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On α-characteristic Equations and α-minimal Polynomial of Rectangular Matrices

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Abstract: *In this paper, we study rectangular matrices which satisfy the criteria of the Cayley-Hamilton theorem for a square matrix.Various results on $\alpha-$characteristic polynomials, $\alpha-$characteristic equations, $\alpha-$ eigenvalues and α-minimal polynomial of rectangular matrices are proved.*
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I. Definitions and introduction:

In [2], define the product of rectangular matrices A and B of order $m\times n$ by $A.B = A\alpha B$, for a fixed rectangular matrix $\alpha_{n\times m}$. With this product, we have $A^2 = A\alpha A,\ A^3 = A^2(\alpha A), A^4 = A^3(\alpha A)$,……., $A^n = A^{n-1}(\alpha A)$. Let us consider a rectangular matrix $A_{m\times n}$. Then we consider a fixed rectangular matrix $\alpha_{n\times m}$ of the opposite order of A. Then αA and $A\alpha$ are both square matrices of order n and m respectively. If $m\le n$, then αA is the highest n^{th} order singular square matrix and $A\alpha$ is the lowest m^{th} order square matrix forming their product. Then the matrix $\alpha A-\lambda I_n$ and $A\alpha-\lambda I_m$ are called the left $\alpha-$characteristic matrix and right $\alpha-$characteristic matrix of A respectively, where λ is an indeterminate. Also the determinant $|\alpha A-\lambda I|$ is a polynomial in λ of degree n, called the left $\alpha-$ characteristic polynomial of A and $|A\alpha-\lambda I|$ is a polynomial in λ of degree m, called the right $\alpha-$characteristic polynomial of A. That is, the characteristic polynomial of singular square matrix αA is called the left $\alpha-$ characteristic polynomial of A and the characteristic polynomial the square matrix $A\alpha$ is called the right $\alpha-$characteristic polynomial of A.

The equations $|\alpha A-\lambda I| = 0$ and $|A\alpha-\lambda I| = 0$ are called the left $\alpha-$characteristic equation and right $\alpha-$characteristicequation of A respectively. Then the rectangular matrix A satisfies the left $\alpha-$characteristic equation, and the left $\alpha-$characteristic equation of A is called the $\alpha-$characteristic equation of A.

For $m\ge n$, the rectangular matrix $A_{m\times n}$ satisfies the right $\alpha-$characteristic equation of A. So, in this case, the equation $|A\alpha-\lambda I| = 0$ is called the $\alpha-$characteristic equation of A. The roots of the $\alpha-$characteristic equation of a rectangular matrix A are called the $\alpha-$eigenvalues of A.
If λ is an $\alpha-$eigenvalue of A, the matrix $\alpha A-\lambda I_n$ is singular. The equation $(\alpha A-\lambda I_n)X = 0$ then possesses a non-zero solution i.e. there exists a non-zero column vector X such that $\alpha AX = \lambda X$. A non-zero vector X satisfying this equation is called a $\alpha-$characteristic vector or $\alpha-$eigenvector of A corresponding to the $\alpha-$eigenvalue λ.

For a rectangular matrix $A_{m\times n}$ over a field K, let $J(A)$ denote the collection of all polynomial $f(\lambda)$ for which $f(A)=0$ (Note that $J(A)$ is nonempty, since the $\alpha-$characteristic polynomial of A belongs to $J(A)$).Let $m_\alpha(\lambda)$ be the monic polynomial of minimal degree in $J(A)$. Then $m_\alpha(\lambda)$ is called the minimal polynomial of A.

II. Main results:

Theorem 2.1: If A is a rectangular matrix of order $m\times n$ and α is a rectangular matrix of order $n\times m$ then the left $\alpha-$ characteristic polynomial of a A and right $\alpha'-$ characteristic polynomial of a A' are same, where A' and α' are the transpose of A and α respectively.

Proof: Since $|\alpha A-\lambda I|=\left|(\alpha A-\lambda I)'\right|$

$$\Rightarrow |\alpha A-\lambda I|=\left|(\alpha A)'-(\lambda I)'\right|$$

$$\Rightarrow |\alpha A-\lambda I|=|A'\alpha'-\lambda I|$$

Which shows that the left $\alpha-$ characteristic polynomial of a A and right $\alpha'-$ characteristic polynomial of a A' are same.

