IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 6, Issue 4 (May. - Jun. 2013), PP 49-58
www.iosrjournals.org

Chained Commutative Ternary Semigroups

[1] G. Hanumanta Rao, [2]A. Anjaneyulu, [3]A. Gangadhara Rao
[1]Department of Mathematics, S.V.R.M. College, Nagaram, Guntur (dt) A.P. India.
[2,3] Department of Mathematics, V.S.R & N.V.R.College, Tenali, A.P. India.

Abstract : *In this paper, the terms chained ternary semigroup, cancellable clement , cancellative ternary semigroup, A-regular element, π- regular element, π- invertible element, noetherian ternary semigroup are introduced. It is proved that in a commutative chained ternary semigroup T, i) if P is a prime ideal of T and* $x \notin P$ *then* $\bigcap_{n=1}^{\infty} x^n PT = P$ *for all odd natural numbers n . ii) T is a semiprimary ternary semigroup. iii) If* $a \in T$ *is a semisimple element of T, then* $< a >^w \neq \phi$. *iv) If* $< a >^w = \phi$ *for all* $a \in T$, *then T has no semisimple elements. v) T has no regular elements, then for any* $a \in T$, $< a >^w = \phi$ *or* $< a >^w$ *is a prime ideal. vi) If T is a commutative chained cancellative ternary semigroup then for every non π-invertible element a,* $< a >^w$ *is either empty or a prime ideal of T. Further it is proved that if T is a chained ternary semigroup with* $T \backslash T^3 = \{ x \}$ *for some* $x \in T$, *then i)* $T \backslash \{ x \}$ *is an ideal of T. ii)* $T = xT^1T^1 = T^1xT^1 = T^1T^1x$ *and* $T^3 = xTT = TxT = TTx$ *is the unique maximal ideal of T. iii) If* $a \in T$ *and* $a \notin < x >^w$ *then* $a = x^n$ *for some odd natural number* $n > 1$. *iv)* $T \backslash < x >^w = \{ x, x^3, x^5, ...\}$ *or* $T \backslash < x >^w = \{x, x^3, . . . , x^r\}$ *for some odd natural number r. v) If* $a \in T$ *and* $a \in < x >^w$ *then* $a = x^r$ *for some odd natural number r or* $a = x^n s_n t_n$ *and* $s_n \in < x >^w$ *or* $t_n \in < x >^w$ *for every odd natural number n. vi) If T contains cancellable elements then x is cancellable element and* $< x >^w$ *is either empty or a prime ideal of T. It is also prove that, in a commutative chained ternary semigroup T, T is archemedian ternary semigroup without idempotent elements if and only if* $< a >^w = \phi$ *for every* $a \in T$. *Further it is proved that if T is a commutative chained ternary semigroup containing cancellable elements and* $< a >^w = \phi$ *for every* $a \in T$, *then T is a cancellative ternary semigroup. It is proved that if T is a noetherian ternary semigroup containing proper ideals then T has a maximal ideal. Finally it is proved that if T is a commutative ternary semigroup such that* $T = < x >$ *for some* $x \in T$, *then the following are equivalent. 1)* $T = \{x, x^2, x^3, ...\}$ *is infinite. 2) T is a noetherian cancellative ternary semigroup with* $x \notin xTT$. *3) T is a noetherian cancellative ternary semigroup without idempotents. 4)* $< a >^w = \phi$ *for all* $a \in T$. *5)* $< x >^w = \phi$. *and if T is a commutative chained ternary semigroup with* $T \neq T^3$, *then the following are equivalent. (1)* $T=\{x, x^3, x^5, ...\}$, *where* $x \in T \backslash T^3$ *(2) T is Noetherian cancellative ternary semigroup without idempotents. (3)* $< a >^w = \phi$ *for all* $a \in T$. *Finally, it is proved that If T is a commutative chained noetherian cancellative ternary semigroup without regular elements, then* $< a >^w = \phi$ *for all* $a \in T$.

Keywords *- chained ternary semigroup, cancellable clement , cancellative ternary semigroup, noetherian ternary semigroup and ternary group.*

I. Introduction :

The algebraic theory of semigroups was widely studied by CLIFFORD and PRESTON [5], [6]; PETRICH [15]. The ideal theory in commutative semigroups was developed by BOURNE [4], HARBANS LAL [10], SATYANARAYANA [19], [20], MANNEPALLI and NAGORE [14]. The ideal theory in duo semigroups was developed by ANJANEYULU [1], [2], HOEHNKE [11] and KAR.S and MAITY. B. K[12], [13]. SANTIAGO [18] developed the theory of ternary semigroups. SARALA. Y, ANJANEYULU. A and MADHUSUDHANA RAO.D [16], [17] introduced the ideal theory in ternary semigroups and characterize the properties of ideals. GIRI and WAZALWAR[7] initiated the study of prime radicals in semigroups. ANJANEYULU. A[1], [2], [3] initiated the study of primary and semiprimary ideals in semigroups. He also introduced chained duo semigroups. HANUMANTHA RAO.G, ANJANEYULU. A and GANGADHARA RAO. A[8], [9] introduced the study of primary and semiprimary ideals in ternary semigroups. In this paper we introduce the notions of chained commutative ternary semigroups, noetherian ternary semigroups and characterize chained commutative ternary semigroups, noetherian ternary semigroups.

II. Priliminaries :

DEFINITION 2.1 : Let T be a non-empty set. Then T is said to be a ***ternary semigroup*** if there exist a mapping from T×T×T to T which maps $(x_1, x_2, x_3) \to [x_1x_2x_3]$ satisfying the condition : $[(x_1x_2x_3)x_4x_5] = [x_1(x_2x_3x_4)x_5] = [x_1x_2(x_3x_4x_5)]$ $\forall\ x_i \in$ T, $1 \le i \le 5$.

NOTE 2.2 : For the convenience we write $x_1x_2x_3$ instead of $[x_1x_2x_3]$

NOTE 2.3 : Let T be a ternary semigroup. If A,B and C are three subsets of T , we shall denote the set ABC = $\{abc : a \in A, b \in B, c \in C\}$.

DEFINITION 2.4 : A ternary semigroup T is said to be ***commutative*** provided $abc = bca = cab = bac = cba = acb$ for all $a,b,c \in$ T.

DEFINITION 2.5 : A nonempty subset A of a ternary semigroup T is said to be ***left ternary ideal*** or ***left ideal*** of T if $b, c \in$ T, $a \in$ A implies $bca \in$ A.

NOTE 2.6 : A nonempty subset A of a ternary semigroup T is a left ideal of T if and only if TTA $\subseteq$ A.

DEFINITION 2.7 : A nonempty subset of a ternary semigroup T is said to be a ***lateral ternary ideal*** or simply ***lateral ideal*** of T if $b, c \in$ T , $a \in$ A implies $bac \in$ A.

NOTE 2.8 : A nonempty subset of A of a ternary semigroup T is a lateral ideal of T if and only if TAT $\subseteq$ A.

DEFINITION 2.9 : A nonempty subset A of a ternary semigroup T is a ***right ternary ideal*** or simply ***right ideal*** of T if $b, c \in$ T , $a \in$ A implies $abc \in$ A

NOTE 2.10 : A nonempty subset A of a ternary semigroup T is a right ideal of T if and only if ATT $\subseteq$ A.

