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Jordan Higher K-Centralizer on Γ-Rings

Salah M. Salih, Ali M.Kamal, Balsam M. Hamad
Department of Mathematics, College of Education, Al-Mustansiriya University

Abstract: *Let M be a semiprime Γ-ring satisfying a certain assumption. Then we prove that every Jordan left higher k-centralizer on M is a left higher k-centralizer on M. We also prove that every Jordan higher k-centralizer of a 2-torsion free semiprime Γ-ring M satisfying a certain assumption is a higher k-centralizer.*
Keywords: *Semiprime Γ-ring, left higher centralizer, higher k-centralizer, Jordan higher k-centralizer.*

I. Introduction:

The definition of a Γ-ring was introduced by Nobusawa [7] and generalized by Barnes [2] as follows:
Let M and Γ be two additive abelian groups. If there exists a mapping $M\times\Gamma\times M \longrightarrow M$ (the image of (a,α,b) being denoted by $a\alpha b$; $a, b \in M$ and $\alpha \in \Gamma$) satisfying for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$,
$a(\alpha + \beta)c = a\alpha c + a\beta c$,
$a\alpha(b + c) = a\alpha b + a\alpha c$

(iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Then M is called a Γ-ring.

In [3] F.J.Jing defined a derivation on Γ-ring, as follows:
An additive map $d: M \longrightarrow M$ is said to be a derivation of M if

$$d(x\alpha y) = d(x)\alpha y + x\alpha d(y), \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma$$

M.Soponci and A.Nakajima in [8] defined a Jordan derivation on Γ-ring, as follows:
An additive map $d: M \longrightarrow M$ is called a Jordan derivation of Γ-ring if

$$d(x\alpha x) = d(x)\alpha x + x\alpha d(x), \text{ for all } x \in M \text{ and } \alpha \in \Gamma$$

A.H.Majeed and S.M.Salih in [6] defined a higher derivation and Jordan higher derivation on Γ-ring as follows:

A family of additive mapping of M, $D = (d_i)_{i\in N}$ is called a higher derivation of M if for every $x, y \in M$, $\alpha \in \Gamma$, $n \in N$

$$d_n(x\alpha y) = \sum_{i+j=n} d_i(x)\alpha d_j(y)$$

D is called Jordan derivation of M if

$$d_n(x\alpha x) = \sum_{i+j=n} d_i(x)\alpha d_j(x)$$

In 2011 M.F.Hoque and A.C.Paul, [5], also B.Zalar in [11] defined a centralizer on Γ-ring, as follows
An additive mapping $T: R \longrightarrow R$ is left (right) centralizer if
$T(x\alpha y) = T(x)\alpha y$ ($T(x\alpha y) = x\alpha T(y)$) holds for all $x, y \in M$ and $\alpha\in\Gamma$.

In [5], defined a Jordan centralizer on Γ-ring,
An additive mapping $T: M \longrightarrow M$ is Jordan left (right) centralizer if

$$T(x\alpha x) = T(x)\alpha x \ (T(x\alpha x) = x\alpha T(x)) \text{ for all } x \in M \text{ and } \alpha\in\Gamma.$$

In [9] Salah M.Salih and Balsam Majid H. defined a higher centralizer on Γ-ring, as follows:
A family of additive mapping of M, such that $t_0 = id_M$ then T is said to be higher centralizer of M if

$$t_n(x\alpha y + y\beta x) = \sum_{i=1}^{n} t_i(x)\alpha y + y\beta t_i(x)$$

for all $x, y \in M$, $\alpha, \beta \in \Gamma$ and $n \in N$.

In [9], defined a Jordan higher centralizer on Γ-ring, as follows:
A family of additive mappings of M, such that $t_0 = id_M$ then T is said to be a Jordan higher centralizer of M if

$$t_n(x\alpha x + x\alpha x) = \sum_{i=1}^{n} t_i(x)\alpha x + x\alpha t_i(x)$$

for all $x \in M$, $\alpha \in \Gamma$ and $n \in N$.