Similarly we can show that If A is a rectangular matrix of order $m\times n$ and α is a rectangular matrix of order $n\times m$ then the right $\alpha-$ characteristic polynomial of a A and left $\alpha'-$ characteristic polynomial of a A' are same.

Theorem 2.2: Let $A_{m\times n}$, $m\le n$ be a rectangular matrix. If $\lambda_1,\lambda_2,\lambda_3,....,\lambda_n$ are $\alpha-$ characteristic roots of A then the $\alpha-$ characteristic roots A^2 are $\lambda_1{}^2,\lambda_2{}^2,\lambda_3{}^2,....,\lambda_n{}^2$.

Proof: Let λ be a $\alpha-$ characteristic root of the rectangular matrix $A_{m\times n}$, $m\le n$.Then there exists at least a nonzero column matrix $X_{n\times 1}$ such that

$$\alpha AX=\lambda X$$
$$\Rightarrow \alpha A(\alpha AX)=\alpha A(\lambda X)$$
$$\Rightarrow \alpha(A\alpha A)X=\lambda(\alpha AX)$$
$$\Rightarrow \alpha A^2X=\lambda(\lambda X)$$
$$\Rightarrow \alpha A^2X=\lambda^2X$$

Therefore λ^2 is a $\alpha-$ characteristic root of A^2 .

Thus we can conclude that if $\lambda_1,\lambda_2,\lambda_3,......,\lambda_n$ are $\alpha-$ characteristic roots of A then, $\lambda_1{}^2,\lambda_2{}^2,\lambda_3{}^2,....,\lambda_n{}^2$ are the $\alpha-$ characteristic roots of A^2 .

Theorem 2.3: Let A be a $m\times n$ $(m\le n)$rectangular matrix and α be a nonzero $n\times m$ rectangular matrix. If $a_m\lambda^m+a_{m-1}\lambda^{m-1}+a_{m-2}\lambda^{m-2}+.....+a_1=0$ is the right $\alpha-$ characteristic equation of A then $a_m\lambda^n+a_{m-1}\lambda^{n-1}+a_{m-2}\lambda^{n-2}+......+a_1\lambda^{n-m}=0$ is the left $\alpha-$ characteristic equation of A.

Proof: Since right $\alpha-$ characteristic equation of A is the characteristic equation of the square matrix $A\alpha$. So $A\alpha$ satisfies the right $\alpha-$characteristic equation of A. That is

$$a_m(A\alpha)^m+a_{m-1}(A\alpha)^{m-1}+a_{m-2}(A\alpha)^{m-2}+......+a_1I=0$$
$$\Rightarrow a_mA^m\alpha+a_{m-1}A^{m-1}\alpha+a_{m-2}A^{m-2}\alpha+......+a_1I=0$$
$$\Rightarrow a_mA^m\alpha A^{n-m}+a_{m-1}A^{m-1}\alpha A^{n-m}+a_{m-2}A^{m-2}\alpha A^{n-m}+.......+a_1IA^{n-m}=0$$
$$\Rightarrow a_mA^n+a_{m-1}A^{n-1}+a_{m-2}A^{n-2}+.......+a_1A^{n-m}=0$$
$$\Rightarrow \alpha(a_mA^n+a_{m-1}A^{n-1}+a_{m-2}A^{n-2}+.......+a_1A^{n-m})=\alpha.0$$
$$\Rightarrow a_m\alpha A^n+a_{m-1}\alpha A^{n-1}+a_{m-2}\alpha A^{n-2}+......+a_1\alpha A^{n-m}=0$$
$$\Rightarrow a_m(\alpha A)^n+a_{m-1}(\alpha A)^{n-1}+a_{m-2}(\alpha A)^{n-2}+......+a_1(\alpha A)^{n-m}=0$$

Therefore $a_m\lambda^n+a_{m-1}\lambda^{n-1}+a_{m-2}\lambda^{n-2}+......+a_1\lambda^{n-m}=0$ is the characteristic equation of the singular square matrix αA. Since the characteristic equation of the singular square matrix αA is the left $\alpha-$ characteristic equation of A. Hence the result.