DEFINITION 2.11 : A nonempty subset A of a ternary semigroup T is a ***two sided ternary ideal*** or simply ***two sided ideal*** of T if $b, c \in$ T , $a \in$ A implies $bca \in$ A, $abc \in$ A.

NOTE 2.12 : A nonempty subset A of a ternary semigroup T is a two sided ideal of T if and only if it is both a left ideal and a right ideal of T .

DEFINITION 2.13 : A nonempty subset A of a ternary semigroup T is said to be ***ternary ideal*** or simply an ***ideal*** of T if $b, c \in$ T , $a \in$ A implies $bca \in$ A, $bac \in$ A, $abc \in$ A.

NOTE 2.14 : A nonempty subset A of a ternary semigroup T is an ideal of T if and only if it is left ideal, lateral ideal and right ideal of T .

DEFINITION 2.15 : An ideal A of a ternary semigroup T is said to be a ***proper ideal*** of T if A $\neq$ T.

DEFINITION 2.16 : An ideal A of a ternary semigroup T is said to be a ***trivial ideal*** provided T\ A is singleton.

DEFINITION 2.17 : An ideal A of a ternary semigroup T is said to be a ***maximal ideal*** provided A is a proper ideal of T and is not properly contained in any proper ideal of T.

DEFINITION 2.18 : An ideal A of a ternary semigroup T is said to be ***a principal ideal*** provided A is an ideal generated by $\{a\}$ for some $a \in$T. It is denoted by J (a) (or) < a >.

DEFINITION 2.19 : An ideal A of a ternary semigroup T is said to be a ***completely prime ideal*** of T provided $x, y, z \in$ T and $xyz \in$ A implies either $x \in$ A or $y \in$ A or $z \in$ A.

DEFINITION 2.20 : An ideal A of a ternary semigroup T is said to be a ***prime ideal*** of T provided X,Y,Z are ideals of T and XYZ $\subseteq$ A $\Rightarrow$ X $\subseteq$ A or Y $\subseteq$ A or Z $\subseteq$ A.

THEOREM 2.21 : Every completely prime ideal of a ternary semigroup T is a prime ideal of T.

THEOREM 2.22 : Let T be a commutative ternary semigroup . An ideal P of T is a prime ideal if and only if P is a completely prime ideal.

DEFINITION 2.23 : An ideal A of a ternary semigroup T is said to be a ***completely semiprime ideal*** provided $x \in$T, $x^n \in$A for some odd natural number n >1 implies $x \in$ A.

THEOREM 2.24 : An ideal A of a ternary semigroup T is semiprime if and only if X is an ideal of T, $X^3 \subseteq$ A implies X ⊆ A.

THEOREM 2.25 : Every prime ideal of a ternary semigroup T is semiprime.

NOTATION 2.26 : If A is an ideal of a ternary semigroup T, then we associate the following four types of sets.

A_1 = The intersection of all completely prime ideals of T containing A.

A_2 = $\{x \in T: x^n \in A$ for some odd natural numbers $n\}$

A_3 = The intersection of all prime ideals of T containing A.

A_4 = $\{x \in T: < x >^n \subseteq$ A for some odd natural number $n\}$

THEOREM 2.27 : If A is an ideal of a ternary semigroup T, then A $\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

THEOREM 2.28 : If A is an ideal of a commutative ternary semigroup T, then $A_1 = A_2 = A_3 = A_4$.

DEFINITION 2.29 : If A is an ideal of a ternary semigroup T , then the intersection of all prime ideals of T containing A is called ***prime radical*** or simply ***radical*** of A and it is denoted by $\sqrt{A}$ or *rad* A.

DEFINITION 2.30 : If A is an ideal of a ternary semigroup T , then the intersection of all completely prime ideals of T containing A is called ***completely prime radical*** or simply ***complete radical*** of A and it is denoted by *c.rad* A.

COROLLARY 2.31 : If $a \in \sqrt{A}$, then there exist an odd positive integer *n* such that $a^n \in$ A.

COROLLARY 2.32 : If A is an ideal of a commutative ternary semigroup T, then *rad* A = *c.rad* A.

DEFINITION 2.33 : An element *a* of a ternary semigroup T is said to be ***regular*** if there exist $x, y \in$ T such that $axaya = a$.

DEFINITION 2.34 : A ternary semigroup T is said to be ***regular ternary semigroup*** provided every element is regular.

DEFINITION 2.35 : An element *a* of a ternary semigroup T is said to be ***left regular*** if there exist $x, y \in$ T such that $a = a^3xy$.

DEFINITION 2.36 : An element *a* of a ternary semigroup T is said to be ***lateral regular*** if there exist $x, y \in$ T such that $a = xa^3y$.

DEFINITION 2.37 : An element *a* of a ternary semigroup T is said to be ***right regular*** if there exist $x, y \in$ T such that $a = xya^3$.

DEFINITION 2.38 : An element *a* of a ternary semigroup T is said to be ***intra regular*** if there exist $x, y \in$ T such that $a = xa^5y$.

DEFINITION 2.39 : An element *a* of a ternary semigroup T is said to be ***semisimple*** if $a \in <a>^3$ i.e. $<a>^3 = <a>$.

THEOREM 2.40 : An element *a* of a ternary semigroup T is said to be ***semisimple*** if $a \in <a>^n$ i.e. $<a>^n = <a>$ for all odd natural number *n*.

DEFINITION 2.41 : A ternary semigroup T is called ***semisimple ternary semigroup*** provided every element in T is semisimple.

DEFINITION 2.42 : An element *a* of a ternary semigroup T is said to be an ***idempotent*** element provided $a^3 = a$.

THEOREM 2.43 : Let T be a ternary semigroup and $a \in$ T. If *a* is idempotent, then *a* is semisimple.

THEOREM 2.44 : Let T be a ternary semigroup. If T has no semisimple elements, then T has no idempotent elements.

DEFINITION 2.45 : A ternary semigroup T is said to be an ***idempotent ternary semigroup*** or ***ternary band*** provided every element of T is an idempotent.

THEOREM 2.46 : If T is a ternary semigroup with unity 1 then the union of all proper ideals of T is the unique maximal ideal of T.

THEOREM 2.47 : If T is a commutative ternary semigroup and A is an ideal of T, then $abc \in$ A if and only if $<a><b><c> \subseteq$ A.

COROLLARY 2.48 : If T is a commutative ternary semigroup and *a, b, c* ∈ T, then $<abc> = <a><b><c>$.

DEFINITION 2.49 : An ideal A of a ternary semigroup T is said to be a ***completely semiprime ideal*** provided $x \in$ T, $x^n \in$ A for some odd natural number $n > 1$ implies $x \in$ A.

DEFINITION 2.50 : An ideal A of a ternary semigroup T is said to be ***semiprime ideal*** provided X is an ideal of T and $X^n \subseteq$ A for some odd natural number *n* implies X ⊆ A.

DEFINITION 2.51 : A ternary semigroup T is said to be an ***archimedean ternary semigroup*** provided for any $a, b \in$ T there exists an odd natural number *n* such that $a^n \in TbT$.