Z.Ullah and M.A. Chaudhay [10] developed the concepts of a K-centralizer on a semiprime Γ-ring and Jordan K-centralizer on Γ-ring as follows:

Let M be a Γ-ring and $K: M \longrightarrow M$ an automorphism such that $K(x\alpha y) = K(x)\alpha K(y)$ for all $x, y \in M$, $\alpha \in \Gamma$. An additive mapping $T: M \longrightarrow M$ is a left (right K-centralizer if $T(x\alpha y) = T(x)\alpha K(y)$ $(T(x\alpha y) = K(x)\alpha T(y))$ holds for all $x, y \in M$, $\alpha \in \Gamma$. T is called a K-centralizer if it is both a left and right K-centralizer.

In this paper, we define and study higher K-centralizer, Jordan higher K-centralizer, and we prove that every Jordan higher K-centralizer of a semiprime Γ-ring is a higher K-centralizer. Throught this paper we denote the set of all natural numbers include zero.

II. Preliminaries

In this section we will introduce the definition of K-higher centralizer, Jordan K-higher centralizer and describe some notions.

Definition (2.1):

Let M be a Γ-ring. An additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U (U\Gamma M \subset U)$. If U is both a left and right ideal, then U is called an ideal of M.

Definition (2.2):

An ideal P of a Γ-ring M is called prime ideal if for any ideals A, B of M, $A\Gamma B \subseteq P$, implies $A \subseteq P$ or $B \subseteq P$.

Definition (2.3):

An ideal P of a Γ-ring M is called semi-prime if for any ideal A of M, $A\Gamma A \subseteq P$, implies $A \subseteq P$.

Definition (2.4):

A Γ-ring M is said to be prime if $a \Gamma M \Gamma b = \{0\}$, $a, b \in M$, implies $a = 0$ or $b = 0$.

Definition (2.5):

A Γ-ring M is said to be semiprime if $a \Gamma M \Gamma a = \{0\}$, $a \in M$, implies $a = 0$.

Definition (2.6):

A Γ-ring M is said to be commutative if $x\alpha y = y\alpha x$ for all $x, y \in M$, $\alpha \in \Gamma$.

Definition (2.7):

A Γ-ring M is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$.

Definition (2.8):

Let M be a Γ-ring. Then the set

$Z(M) = \{x \in M: x\alpha y = y\alpha x \text{ for all } y \in M, \alpha \in \Gamma\}$

is called the center of the Γ-ring M.

III. The Higher K-Centralizer of Semiprime Γ-Ring

Now we will introduce the definition of left (right) higher K-centralizer and higher K-centralizer, Jordan higher K-centralizer on Γ-ring and other concepts which be used in our work.

Definition (3.1):

Let M be a Γ-ring and $T = (t_i)_{i\in N}$ be a family of additive mappings of M, such that $t_0 = id_M$ and $K = (k_i)_{i\in N}$ a family of automorphism. Then T is said to be left (right) higher K-centralizer if

$$T_n(x\alpha y) = \sum_{i+j=n} t_i(x)\alpha k_j(y) \quad \left(T_n(x\alpha y) = \sum_{i+j=n} k_i(x)\alpha t_j(y)\right)$$

holds for all $x, y \in M$, $\alpha \in \Gamma$. T_n is called a higher K-centralizer if it is both a left and a right K-centralizer.

For any fixed $a \in M$ and $\alpha \in \Gamma$, the mapping $T_n(x) = \sum_{i=n} a\alpha k_i(x)$ is a left higher K-centralizer and $T_n(x) = \sum_{i=n} k_i(x)\alpha a$ is a right K-centralizer.