Corollary 2.4: Let A be a $m\times n$ $(m\le n)$rectangular matrix and α be a nonzero $n\times m$ rectangular matrix. If $\lambda_1,\ \lambda_2,\ \lambda_3,\ ......,\ \lambda_m$ are right $\alpha-$ characteristic roots of A then $\lambda_1,\ \lambda_2,\ \lambda_3,\ ......,\ \lambda_m$, $\underbrace{0,0,.......0}_{(n-m)copies}$ are the left $\alpha-$ characteristic roots of A. That is,if A is a $m\times n$ $(m\le n)$rectangular matrix and

α is a nonzero $n\times m$ rectangular matrix then A has at most m number of nonzero left $\alpha-$ characteristic roots of A.

Corollary 2.5: Let A be a $m\times n$ $(m\geq n)$rectangular matrix and α be a nonzero $n\times m$ rectangular matrix. If $a_n\lambda^n+a_{n-1}\lambda^{n-1}+a_{n-2}\lambda^{n-2}+.....+a_1=0$ is the left $\alpha-$ characteristic equation of A then $a_n\lambda^m+a_{n-1}\lambda^{m-1}+a_{n-2}\lambda^{m-2}+......+a_1\lambda^{m-n}=0$ is the right $\alpha-$ characteristic equation of A.

nd α is a nonzero $n\times m$ rectangular matrix then A satisfies its $\alpha-$ characteristic equation.

Corollary 2.6: If A is a rectangular matrix of order $m\times n$ and α is a rectangular matrix of order $n\times m$ such that n=m+1 then $\alpha-$characteristic equation of A is $a_n\lambda^n+a_{n-1}\lambda^{n-1}+a_{n-2}\lambda^{n-2}+......+a_1\lambda=0$ and the rectangular matrix A satisfies it.

Theorem 2.7: If $m_\alpha(\lambda)$ be a $\alpha-$minimal polynomial of a rectangular matrix A of order $m\times n$ $(m\leq n)$, then the $\alpha-$ characteristic equation of A divides $[m_\alpha(\lambda)]^n$.

Proof: Suppose $m_\alpha(\lambda)=\lambda^r+c_1\lambda^{r-1}+c_2\lambda^{r-2}+......+c_{r-1}\lambda$. Consider the following matrices:

$$B_0=I$$
$$B_1=\alpha A+c_1I$$
$$B_2=(\alpha A)^2+c_1\alpha A+c_2I$$
$$B_3=(\alpha A)^3+c_1(\alpha A)^2+c_2\alpha A+c_3I$$

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$$B_{r-1}=(\alpha A)^{r-1}+c_1(\alpha A)^{r-2}+c_2(\alpha A)^{r-3}+.....+c_{r-1}I$$
$$B_r=(\alpha A)^r+c_1(\alpha A)^{r-1}+c_2(\alpha A)^{r-2}+c_3(\alpha A)^{r-3}+.....+c_rI$$

Then $B_0=I$

$$B_1-\alpha AB_0=c_1I$$
$$B_2-\alpha AB_1=c_2I$$
$$B_3-\alpha AB_2=c_3I$$

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$$B_{r-1}-\alpha AB_{r-2}=c_{r-1}I$$

And $-\alpha AB_{r-1}=c_rI-B_r$

$$=c_rI-[(\alpha A)^r+c_1(\alpha A)^{r-1}+c_2(\alpha A)^{r-2}+c_3(\alpha A)^{r-3}+......+c_rI]$$
$$=-[\alpha A^r+c_1\alpha A^{r-1}+c_2\alpha A^{r-2}+c_3\alpha A^{r-3}+......+c_{r-1}\alpha A]$$
$$=-\alpha(A^r+c_1A^{r-1}+c_2A^{r-2}+c_3A^{r-3}+......+c_{r-1}A)$$
$$=-\alpha.m_\alpha(A)$$
$$=-\alpha.0$$
$$=0$$

Set $B(\lambda)=\lambda^{r-1}B_0+\lambda^{r-2}B_1+\lambda^{r-3}B_2+.......+\lambda B_{r-2}+B_{r-1}$

