(x,y) < 0$
(iii) There exists a constant $k_f > 0$ such that for every function $g \in G$,
$\min_{x,y \in E(f-g_0)} (f(x,y) - g_0(x,y)) g(x,y) \leq -k_f \| f - g_0 \|_\infty \| g \|_\infty$.

Proof. (iii) $\Rightarrow$ (i). We shall suppose that (iii) holds and let $g \in G$. Then by (iii) there exist the points $x, y \in E(f - g_0)$ such that
$(f(x,y) - g_0(x,y))(g(x,y) - g_0(x,y)) \leq -k_f \| f - g_0 \|_\infty \| g - g_0 \|_\infty$. This implies that

$\| f - g \|_\infty \geq |f(x,y) - g(x,y)|$
$= |f(x,y) - g_0(x,y) - (g(x,y) - g_0(x,y))|$
$= |f(x,y) - g_0(x,y)| + |g(x,y) - g_0(x,y)| \geq \| f - g_0 \|_\infty + \dfrac{k_f \| f - g_0 \|_\infty \| g - g_0 \|_\infty}{|f(x,y) - g_0(x,y)|}$

$= \| f - g_0 \|_\infty + k_f \| g - g_0 \|_\infty .$

(i) $\Rightarrow$ (iii). Suppose that (iii) fails, i.e. there exists a function $g_1 \in G$ such that for all $x, y \in E(f - g_0)$,

$(f(x, y) - g_0(x, y))g_1(x, y) > -k_f \| f - g_0 \|_\infty \| g_1 \|_\infty .$

Since $E(f - g_0)$ is compact, there exists an open neighborhood U of $E(f - g_0)$ such that for all $x, y \in U$

$(f(x, y) - g_0(x, y))g_1(x, y) > -k_f \| f - g_0 \|_\infty \| g_1 \|_\infty$

and

$$|f(x, y) - g_0(x, y)| \geq \frac{1}{2} \| f - g_0 \|_\infty . \qquad (6)$$

Further, we can choose U sufficiently small such that for all $x, y \in U$ with $(f(x, y) - g_0(x, y))g_1(x, y) < 0$

$$|g_1(x, y)| < k_f \| g_1 \|_\infty . \qquad (7)$$

Since $[a, b] \times [c, d] \backslash U$ is compact, there exists a real number $c > 0$ such that for all $x, y \in [a, b] \times [c, d] \backslash U$,

$$|f(x, y) - g_0(x, y)| \leq \| f - g_0 \|_\infty - c. \qquad (8)$$

We may assume that without loss of generality

$$\| g_1 \|_\infty \leq \min\left\{c, \frac{1}{2}\| f - g_0 \|_\infty\right\} \qquad (9)$$

Let $g_2 = g_0 + g_1$. Then by (8) and (9) for all $x, y \in [a, b] \times [c, d] \backslash U$,

$|f(x, y) - g_2(x, y)| = |f(x, y) - g_0(x, y) - g_1(x, y)|$
$\leq \| f - g_0 \|_\infty - c + \| g_1 \|_\infty$
$\leq \| f - g_0 \|_\infty .$

Again, by (6) and (7), for all $x, y \in U$ with $(f(x, y) - g_0(x, y))g_1(x, y) < 0$,

$|f(x, y) - g_2(x, y)| = |(f(x, y) - g_0(x, y)) - g_1(x, y)|$
$= |f(x, y) - g_0(x, y)| + |g_1(x, y)|$
$\leq \| f - g_0 \|_\infty + k_f \| g_1 \|_\infty$
$= \| f - g_0 \|_\infty + k_f \| g_2 - g_0 \|_\infty .$

By (6) and (9) for all $x, y \in U$ with $(f(x, y) - g_0(x, y))g_1(x, y) \geq 0$,

$|f(x, y) - g_2(x, y)| = |(f(x, y) - g_0(x, y)) - g_1(x, y)|$
$= |f(x, y) - g_0(x, y)| - |g_1(x, y)|$
$\leq \| f - g_0 \|_\infty$

$\Rightarrow \| f - g_2 \|_\infty < \| f - g_0 \|_\infty + k_f \| g_2 - g_0 \|_\infty$,