Definition (3.2):

Let M be a Γ-ring and T = $(t_i)_{i\in N}$ be a family of additive mappings of M, such that $t_0 = id_M$ and K = $(k_i)_{i\in N}$ a family of automorphism. Then T is said Jordan left (right) higher K-centralizer if

$$T_n(x\alpha x) = \sum_{i+j=n} t_i(x)\alpha k_j(x) \quad \left(T_n(x\alpha x) = \sum_{i+j=n} k_i(x)\alpha t_j(x)\right)$$

holds for all $x \in M$, $\alpha \in \Gamma$.

Definition (3.3):

Let M be a Γ-ring and T = $(t_i)_{i\in N}$ be a family of additive mappings of M, such that $t_0 = id_M$ and K = $(k_i)_{i\in N}$ a family of automorphism. Then T is said Jordan higher K-centralizer if

$$T_n(x\alpha y + y\alpha x) = \sum_{i+j=n} t_i(x)\alpha k_j(x) + k_j(y)\alpha t_i(x)$$

holds for all $x, y \in M$, $\alpha \in \Gamma$.

Lemma (3.4): [5]

Let M be a semiprime Γ-ring. If $a, b \in M$ and $\alpha, \beta \in \Gamma$ are such that $a\alpha x\beta b = 0$ for all $x \in M$ then $a\alpha b = b\alpha a = 0$.

Lemma (3.5): [5]

Let M be a semiprime Γ-ring and A: M×M ⟶ M a additive mapping. If $A(x,y)\alpha w\beta(x,y) = 0$ for all x, y, w ∈ M and α, β ∈ Γ, then $A(x,y)\alpha w\beta(u,v) = 0$ for all x, y, u, v ∈ M and α, β ∈ Γ.

Lemma (3.6): [5]

Let M be a semiprime Γ-ring satisfying the assumption $x\alpha y\beta z = x\beta y\alpha z$ for all x, y, z ∈ M and α, β ∈ Γ. If $a \in M$ is a fixed element such that $a\alpha[x,y]\beta = 0$ for all x, y ∈ M and α, β ∈ Γ, then there exists an ideal U of M such that $a \in U \subset Z(M)$.

Lemma (3.7): [5]

Let M be a semiprime Γ-ring satisfying the assumption $x\alpha y\beta z = x\beta y\alpha z$ for all x, y, z ∈ M and α, β ∈ Γ. Let D be a derivation of M and α ∈ M, a fixed element

(i) If $D(x)\alpha D(y) = 0$ for all x, y ∈ M and α, β ∈ Γ, then D = 0.

(ii) If $a\alpha x - x\alpha a \in Z(M)$ for all $x \in M$ and $\alpha \in \Gamma$, then $a \in Z(M)$.

Lemma (3.8): [5]

Let M be a semiprime Γ-ring satisfying the assumption $x\alpha y\beta z = x\beta y\alpha z$ for all x, y, z ∈ M and α, β ∈ Γ. Let $a, b \in M$ be two fixed elements such that $a\alpha x = x\alpha b$ for all x ∈ M and α ∈ Γ. Then $a = b \in Z(M)$.

Lemma (3.9):

Let M be a semiprime Γ-ring satisfying the assumption $x\alpha y\beta z = x\beta y\alpha z$ for all x, y, z ∈ M and α, β ∈ Γ. Let T: M ⟶ M be a Jordan left higher K-centralizer, then

(a) $T_n(x\alpha y + y\alpha x) = \sum_{i+j=n} t_i(x)\alpha k_j(y) + t_i(y)\alpha k_j(y)$

(b) $T_n(x\alpha y\beta x + x\beta y\alpha x) = \sum_{i+j+s=n} t_i(x)\alpha k_j(y)\beta k_s(x) + t_i(x)\beta k_j(y)\alpha k_s(x)$

(c) If M is a 2-torsion free Γ-ring satisfying the above assumption, then

(i) $T_n(x\alpha y\beta x) = \sum_{i+j+s=n} t_i(x)\alpha k_j(y)\beta k_s(x)$

(ii) $T_n(x\alpha y\beta z + z\beta y\alpha x) = \sum_{i+j+s=n} t_i(x)\alpha k_j(y)\beta k_s(z) + t_i(z)\beta k_j(y)\alpha k_s(z)$

Proof:

Since T_n is a Jordan left higher K-centralizer, therefore

(1) $T_n(x\alpha x) = \sum_{i+j=n} t_i(x)\alpha k_j(x)$.