DEFINITION 2.52 : A ternary semigroup T is said to be a ***strongly archimedean ternary semigroup*** provided for any $a, b \in$ T, there exist an odd natural number *n* such that $<a>^n \subseteq <b>$.

THEOREM 2.53 : Every strongly archimedean ternary semigroup is an archimedean ternary semigroup.

THEOREM 2.54 : If T is a commutative ternary semigroup, then the following are equivalent.

1) T is a strongly archimedean semigroup.
2) T is an archimedean semigroup.
3) T has no proper completely prime ideals.
4) T has no proper prime ideals.

THEOREM 2.55 : An ideal Q of ternary semigroup T is a semiprime ideal of T if and only if $\sqrt{Q}$ =Q.

DEFINITION 2.56 : An ideal A of a ternary semigroup T is said to be ***semiprimary*** if $\sqrt{A}$ is a prime ideal
DEFINITION 2.57 : A ternary semigroup T is said to be ***semiprimary ternary semigroup*** if every ideal of T is a semi primary ideal.
DEFINITION 2.58 : A ternary semigroup T is said to be ***simple ternary semigroup*** if T is its only ideal.
THEOREM 2.59 : If T is a left simple ternary semigroup (or) a lateral simple ternary semigroup (or) a right simple ternary semigroup then T is a simple ternary semigroup.
THEOREM 2.60 : If T is a commutative ternary semigroup such that T^3 = T, then every maximal ideal of T is a prime ideal of T.
THEOREM 2.61 : T is a commutative ternary semigroup such that T^3 = Tand T having maximal ideals then T contains regular elements.

III. Chained Commutative Ternary Semigroups

DEFINITION 3.1 : A ternary semigroup T is said to be a ***chained ternary semigroup*** if the ideals in T are linearly ordered by set inclusion.
NOTE 3.2 : An ideal P of a commutative ternary semigroup T is prime if and only if it is completely prime. i.e., P is prime if and only if $x, y, z \in T$, $xyz \in P \Rightarrow$ either $x \in P$ or $y \in P$ or $z \in P$.

NOTATION 3.3 : If A is any ideal of a ternary semigroup T, then denote $A^w = \bigcap_{n=1}^{\infty} A^n$ where n is odd natural number**.**
THEOREM 3.4 : Let T be a commutative chained ternary semigroup and P is a prime ideal of T and $x \notin P$ then $\bigcap_{n=1}^{\infty} x^n$ PT= P for all odd natural numbers *n* .

***Proof* :** Since $x \notin P$ and P is prime , $x^n \notin P$ for all odd natural numbers *n*. Since $x^n \in T$ and P is an ideal of T, $x^n PT \subseteq P$ for all odd natural numbers *n*.

Therefore $\bigcap_{n=1}^{\infty} x^n PT \subseteq P$ for all $x \in T$. Since T is a commutative ternary semigroup, $x^n T^1 T^1$ is an ideal of T. Since $x^n \notin P$, $x^n T^1 T^1 \not\subseteq P$. Since T is a chained ternary semigroup, $P \subseteq x^n T^1 T^1$ for all odd natural numbers *n*. Let $y \in P$. Then $y \in x^n T^1 T^1 \Rightarrow y = x^n st$ for some $s, t \in T^1$. Now $x^n st \in P$, $x^n \notin P$. Since P is prime, $s \in P$ or $t \in P$. Therefore $y = x^n st \in x^n PT$ for all odd natural number *n* and hence $P \subseteq x^n PT$ for all odd natural number n. Hence $P \subseteq \bigcap_{n=1}^{\infty} x^n PT$ for all odd natural numbers $n \in N$. Therefore $P = \bigcap_{n=1}^{\infty} x^n PT$.

THEOREM 3.5 : If T is a commutative chained ternary semigroup, then T is a semiprimary ternary semigroup.

***Proof* :** Let A be an ideal of T. We have $\sqrt{A} = \bigcap_{n=1}^{\infty} P_\alpha$ = Intersection of all prime ideals of T containing A. Since T is commutative chained terinary semigroup, we have { $P_\alpha : \alpha \in \Delta$ } forms a chain. By Zorns Lemma, { $P_\alpha : \alpha \in \Delta$ } has minimal element say P_β. Therefore $\sqrt{A} = P_\beta$ and P_β is a prime ideal of T, and hence $\sqrt{A}$ is prime. Therefore A is a semiprimary ideal of T and hence T is a semiprimary ternary semigroup.

THEOREM 3.6 : Let T be a commutative chained ternary semigroup. If $a \in T$ is a semisimple element of T, then $< a >^w \neq \phi$.

***Proof* :** Suppose that *a* is a semisimple element of T. Therefore $a \in < a >^3$, implies that $< a > = < a >^3$. Therefore $a \in < a > = < a >^n$ for all odd natural numbers *n* and hence $a \in \bigcap_{n=1}^{\infty} < a >^n = < a >^w$ and hence $< a >^w \neq \phi$.

COROLLARY 3.7 : Let T be a commutative chained ternary semigroup. If $< a >^w = \phi$ for all $a \in T$, then T has no semisimple elements.

***Proof* :** Suppose that $< a >^w = \phi$ for all $a \in T$. Suppose if possible T has a semisimple element *x*. By theorem 3.6, $< x >^w \neq \phi$. It is a contradiction. Therefore T has no semisimple elements.

COROLLARY 3.8 : Let T be a commutative chained ternary semigroup. If $< a >^w = \phi$ for all $a \in T$, then T has no idempotent elements.

***Proof* :** Suppose that $< a >^w = \phi$ for all $a \in T$. By theorem 3.7, T has no semisimple elements. By theorem 2.44, T has no idempotent elements.

THEOREM 3.9 : Let T be a commutative ternary semigroup and $a \in T$. Then *a* is semisimple if and only if *a* is left, right, lateral regular and regular.

Proof **:** Suppose that a is semisimple in T. Therefore $a \in < a >^3$. Since T is commutative, $a \in < a >^3 = < a^3 >$. Therefore $a = a^3st$ for some $s, t \in$ T. Hence a is left regular. Since T is commutative, $a = a^3st = sa^3t = st\, a^3 = asata$. Therefore a is left, right, lateral regular and regular.
Conversely suppose that a is left, right, lateral regular and regular.
Therefore $a = a^3st = sa^3t = sta^3 = asata$ for some $s, t \in$ T. Now $a = a^3st \in < a^3 >= < a >^3$. Hence a is semisimple.

THEOREM 3.10 : Let T be a chained commutative ternary semigroup. If T has no regular elements, then for any $a \in$ T, $< a >^w = \phi$ or $< a >^w$ is a prime ideal.