Then $(\lambda I-\alpha A).B(\lambda)=(\lambda I-\alpha A).(\lambda^{r-1}B_0+\lambda^{r-2}B_1+\lambda^{r-3}B_2+.......+\lambda B_{r-2}+B_{r-1})$

$$=\lambda^rB_0+\lambda^{r-1}(B_1-\alpha AB_0)+\lambda^{r-2}(B_2-\alpha AB_1)+.......+\lambda(B_{r-1}-\alpha AB_{r-2})-\alpha AB_{r-1}$$
$$=\lambda^rI+c_1\lambda^{r-1}I+c_2\lambda^{r-2}I+......+c_{r-1}\lambda I$$
$$=(\lambda^r+c_1\lambda^{r-1}+c_2\lambda^{r-2}+......+c_{r-1}\lambda)I$$
$$=m_\alpha(\lambda)I$$

The determinant on both sides gives

$|\lambda I - \alpha A|.|B(\lambda)| = |m_\alpha(\lambda)I|$

$= [m_\alpha(\lambda)]^n.$

Since $|B(\lambda)|$ is a polynomial, $|\lambda I - \alpha A|$ divides $[m_\alpha(\lambda)]^n$. That is the $\alpha -$ characteristic polynomial of A divides $[m_\alpha(\lambda)]^n$.

Example: Find the $\alpha -$characteristic equation, $\alpha -$ eigenvalues, $\alpha -$ eigenvectors, $\alpha -$ minimal polynomial of the rectangular matrix $A = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 5 \end{pmatrix}$, where $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$.

Solution: The $\alpha -$characteristic equation of A is given $|\alpha A - \lambda I| = 0$, where I is the unit matrix of order three.

That is $\left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$

$\Rightarrow \begin{vmatrix} 9-\lambda & 0 & 0 \\ o & 4-\lambda & 5 \\ 0 & 4 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda(\lambda - 9)^2 = 0$

$\Rightarrow \lambda = 0,9,9$

Thus the $\alpha -$eigenvalues of A are 0, 9, 9.

Now, we find the $\alpha -$ eigenvectors of A corresponding to each $\alpha -$eigenvalue in the real field.

The $\alpha -$ eigenvectors of A corresponding to the $\alpha -$eigenvalue $\lambda = 0$ are nonzero column vector X given by the equation, $(\alpha A - 0.I)X = 0$

$\Rightarrow \alpha AX = 0$

$\Rightarrow \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$

$\Rightarrow \begin{pmatrix} 9x_1 \\ 4x_2 + 5x_3 \\ 4x_2 + 5x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow 9x_1 = 0, 4x_2 + 5x_3 = 0$

Therefore the $\alpha -$ eigenvectors corresponding to the $\alpha -$ eigenvalue $\lambda = 0$ are given by

$X = \begin{pmatrix} o \\ -5k \\ 4k \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 5k \\ -4k \end{pmatrix}$, where k is a real number.

Similarly, the $\alpha -$ eigenvectors of A corresponding to the $\alpha -$ eigenvalue $\lambda = 9$ can be calculated.

The $\alpha -$minimal polynomial $m_\alpha(\lambda)$ must divide $|\alpha A - \lambda I|$. Also each factor of $|\alpha A - \lambda I|$ that is λ and $\lambda - 9$ must also be a factor of $m_\alpha(\lambda)$. Thus $m_\alpha(\lambda)$ is exactly only one of the following:

$f(\lambda) = \lambda(\lambda - 9)$ or $\lambda(\lambda - 9)^2$. Testing $f(\lambda)$ we have

$f(A) = A^2 - 9A$

$$=\begin{pmatrix}9&0&0\\0&4&5\end{pmatrix}\begin{pmatrix}1&0\\0&1\\0&1\end{pmatrix}\begin{pmatrix}9&0&0\\0&4&5\end{pmatrix}-9\begin{pmatrix}9&0&0\\0&4&5\end{pmatrix}$$

$$=\begin{pmatrix}81&0&0\\0&36&45\end{pmatrix}-\begin{pmatrix}81&0&0\\0&36&45\end{pmatrix}$$

$$=0$$

Thus $f(\lambda)=m_{\alpha}(\lambda)=\lambda(\lambda-9)=\lambda^2-9\lambda$ is the $\alpha-$minimal of A.

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