(a) Replacing x by $x + y$ in (1), we get

(2) $T_n(x\alpha y + y\alpha x) = \sum_{i+j=n} t_i(x)\alpha k_j(y) + t_i(y)\alpha k_j(x)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

(b) Replacing y by $x\alpha y + y\alpha x$ and α by β in (2), we get

$$T_n(x\beta(x\alpha y + y\alpha x) + (x\alpha y + y\alpha x)\beta x) = \sum_{i+j=n} t_i(x)\beta k_j(x\alpha y + y\alpha x) + t_i(x\alpha y + y\alpha x)\alpha k_j(x)$$

The last relation along with (2) implies

$$T_n(x\beta x\alpha y + x\beta y\alpha x + x\alpha y\beta x + y\alpha x\beta x) = \sum_{i+j+s=n} t_i(x)\beta k_j(x)\alpha k_s(y) + t_i(x)\beta k_j(y)\alpha k_s(x) + t_i(x)\alpha k_j(y)\beta k_s(x) + t_i(y)\alpha k_j(x)\beta k_s(x)$$

which gives

$$T_n(x\beta x\alpha y + y\alpha x\beta x) + T_n(x\beta y\alpha x + x\alpha y\beta x) = \sum_{i+j+s=n} t_i(x)\beta k_j(x)\alpha k_s(y) + t_i(x)\beta k_j(y)\alpha k_s(x) + t_i(x)\alpha k_j(y)\beta k_s(x) + t_i(y)\alpha k_j(x)\beta k_s(x)$$

the last relation along with (2) implies

(3) $T_n(x\beta y\alpha x + x\alpha y\beta x) = \sum_{i+j+s=n} t_i(x)\beta k_j(y)\alpha k_s(x) + t_i(x)\alpha k_j(y)\beta k_s(x)$

(c) Using the assumption $x\alpha y\beta z = x\beta y\alpha z$ and 2-torsion freeness of M, from (3) we get

(4) $T_n(x\beta y\alpha x) = \sum_{i+j+s=n} t_i(x)\beta k_j(y)\alpha k_s(x)$

Replacing x by $x + z$ in (4), we get

$$T_n((x+z)\beta y\alpha(x+z)) = \sum_{i+j+s=n} t_i(x+z)\beta k_j(y)\alpha k_s(x+z)$$

Which implies

$$T_n(x\beta y\alpha z + z\beta y\alpha x) = \sum_{i+j+s=n} t_i(x)\beta k_j(y)\alpha k_s(z) + t_i(z)\beta k_j(y)\alpha k_s(x)$$

The last relation along with the assumption $x\alpha y\beta z = x\beta y\alpha z$ gives

(5) $T_n(x\alpha y\beta z + z\beta y\alpha x) = \sum_{i+j+s=n} t_i(x)\alpha k_j(y)\beta k_s(z) + t_i(z)\beta k_j(y)\alpha k_s(x)$.

Theorem (3.10):

Let M be a semiprime Γ-ring satisfying the assumption $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Let $T_n: M \longrightarrow M$ be a Jordan left higher K-centralizer. Then T_n is a left higher K-centralizer.