Proof **:** Suppose that T has no idempotent elements and $a \in$ T.
We have $< a >^w = \bigcap_{n=1}^{\infty} < a >^n$.
Assume that $< a >^w \neq \phi$. If possible, suppose that $< a >^w$ is not prime. Then there exist $x, y, z \in$ T such that $xyz \in < a >^w$ and $x, y, z \notin < a >^w$. By theorem 2.47, $< x > < y >< z > = < xyz > \subseteq < a >^w$.
Now $x, y, z \notin < a >^w$, implies that there exists odd natural numbers n, m, p such that $x \notin < a >^n$, $y \notin < a >^m$ and $z \notin < a >^p$.
Consider $k = \min \{n, m, p\}$. Then $x, y, z \notin < a >^k$. Since T is commutative chained ternary semigroup, we have $< a >^k \subseteq < x >$, $< a >^k \subseteq < y >$ and $< a >^k \subseteq < z >$.
Therefore $< a >^{3k} = < a >^k< a >^k< a >^k \subseteq < x > < y >< z > = < xyz > \subseteq < a >^w \subseteq < a >^{9k}$.
Then $< a >^{3k} \subseteq < a >^{9k} = < a >^{3k} < a >^{3k}< a >^{3k}$ and hence $a^{3k} \in < a^{3k} >^3$.
Therefore a^{3k} is a semisimple element of T. By theorem 3.9, a^{3k} is a regular element of T.
It is a contradiction. Hence $< a >^w$ is a prime ideal of T.

DEFINITION 3.11 : Let T be ternary semigroup and $a \in$ T. Then a is said to be a
left cancellable clement if $aax = aay \Rightarrow x = y$,
lateral cancellable clement if $axa = aya \Rightarrow x = y$,
right cancellable clement if $xaa = yaa \Rightarrow x = y$ holds for all $x, y \in$ T.

DEFINITION 3.12 : Let T be ternary semigroup and $a \in$ T. Then a is said to be ***cancellable clement*** if it is left, lateral and right cancellable element.

DEFINITION 3.13 : A ternary semigroup T is said to be a
left cancellative if $abx = aby \Rightarrow x = y$ for all $a, b \in$ T
lateral cancellative if $axb = ayb \Rightarrow$ x = y for all $a, b \in$ T
right cancellative if $xab = yab \Rightarrow$ x = y for all $a, b \in$ T.

DEFINITION 3.14 : A ternary semigroup T is said to be ***cancellative ternary semigroup*** if T is left, lateral and right cancellative.

THEOREM 3.15 : In a ternary semigroup T, the following are equivalent.
1. T is lateral cancellative.
2. T is left and right cancellative.
3. T is cancellative

Proof : (1) ⇒ (2) : Suppose that ternary semigroup T is lateral cancellative. Therefore $axb = ayb \Rightarrow x = y$.
Let $a, b, x, y \in$ T such that $xab = yab$.
Now $ab[xab] = ab[yab]] \Rightarrow a[bxa]b = a[bya]b \Rightarrow bxa = bya \Rightarrow x = y$.
Thus T is right cancellative. Similarly we can prove that T is left cancellative.
(2) ⇒ (3) : Suppose that ternary semigroup T is left and right cancellative.
Let $a, b, x, y \in$ T such that $axb = ayb$.
Now $axb = ayb \Rightarrow a[axb]b = a[ayb]b \Rightarrow [aax]bb = [aay]bb \Rightarrow aa[x]bb = aa[y]bb$. Since T is left and right cancellative, we get $x = y$. Thus T is lateral cancellative.
(3) ⇒ (1) : Suppose that ternary semigroup T is cancellative. By the definition 3.13, T is lateral cancellative.

DEFINITION 3.16 : Let T be a ternary semigroup and $a \in$ T. Then a is said to be ***strongly regular element*** if there exists $x \in$ T such that $axaxa = a$.

THEOREM 3.17: Let T be a ternary semigroup and $a \in$ T. Then a regular element in T if and only if a is *strongly* regular element in T.

Proof **:** Suppose that a regular element in T. Therefore there exists $x, y \in$ T such that $axaya = a$. Now $axayaxaya = axaya = a \Rightarrow a(xay)a(xay)a = a$. That is $asasa = a$ where $(xay) = s$. Hence a is strongly regular.
Conversely, suppose that a is strongly regular element in T. Therefore there exists $x \in$ T such that $axaxa = a$. Hence a is regular in T.

DEFINITION 3.18 : Let T be a ternary semigroup and $a \in$ T. Then a is said to be ***π- regular*** if there exists $x \in$ T such that $a^nxa^nxa^n = a^n$ for some odd natural number n.

DEFINITION 3.19 : Let T be a ternary semigroup and $a \in$ T. Then a is said to be ***π- invertible element*** if there exists $x \in$ T such that $a^n x a^n x a^n = a^n$ and $x a^n x a^n x = x$ for some odd natural number n.

THEOREM 3.20 : If T is a commutative chained cancellative ternary semigroup then for every non π-invertible element a, $< a >^w$ is either empty or a prime ideal of T.

Proof **:** Suppose that a is a non π-invertible element in T. If $< a >^w = \phi$ then theorem is trivial. Let $< a >^w \neq \phi$. If possible, suppose that $< a >^w$ is not prime.

Then there exist $x, y, z \in$ T such that $xyz \in < a >^w$ and $x, y, z \notin < a >^w$.

By theorem 2.48, $< x > < y >< z > = < xyz >$. Now $x, y, z \notin < a >^w$, implies that there exists odd natural numbers n, m, p such that $x \notin < a >^n$, $y \notin < a >^m$ and $z \notin < a >^p$.

Consider k = min { n , m, p }. Then $x, y, z \notin < a >^k$. Since T is chained ternary semigroup, we have $< a >^k \subseteq < x >$, $< a >^k \subseteq < y >$ and $< a >^k \subseteq < z >$.

Therefore $< a >^{3k} = < a >^k < a >^k < a >^k \subseteq < x > < y >< z > = < xyz > \subseteq < a >^{9k} \subseteq < a >^w$.

Then $< a >^{3k} \subseteq < a >^{9k} = < a >^{3k} < a >^{3k} < a >^{3k}$ and hence $a^{3k} \in < a^{3k} >^3$. Therefore a^{3k} is a semisimple element of T. By theorem 3.9, a^{3k} is a regular element of T. By theorem 3.17, a^{3k} is a strogly regular element of T. Therefore $a^{3k} = a^{3k} x a^{3k} x a^{3k}$ for some $x \in$ T.

Now $a^{3k} x a^{3k} x a^{3k}\ x a^{3k} = a^{3k} x a^{3k}$. Since T is cancellative, $x a^{3k} x\ a^{3k} x = x$. Hence a is a π-invertible element in T. It is a contradiction. Thus $< a >^w$ is a prime ideal of T.

Hence $< a >^w = \phi$ or $< a >^w$ is prime ideal of T.

THEOREM 3.21 : Let T be a chained ternary semigroup. If T≠T^3 then T\T^3 = { x } for some $x \in$ T.

Proof **:** Suppose if possible $x, y \in$ T\T^3 and $x \neq y$. Since T is a chained ternary semigroup, $< x > \subseteq < y >$ or $< y > \subseteq < x >$. If $< x > \subseteq < y >$, then $x \in < y >$ and hence $x = yst$ for some $s, t \in$ T. Therefore $x \notin$ T^3, which is not true. If $< y > \subseteq < x >$, then $y \in < x >$ and hence $y = xpq$ for some $p, q \in$T. Therefore $y \in$ T^3, which is not true. It is a contradiction . Therefore $x = y$. So there exists unique $x \in$ T such that $x \notin$ T^3. Therefore T\T^3 = { x }.