$\Rightarrow$(i) fails. (ii) $\Rightarrow$ (iii) suppose that (ii) holds.
Let F: $\{g \in G : \| g \|_\infty = 1\} R \to R$ be the mapping, defined by

$$F(g) = \min_{x,y \in E(f-g_0)} \frac{(f(x,y) - g_0(x,y))g(x,y)}{\|f - g_0\|_\infty}$$

Since G is finite dimensional, the set $\{g \in G : \| g \|_\infty = 1\}$ is compact. Therefore, since by (ii) F (g) < 0 for all $\{g \in G : \| g \|_\infty = 1\}$, there exists a constant $k_f > 0$

Such that $F\left(\frac{g}{\|g\|_\infty}\right) \le -k_f$ for all $g \in G$, which proves (iii)

$\therefore (iii) \Rightarrow (ii)$ is obvious
Hence the proof of the theorem is complete.

Theorem 3.7. Let G be a finite dimensional subspace of a real 2-Hilbert space X . Then for every $x \in X$, there exists a unique best approximation from G .
Proof. Let (X, $\| \cdot, \cdot \|$) be a 2-Hilbert space and let G be a finite dimensional subspace of X .

Let $x, y \in X$ and $x \neq y$ then by parallelogram law

$$\| x + y, z \|^2 + \| x - y, z \|^2 = 2(\| x, z \|^2 + \| y, z \|^2). \qquad (10)$$

Let $\| x, z \| = \| y, z \| = 1$.

Then by (10) $\| x + y, z \|^2 = 4 \| x - y, z \|^2 < 4$

$$\Rightarrow \left\| \frac{x+y}{2}, z \right\|^2 < 1$$

$\Rightarrow$ X is strictly convex. Therefore, by Theorem 3.2 there exists a unique best approximation to $x \in X \setminus G$ from G .
Theorem 3.8. Let G be a subspace of a real 2-Hilbert space X , $x \in X \setminus G$ and $g_0 \in G$. Then the following statements are equivalent:

(i) The element g_0 is a best approximation of x from G . (ii) For all $g \in G$, $(x - g_0, g/z) = 0$ $z \in X \setminus V(x, G)$.
Proof. (ii) $\Rightarrow$ (i). Suppose that (ii) holds and let $g \in G$. Then by (ii) $\| x - g, z \|^2 - \| x - g_0, z \|^2$
$= (x - g, x - g/z) - (x - g_0, x - g_0/z)$
$= (x, x/z) + (g, g/z) - 2(x, g/z) - (x - x/z) - (g_0, g_0/z) + 2(x, g_0/z)$
≥ 0
$= (g - g_0, g - g_0/z) + 2(x - g_0, g_0 - g/z) \ge 0$
$= \| g - g_0, z \|^2 \ge 0$

That is $\| x - g, z \| \ge \| x - g_0, z \|$ which proves (i)

(i) $\Rightarrow$ (ii). Suppose that (ii) fails, i.e., there exists a function $g' \in G$ such that $(x - g_0, g') \neq 0$.

$$\left\| x - \left(g_0 + \frac{(x - g_0, g'/z)}{(g', g'/z)} g'/z \right) \right\|^2$$

$$= \|x - g_0/z\|^2 - \frac{(x - g_0, g'/z)^2}{(g', g'/z)}$$

$< \|x - g_0/z\|^2$ Which implies that g_0 is not a best approximation of x.

Hence the proof.

Corollary 3.9. Let $G = \text{span}(g_1, g_2, \cdots, g_n)$ be an n-dimensional subspace of a real 2-Hilbert space X, $x \in X \backslash G$ and $g_0 = \sum_{i=1}^{n} a_i g_i \in G$. Then the following statements are equivalent:

(i) The element g_0 is a best approximation of x from G.

(ii) The coefficients $a_1, a_2, \cdots a_n$ satisfy the following system of linear equations

$$\sum_{i=1}^{n} a_i (g_i, g_j/z) = (x, g_j/z), \quad j = 1, 2, \cdots, n.$$

Proof. The condition $(f - g_0, g/z) = 0$ in Theorem 3.8 is equivalent to $(f - g_0, g/z) = 0, j = 1, 2, \ldots, n$.

Since $g_0 = \sum_{i=1}^{n} a_i g_i$, $\sum_{i=1}^{n} a_i (g_i, g_j/z) = (f, g_j/z)$ is equivalent to $(f - g_0, g_j) = 0, j = 1, 2, \ldots, n$.

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