Proof:

Using lemma (3.9-c(i)), we have

(6) $$T_n(x\alpha y\beta zYy\alpha x + y\alpha x\beta zYx\alpha y) = \sum_{i+j+s+t+r=n} t_i(x)\alpha k_j(y)\beta k_s(z)Yk_t(y)\alpha k_r(x) + t_i(y)\alpha k_j(x)\beta k_s(z)Yk_t(x)\alpha k_r(y)$$

Moreover, lemma (3.9-c(ii)) gives

(7) $T_n(x\alpha y\beta zYy\alpha x + y\alpha x\beta zYx\alpha y) = \sum_{i+j+s+t=n} t_i(x\alpha y)\beta k_j(z)Yk_s(y)\alpha k_t(x) + t_i(y\alpha x)\beta k_j(z)Yk_s(x)\alpha k_t(y)$

Subtracting (6) from (7), we get

$$\left(\sum_{i+s+t+r=n} t_i(x\alpha y) - t_j(x)\alpha k_s(y)\right)\beta k_s(z)Yk_t(y)\alpha k_r(x) +$$

$$\left(\sum_{i+j+s+t+r=n} t_i(y\alpha x) - t_j(y)\alpha k_s(x)\right)\beta k_s(z)Yk_t(x)\alpha k_r(y) = 0$$

Which implies

(8) $H(x,y)\beta \sum_{s+t+r=n} k_s(z)Yk_t(y)\alpha k_r(x) + H(y,x)\beta \sum_{s+t+r=n} k_s(z)Yk_t(x)\alpha k_r(y) = 0$

When $H(x,y) = \sum_{i=n} t_i(x\alpha y) - \sum_{i+j=n} t_i(x)\alpha y k_j(y)$

Which along with (2) implies $H(x,y) = - H(y,x)$

Using the last relation, from (8), we get

$H(x,y)\beta k_s(z)Y[k_t(x)k_r(y)]_\alpha = 0$

Replacing x by $k_t^{-1}(x)$, y by $k_r^{-1}(y)$ and z by $k_s^{-1}(z)$ in the last relation, we get

$$H(k_t^{-1}(x), k_r^{-1}(y))\beta zY[x,y]_\alpha = 0$$

The last relation along with lemma 3.5 implies

$$H(k_t^{-1}(x), k_r^{-1}(y))\beta zY[u,v]_\alpha = 0.$$

Replacing x by $k_t(x)$ and y by $k_r(y)$ in the last relation, we get

(9) $H(x,y)\beta zY[u,v]_\alpha = 0.$

Using lemma 3.4 in (9), we get

(10) $H(x,y)\beta\ [u,v]_\alpha = 0.$

We now fix some $x, y \in M$ and denote $H(x,y)$ by H. Using lemma 3.6 we get the existence of an ideal U such that $H \in U \subseteq Z(M)$.

In particular, $b\alpha H$, $H\alpha b \in Z(M)$ for all $b \in M$, then

$x\alpha(H\beta HYy) = (H\beta HYy)\alpha x = (yYH\beta H)\alpha x = yY(H\beta H\alpha x) = (H\beta H\alpha x)Yy$

which implies

$4T_n(x\alpha(H\beta HYy) = 4T_n(yY(H\beta H\alpha x))$

Which gives

$2T_n(x\alpha H\beta HYy + x\alpha H\beta HYy) = 2T_n(yYH\beta H\alpha x + yYH\beta H\alpha x) =$

$2T_n(x\alpha H\beta HYy + H\beta HYy\, x\alpha) = 2T_n(yYH\beta H\alpha x + H\beta H\alpha xYy)$

Using (2) in the last relation, we get

$$2\sum_{i+j+s+t=n} t_i(x)\alpha k_j(H)\beta k_s(H)Yk_t(y) + 2\sum_{i+j=n} t_i(H\beta HYy)\alpha k_j(x) =$$

$$2\sum_{i+j+s+t=n} t_i(y)Yk_j(H)\beta k_s(H)\alpha k_t(y) + 2\sum_{i+j=n} t_i(H\beta H\alpha x)Yk_j(y)$$