THEOREM 3.22 : Let T be a chained ternary semigroup with T\T^3 = { x } for some $x \in$ T. Then T\ { x } is an ideal of T.

Proof **:** Let $a \in$ T\{x}and $s, t \in$ T. we have $ast \in$ T^3. Since $x \notin$ T^3, we have $ast \neq x$ and hence $ast \in$ T\{ x }. Hence T\{ x } is a right ideal of T. similarly, we can get $sta, sat \in$ T\{ x }. Therefore T\ { x } is an ideal of T.

THEOREM 3.23 : Let T be a commutative chained ternary semigroup. If T≠T^3 such that T\T^3 = { x } for some $x \in$ T, then T = xT^{1}T^1 = T^1xT^1 = T^{1}T^1x and T^3 = xTT = TxT = TTx is the unique maximal ideal of T.

Proof **:** Since T\T^3 ={ x }, T^3 = T\{x}. Now xT^{1}T^1 is an ideal of T and T^3 is an ideal of T. Since $x \notin$ T^3 and T is a chained ternary semigroup, T$^3 \subseteq x$ T^{1}T^1. Clearly, xTT $\subseteq$ T^3. Hence T^3 = TTx = TxT = xTT.

Since T^3 is trivial, T^3 = xTT = TxT = TTx is the unique maximal ideal of T.

THEOREM 3.24 : Let T be a commutative chained ternary semigroup with T ≠ T^3 such that T\T^3 = { x } for some $x \in$ T. If $a \in$ T and $a \notin < x >^w$ then $a = x^n$ for some odd natural number n > 1

Proof **:** Since T is a commutative chained ternary semigroup with T ≠ T^3 such that T\T$^3 \neq$ { x }. By theorem 3.23, T^3 = TTx = xTT = T\ { x }. Since $a \notin < x >^w$, there exists a odd natural number k such that $a \notin < x >^k$. Let n be the least odd positive integer such that $a \in < x >^{n-2}$ and $a \notin < x >^n$. Therefore $a \in x^{n-2}$TT \ x^n TT and hence $a = x^{n-2} st$ for some $s, t \in$ T.

If $s, t \in x$ TT then $a = x^n\ s_n s^1{}_n t_n t^1{}_n \in x^n$ TT = $< x^n >$ = $< x >^n$. It is a contradiction. Hence $s, t \notin x$ TT. Therefore $s = x$ and $t = x$. Thus $a = x^n$ for some odd natural number n. If n = 1 then $a = x \in < x >$. It is a contradiction. Therefore n > 1.

THEOREM 3.25 : Let T be a commutative chained ternary semigroup with T\T^3 = { x }. Then T\ $< x >^w$ = { x, x^3, x^5, ...} or T\ $< x >^w$ ={x, x^3, . . . , x^r} for some odd natural number r.

Proof **:** By theorem 3.24, T\ $< x >^w \subseteq$ { x, x^3, x^5, ...}. If $x^n \in$ T\ $< x >^w$ for all odd natural number n, then T \ $< x >^w$ = { x, x^3, x^5, ... } . If $x^n \notin$ T \ $< x >^w$ for some odd natural number n, then we can choose the least odd positive integer r is such that $x^{r+2} \notin$ T \ $< x >^w$. Therefore $x, x^3, . . . x^r \in$ T \ $< x >^w$ for all $n > r$. Therefore T \ $< x >^w$ ={ $x, x^3, x^5, ..., x^r$}.

THEOREM 3.26 : Let T be a commutative chained ternary semigroup with $T \neq T^3$ such that $T \backslash T^3 = \{ x \}$. If $a \in T$ and $a \in \langle x \rangle^w$ then $a = x^r$ for some odd natural number r or $a = x^n s_n t_n$ and $s_n \in \langle x \rangle^w$ and $t_n \in \langle x \rangle^w$ for every odd natural number n or $a = x^m z$ where $z \in \langle x \rangle^w$ for some even natural number m.

Proof **:** Since T is a commutative chained ternary semigroup with $T \neq T^3$ such that $x \in T \backslash T^3$. By theorem 3.23, $T^3 = TTx = xTT = T \backslash \{ x \}$.

Let $a \in T$. Suppose that $a \in \langle x \rangle^w$. Now $a \in \langle x \rangle^w$ implies that $a \in \bigcap_{n=1}^{\infty} \langle x \rangle^n$.

Therefore $a \in \langle x \rangle^n = \langle x^n \rangle$ for every odd natural number n. Therefore $a = x^n s_n t_n$ for some $s_n, t_n \in T$ for every odd natural number n.

Case 1 : If $s_n, t_n \notin \langle x \rangle^w$ for some odd natural number n. By theorem 3.24, $s_n = x^r$, $t_n = x^p$ for some odd natural number $r, p > 1$ and hence $a = x^{n+r+p}$ for some odd natural number $n + r + p$.

Case 2 : If $s_n, t_n \in \langle x \rangle^w$, then $a = x^n s_n t_n$ where $s_n, t_n \in \langle x \rangle^w$.

Case 3 : If only one of the s_n or $t_n \in \langle x \rangle^w$. Suppose that $s_n \in \langle x \rangle^w$ and $t_n \notin \langle x \rangle^w$ then $t_n = x^p$ for some odd natural number p. Therefore $a = x^n s_n t_n = a = x^{n+p} s_n$ where $n + p$ is even.

Hence $a = x^m z$ *where* $z \in \langle x \rangle^w$ for some even natural number m.

THEOREM 3.27 : Let T be a commutative chained ternary semigroup with $T \backslash T^3 = \{ x \}$. If T contains cancellable elements then x is cancellable element and $\langle x \rangle^w$ is either empty or a prime ideal of T.

Proof **:** Suppose , if possible x is not cancellable in T. Let Z be the set of all non cancellable elements of T. Clearly $x \in Z$. So Z is non empty subset of T. Let $a \in Z$ and $s, t \in T$.

Since $a \in Z$, a is not cancellable in T. So there exists $b, c \in T$ such that $aab = aac$ and $b \neq c$.

Now $aab = aac \Rightarrow stst(aab) = stst(aac) \Rightarrow (sta)(sta)b = (sta)(sta)c$ and $b \neq c$.

Hence $sta \in Z$ and hence Z is a left ideal of T. Since T is a commutative ternary semigroup, Z is an ideal of T.

Since $T \backslash T^3 = \{ x \}$, by theorem 3.23, $T = x T^1 T^1$. Since $x \in Z$, Z is an ideal of T,

$x T^1 T^1 \subseteq Z$. Thus $T \subseteq Z$ and hence T = Z. Therefore every element of T is non cancellable. It is a contradiction. Therefore x is cancellable element in T.

Suppose that $\langle x \rangle^w \neq \phi$. Let $a, b, c \in T$ such that $abc \in \langle x \rangle^w$.

Suppose if possible $a \notin \langle x \rangle^w$, $b \notin \langle x \rangle^w$ and $c \notin \langle x \rangle^w$. Now $a, b, c \notin \langle x \rangle^w$, implies that by theorem 3.24, $a = x^n$, $b = x^m$ and $c = x^p$ for some odd natural numbers n, m, p.

Therefore $x^{n+m+p} = abc \in \langle x \rangle^w \subseteq \langle x \rangle^{n+m+p+2}$, implies that $x^{n+m+p} = x^{n+m+p+2} st$ for some $s, t \in T$.

Now $x^{n+m+p} = x^{n+m+p+2} st$ and x is cancellative, implies that $x = x^3 st$ for some $s, t \in T$.

Therefore $x = x^3 st \in T^3$. It is a contradiction. Therefore either $a \in \langle x \rangle^w$ or $b \in \langle x \rangle^w$ or $c \in \langle x \rangle^w$ and hence $\langle x \rangle^w$ is a prime ideal. Therefore $\langle x \rangle^w$ is either empty or a prime ideal of T.

THEOREM 3.28 : Let T be a commutative chained ternary semigroup. Then T is archemedian ternary semigroup without idempotent elements if and only if $\langle a \rangle^w = \phi$ for every $a \in T$.

Proof **:** Suppose that T is an archemedian ternary semigroup without idempotents. If possible, suppose that $\langle a \rangle^w \neq \phi$ for some $a \in T$. By theorem 3.10, $\langle a \rangle^w$ is a prime ideal of T. Since T is an archemedian commutative ternary semigroup, by theorem 2.54, T has no proper prime ideals. Therefore $\langle a \rangle^w = T$. Now $a \in \langle a \rangle^w \subseteq \langle a \rangle^3$ and hence a is semisimple. By theorem 3.9, a is regular. So T has idempotent elements. It is a contradiction. Hence $\langle a \rangle^w = \phi$ for every $a \in S$. Conversely suppose that $\langle a \rangle^w = \phi$ for every $a \in S$. Since $\langle a \rangle^w = \phi$ for every $a \in T$, By corollary 3.7, T has no semisimple elements. By theorem 2.44, T has no idempotent elements. If possible, suppose that P is proper prime ideal of T.

Let $x \in T$ such that $x \notin P$. Since $x \notin P$, by theorem 3.3, $P = \bigcap_{n=1}^{\infty} x^n PT \subseteq \langle x \rangle^w$. Therefore $P \subseteq \langle x \rangle^w = \phi$. It is a contradiction. Hence T has no proper prime ideals. By theorem 2.54, T is an archemedian ternary semigroup.

THEOREM 3.29 : If T is a commutative chained ternary semigroup containing cancellable elements and $\langle a \rangle^w = \phi$ for every $a \in T$, then T is a cancellative ternary semigroup.

Proof **:** Let T be a commutative chained ternary semigroup containing cancellable elements. Suppose that $\langle a \rangle^w = \phi$ for every $a \in T$.

Let Z be the set of all noncancellative elements in T. If possible, suppose that Z is a nonempty subset of T. If $x \in$ Z, then there exists $y, z \in$ T such that $xxy = xxz$ and $y \neq z$. Therefore for any $s, t \in$ T, $stst(xxy) = stst(xxz)$ implies that $(stx)(stx)y = (stx)(stx)z$ and $y \neq z$. Hence $(stx) \in$ Z. Therefore Z is a left ideal of T and T is commutative, implies that Z is an ideal of T.

If possible, suppose that Z is not prime. Then there exists $a, b, c \in$ T such that $abc \in$ Z and $a, b, c \notin$ Z. Now $abc \in$ Z, implies that $(abc)(abc)x = (abc)(abc)y$ for some $x, y \in$ T and $x \neq y$. Hence $aa(bc)(bc)x = aa(bc)(bc)y$ and $a \notin$ Z, implies that $(bc)(bc)x = (bc)(bc)y$. Similarly $b, c \notin$ Z, $x = y$. It is a contradiction. Therefore Z is a prime ideal of T. Since $< a >^w = \phi$ for every $a \in$ T, by theorem3.28, we have T is an archemedian ternary semigroup without idempotents. Therefore by theorem 2.54, T has no proper prime ideals and hence Z = T.

It is contradiction to T contains cancellable elements. Hence Z = ϕ. Thus T is cancellative ternary semigroup.

DEFINITION 3.30 : A ternary semigroup T is said to be ***ternary group*** if for all $a, b, c \in$ T, there exists $x, y, z \in$ G such that $[xab] = [ayb] = [abz] = c$.

THEOREM 3. 31 :If T is a ternary semigroup and *a, b* ∈ T, then *ab*T = { *abt* : *t* ∈ T } is a right ideal of T.

Proof **:** Let $x \in ab$T and $s, t \in$ T. Now $x \in ab$T, implies that $x = abu$ for some $u \in$T.

Since $s, t, u \in$ T, we have $ust \in$ T. Therefore $abust \in ab$T . That is $xst \in ab$T. Hence *ab*T is a right ideal of T.

COROLLARY 3.32 : If T is a ternary semigroup and *a, b* ∈ T, then T*ab* = { *tab* : *t* ∈ T } is a left ideal of T.

Proof **:** The proof of the theorem follows the above theorem.

COROLLARY 3.33 : If T is a commutative ternary semigroup and *a, b* ∈ T, then T*ab* = { *tab* : *t* ∈ T } is an ideal of T.

COROLLARY 3.34 : If T is a commutative ternary semigroup and *a, b* ∈ T, then *ab*T = { *abt* : *t* ∈ T } is an ideal of T.

COROLLARY 3.35 : If T is a commutative ternary semigroup and *a, b* ∈ T, then *a*T*b* = { *atb* : *t* ∈ T } is an ideal of T.

COROLLARY 3.36 : If T is a ternary group and *a* ∈ T, then T*aa* = { *taa* : *t* ∈ T } is a left ideal of T.

COROLLARY 3.37 : If T is a ternary group and *a* ∈ T, then *aa*T = { *aat* : *t* ∈ T } is a left ideal of T.

THEOREM 3.38 : Let T be a commutative chained ternary semigroup. Then T is ternary group if and only if T is simple ternary semigroup.

Proof **:** Suppose that T is a ternary group. Let A be an ideal of T. Clearly, A ⊆ T. Let $t \in$ T and $a \in$ A.

Now $t, a \in$ T and T is ternary group, implies that the equation $axa = t$ has solution in T.

Therefore there exists $s \in$T, such that $asa = t$. Hence $t = asa \in < a > \subseteq$ A. Therefore A = T.

Thus T has no proper ideals. Hence T is simple ternary group.

Conversely, suppose that T is simple ternary semigroup. Therefore T has no proper ideals. Let a, b, c ∈ T.

By theorem 3.31, we have *ab*T = {*abt* : *t* ∈ T} is a right ideal of T. Since T is commutative, *ab*T is an ideal of T. Since T has no proper ideals, we have *ab*T = T. Therefore c ∈ T = *ab*T. Therefore, there exists *s* ∈ T, such that *c* = *abs*. Hence the equation *abx* = *c* has a solution in T. Similarly, we can prove the equations *axb* = *c* and *xab* = *c* has solution in T. Thus T is a ternery group.

COROLLARY 3.39 : If T is a commutative ternary group, then *ab*T = T*ab* = *a*T*b* = T for all *a, b* ∈ T.

COROLLARY 3.40 : If T is a commutative ternary group, then *aa*T = T*aa* = *a*T*a* = T for all *a* ∈ T.

THEOREM 3.41 : If T is a ternary group, then every element of T is regular element in T.

Proof **:** Suppose that T is a ternary group and *a* ∈ T. By corollary 3.40, we have *a*T*a* = T. Now *a* ∈ T and *a*T*a* = T, implies that *a* ∈ *a*T*a*. Therefore, $a = axa$ for some $x \in$ T.

Hence $axaxa = axa = a$. Therefore *a* is strongly regular and hence regular in T. Thus every element of T is regular element in T.

THEOREM 3.42 : If T is a commutative cancellative archemedian chained ternary semigroup with $< a >^w \neq \phi$ for some *a*∈ T, then T is a ternary group.

Proof **:** Let T be a commutative cancelative archemedian chained ternary semigroup with $< a >^w \neq \phi$ for some $a \in$ *T*. If possible, suppose that T has no idempotent elements. Since $< a >^w \neq \phi$, then by theorem 3.10, $< a >^w$ is a prime ideal of T. Since T is an archemedian commutative ternary semigroup by theorem 2.53, T has no proper prime ideals. It is a contradiction. Hence T has idempotent elements. Let *e* be an idempotent element in T. Then $xe^3 = xe$ for every $x \in$ T. Since T is cancellative , we have $xee = x$ for every $x \in$ T. Since T is commutative, $eex = exe = xee = x$ for every $x \in$T. Let $a, b, c \in$ T. Now $e, b, a \in$T and T is archemedian ternary semigroup, implies that $e^n \in < a >$ and $e^n \in < b >$ for some odd natural number n. Since T is commutative, $e \in a$TT and $e \in$ TTb. Therefore $e = axy$ and $e = pqb$ for some $x, y, p, q \in$ T. Now $c = ece = (axy)c(pqb)$, implies that $c = a(xycpq)b$. Therefore $s = xycpq$ is the solution of

$c = asb$. Since T is commutative, the equations $axb = abx = xab = c$ has solution in T. Therefore T is a ternary group.

DEFINITION 3.43 : A ternary semigroup T is said to be a ***noetherian ternary semigroup*** if every ascending chain of ideals becomes stationary ; i.e., If $A_1 \subseteq A_3 \subseteq A_3 \subseteq$....... is an ascending chain of ideals of T, then there exists a odd natural number m such that $A_m = A_n$ for all natural numbers $n \geq m$.

THEOREM 3.44 : If T is a noetherian ternary semigroup containing proper ideals then T has a maximal ideal.

Proof : Let A_1 be a proper ideal of T. If A_1 is not a maximal ideal, then there exists a proper ideal A_2 of T such that $A_1 \subseteq A_2$. If A_2 is not a maximal ideal, then there exists a proper ideal A_3 of T such that $A_1 \subseteq A_2 \subseteq A_3$. By continuing this process we get an ascending chain of proper ideals of T. Since T is noetherian, The chain $A_1 \subseteq A_2 \subseteq A_3$ . . . . is stationary. Therefore there exists a odd natural number n such that $A_n = A_{n+1} = A_{n+2} = \ldots$ Therefore A_n is maximal ideal of T. Hence T has a maximal ideal.

THEOREM 3.45 : If T is a commutative ternary semigroup such that T = < x > for some $x \in$S, then the following are equivalent.

1) T = {$x, x^3, x^5, \ldots$ } is infinite.

2) T is a noetherian cancellative ternary semigroup with $x \notin x$TT.

3) T is a noetherian cancellative ternary semigroup without regular elements.

4) $< a >^w = \phi$ for all $a \in$ T.

5) $< x >^w = \phi$.

Proof : (1) $\Rightarrow$ (2) : Suppose that T = {$x, x^3, x^5, \ldots$} is infinite. Therefore T = < x > and $x \in$ T. Therefore every ideal of T principle ideal of T. Let $A_1 \subseteq A_2 \subseteq A_3$ .... be an ascending chain of ideals of T.

Therefore A = $\bigcup_{i=1}^{\infty} A_i$ also an ideal of T and A is a principle ideal of T. Suppose that A = < a > for some $a \in$ T.

Then $a \in \bigcup_{i=1}^{\infty} A_i$ and hence $a \in A_t$ for some odd natural number t.

Therefore A = $\bigcup_{i=1}^{\infty} A_i = < a > \subseteq A_t \subseteq \bigcup_{i=1}^{\infty} A_i$, and hence A = $\bigcup_{i=1}^{\infty} A_i = A_t$.

Therefore $A_t = A_{t+1} = A_{t+3}$ =....... and hence T is a noetherian ternary semigroup.

Let $a, b, c, d \in$S such that $abc = abd$. Now $a, b, c, d \in$S = < x > imples that $a = x^n, b = x^m, c = x^s , d = x^p$ for some odd natural numbers $n, m, s, p \in$ N. Now $abc = abd$, implies that $x^n x^m x^s = x^n x^m x^p \Rightarrow x^{n+m+s} = x^{nm+ps}$. Since T is infinite set, $n + m + s = n + m + p$ and hence $s = p$. Therefore $x^s = x^p \Rightarrow b = c$. Hence T is cancellative. Suppose that $x \in x$TT. Therefore $x = xx^n x^m$ for some odd natural numbers n, m. Thus $x^{n+m+1} = x$ and hence T is finite. It is a contradiction. So $x \notin x$TT. Therefore T is a noetherian, cancellative ternary semigroup and $x \notin x$TT.

(2) $\Rightarrow$(3) : Suppose that T is a noetherian cancellative ternary semigroup and $x \notin x$TT. If possible, suppose that T has idempotent elements. Let e be an idempotent element in T. Therefore $xe^3 e = xee$. Thus $(xe^2) ee = xee \Rightarrow xe^2 = x$. Since T is a cancellative ternary semigroup $x = xe^2 \in x$TT. It is a contradiction. Hence T has no idempotent elements. Therefore T is a noetherian, cancellative without idempotents.

(3)$\Rightarrow$ (4) : Suppose that T is a noetherian cancellative ternary semigroup without idempotents. Let $a \in$ T. If possible, suppose that $< a >^w = \phi$. Then there exists $b \in$ T such that $b \in < a >^w$. Now $b \in \bigcap_{i=1}^{\infty} < a >^n$, implies that $b \in < a >^n$ for all odd natural numbers n and hence $b = a^i s_i t_i$ for some $s_i , t_i \in$ T for all i = 1, 3, 5, ...

Therefore $b = a^i s_i t_i = a^{i+2} s_{i+1} t_{i+1}$ for i = 1, 3, 5, ... Now consider $y_i = a^i s_i t_i$ for all i = 1, 3, 5, ..., then $y_i = a^2 y_{i+2}$ for all i = 1, 3, 5, ... Therefore $< y_i > \subseteq < y_{i+2} >$ for each i = 1, 3, 5, ... Since T is Noetherian, the chain $< y_1 > \subseteq < y_3 > \subseteq < y_5 > \subseteq$ . . . . . . is stationary. Therefore there exists a odd natural number n such that $< y_n > = < y_{n+2} > = < y_{n+5} >$ = . . . Thus $y_{n+2} = st\, y_n$ for some $s, t \in$ T. Now $a^2 y_{n+2} = y_n$, implies that $a^2 sts_n = s_n$ and hence $(a^3 st)\, s_n = (a)\, s_n$, implies that $(a^3 st) s_n s_n = a\, s_n s_n$. By cancellative law, we have $a^3 s = a$. Thus a is left regular clement in T. Since T is commutative, a is regular element in T. Therefore T has regular elements. It is a contradiction. Therefore $< a >^w = \phi$..

(4)$\Rightarrow$ (5) : Suppose that $< a >^w = \phi$. for every $a \in$ T. Since $x \in$S, clearly $< x >^w = \phi$.

(5)⇒(1) : Suppose that $< a >^w = \phi$. Let $a \in$ T. If possible, suppose that $a \neq x^n$ for any odd natural numbers n. Therefore $a = xs_1t_1$, s_1 , $t_1 \in$S and $s_1 \neq x^p$, $t_1 \neq x^q$ for any odd natural numbers p, q. Similarly $s_1 = xs_2 s^l_2$,and $t_1 = xt_2t^l_2$ where s_2 , s^l_2, t_2 , $t^l_2 \in$S and each of them not eaqual to x^p for any odd natural number. Hence $a = xs_1t_1 = x^3s_3t_3$. By continuing this process, we get $a = xs_1t_1 = x^3s_3t_3 = x^5s_5 t_5 = . . .$, therefore $a \in < x^n > = < x >^n$ for all odd natural numbers n and hence $a \in \bigcap_{i=1}^{\infty} < x >^n = < x >^w = \phi$. It is a contradiction.

Therefore $a = x^n$ for some odd natural numbers n. Hence S={x, x^3, x^3, ...}. If T is finite then T = {x, x^3, x^5,. . .., x^m} for some odd natural numbers m.
Now $< x^m > \subseteq < x^{m-2} > \subseteq . . . . \subseteq < x >$ and hence $< x >^m \subseteq < x >^{m-2} \subseteq < x >^{m-4} \subseteq . . . . < x >$.
Also $< x >^{m+r} = < x >^m$ for all odd natural numbers r. So $x^m \in \bigcap_{i=1}^{\infty} < x >^n = < x >^w = \phi$. It is a contradiction.

Therefore T is infinite.

COROLLARY 3.46 : If T is a commutative chained ternary semigroup with T ≠ T^3 , then the following are equivalent.
(1) S={x, x^3, x^5, ...}, where $x \in$ T\\T^3
(2) T is Noetherian cancellative ternary semigroup without regular elements.
(3) $< a >^w = \phi$ for all $a \in$ T.
***Proof* :** The proof of the theorem follows the above theorem.

THEOREM 3.47: If T is a commutative chained noetherian cancellative ternary semigroup without regular elements, then $< a >^w = \phi$ for all $a \in$T.

***Proof* :** Suppose if possible, T has no proper ideals. Since T is commutative, T has neither proper left/ right/ lateral ideals. Therefore by theorem 3.38, T is a ternary group. By theorem 3.41, every element of T is regular. It is contradiction to T has no regular elements. Hence T has proper ideals. Since T is noetherian ternary semigroup and T has proper ideals implies that T contains maximal ideals. Suppose if possible T = T^3. Since T contains maximal ideals and T = T^3, implies by theorem 2.61, T contains regular elements. It is contradiction. Thus T ≠ T^3, by theorem 3.10, $< a >^w = \phi$ for all $a \in$T.

Acknowledgement : The author would like to thank our college vice-president Dr. S.R.K.Prasad for encouraging me to do this research work.

References

[1] **Anjaneyulu. A**., ***On primary ideals in Semigroups*** - Semigroup Forum, Vol .20(1980), 129-144.
[2] **Anjaneyulu. A**., ***On Primary semigroups*** – Czechoslovak Mathematical Journal., 30(105)., (1980), Praha.
[3] **Anjaneyulu. A**., ***Structure and ideal theory of duo semigroups*** – semigroup forum., Vol.22(1981) 151-158
[4] **Bourne S.G**., ***Ideal theory in a commutative semigroup*** – Dessertation, John Hopkins University(1949).
[5] **Clifford A.H** and **Preston G.B**., ***The Algebroic Theory of Semigroups*** , Vol – I, American Math. Society, Province (1961).
[6] **Clifford A.H** and **Preston G.B**., ***The Algebroic Theory of Semigroups*** , Vol – II, American Math. Society, Province (1967).
[7] **Giri. R. D and Wazalwar. A. K., *Prime ideals and prime radicals in non-commutative semigroups* –** Kyungpook Mathematical Journal, Vol.33, No.1, 37-48, June 1993.
[8] **Hanumanta Rao.G, Anjaneyulu. A and Madhusudhana Rao. D., *Primary ideals inTernary semigroups*** - International eJournal of Mathematics and Engineering 218 (2013) 2145 – 2159.
[9] **Hanumanta Rao.G, Anjaneyulu. A and Gangadhara Rao. A., Semi*primary ideals inTernary semigroups*** - International Journal of Mathematical Sciences, Technology and Humanities 91 (2013) 1010 – 1025
[10] **Harbans lal**., ***Commutative semiprimary semigroups*** – Czechoslovak Mathematical Journal., 25(100)., (1975), 1-3.
[11] **Hoehnke H. J**., ***Structure of semigroups*** –Canadian Mathematical Journal, 18(1966), 449-491.
[12] **Kar. S**. ,***On Ideals in Ternary Semigroups*** . Int. J. Math. Gen. Sci., 18 (2003) 3013- 3023.
[13] **Kar. S** and **Maity.B.K**., ***Some Ideals of Ternary Semigroups*** . Analele Stintifice Ale Universitath "Ali Cuza" Din Iasi(S.N) Mathematica, Tumul Lvii. 2011-12.
[14] **Mannepalli V.L** and **Nagore C. S. H**., ***Generalized Commutative Semigroups*** – Semegroup forum 17(1)(1979), 65-73.
[15] **Petrch.M**., ***Introduction to Semigroups*** , Merril Publishing Company, Columbus, Ohio (1973)
[16] **Sarala. Y, Anjaneyulu. A** and **Madhusudhana Rao.D**., ***Ideals in Ternary Semigroups*** - International Ejournal Of Mathematics And Engineering 203 (2013) 1950-1968.
[17] **Sarala. Y, Anjaneyulu. A** and **Madhusudhana Rao.D.**, ***Prime Radicals in Ternary Semigroups*** - International Organization Of Scientific Research Journal Of Mathematics Issn: 2278-5728. Volume 4, Issue 5 (Jan.-Feb. 2013), Pp 43-53.
[18] **Santiago. M. L. And Bala S.S**., ***Ternary Semigroups*** - Semigroups Forum,Vol. 81,No. 2, Pp. 380-388, 2010.
[19] **Satyanarayana.M**., ***Commutative semigroups in which primary ideals are prime***., Math.Nachr., Band 48 (1971), Heft (1-6, 107-111)
[20] **Satyanarayana.M**., ***Commutative primary semigroups*** – Czechoslovak Mathematical Journal. 22(97), 1972 509-516.