Which implies

$$2\sum_{i+j+s+t=n} t_i(x)\alpha k_j(H)\beta k_s(H)Yk_t(y) + \sum_{i+j=n} t_i(H\beta HYy + yYH\beta H)\alpha k_j(x) =$$

$$2\sum_{i+j+s+t=n} t_i(y)Yk_j(H)\beta k_s(H)\alpha k_t(x) + \sum_{i+j=n} t_i(H\beta H\alpha x + x\alpha H\beta H)Yk_j(y)$$

The last relation along with (2) gives

$$2\sum_{i+j+s+t=n} t_i(x)\alpha k_j(H)\beta k_s(H)Yk_t(y) + \sum_{i+j=n}\left(\sum_{r+s+t=i} t_r(H)\beta k_s(H)Yk_t(y) + \sum_{r+s+t=i} t_r(y)Yk_s(H)\beta k_t(H)\alpha k_j(y)\right) = 2\sum_{i+j+s+t=n} t_i(y)Yk_j(H)\beta k_s(H)\varepsilon k_t(x) + \sum_{i+j=n}\left(\sum_{r+s+t=i} t_r(H)\beta k_s(H)\alpha k_t(x)Yk_j(y) + \sum_{r+s+t=i} t_r(x)\alpha k_s(H)\beta k_t(H)Yk_j(y)\right)$$

So, we have

$$2\sum_{i+j+s+t=n} t_i(x)\alpha k_j(H)\beta k_s(H)Yk_t(y) + \sum_{r+s+t+j=n} t_r(H)\beta k_s(H)Yk_t(y)\alpha k_j(x) + \sum_{r+s+t+j=n} t_r(y)Yk_s(H)\beta k_t(H)\alpha k_j(x) = 2\sum_{i+j+s+t=n} t_i(y)Yk_j(H)\beta k_s(H)\alpha k_t(x) + \sum_{r+s+t+j=n} t_r(H)\beta k_s(H)\alpha k_t(x)Yk_j(y) + \sum_{r+s+t+j=n} t_r(x)\alpha k_s(H)\beta k_t(H)Yk_j(y)$$

Which implies

$$\sum_{i+j+s+t=n} t_i(x)\alpha k_j(H)\beta k_s(H)Yk_t(y) + \sum_{r+s+t+j=n} t_r(H)\beta k_s(H)Yk_t(y)\alpha k_j(x) = \sum_{i+j+s+t=n} t_i(y)Yk_j(H)\beta k_s(H)\alpha k_t(x) + \sum_{r+s+t+j=n} t_r(H)\beta k_s(H)\alpha k_t(x)Yk_j(y)$$

Replacing H by $k_w^{-1}(H)$, where w_j = s or t or j we get

$$\sum_{i+t=n} t_i(x)\alpha H\beta HYk_t(y) + \sum_{r+t+j=n} t_r(k_r^{-1}(H))\beta HYk_t(y)\alpha k_j(x) = \sum_{i+t=n} t_i(y)YH\beta H\alpha k_t(x) + \sum_{r+t+j=n} t_r(k_r^{-1}(H))\beta H\alpha k_t(x)Yk_j(y)$$

Since $H \in U \subseteq Z(M)$ and $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, therefore
$HYk_t(y)\alpha k_j(x)=HYk_t(y)\alpha k_j(x)= k_j(x)\alpha(Hyk_t(y)) = (k_j(x)\alpha H)Yk_t(y) = H\alpha k_j(x)Yk_t(y)$
Using this in the last relation we get

(11) $$\sum_{i+t=n} t_i(x)\alpha k_t(y)YH\beta H = \sum_{i+t=n} t_i(y)YH\beta H\alpha k_t(x) .$$

Now since $H \in U \subseteq Z(M)$, one has
$x\alpha yYH\beta H = x\alpha(yYH)\beta H = (x\alpha H)Y(y\beta H) = (H\alpha x)Y(H\beta y)$, therefore
$4T_n(x\alpha y)YH\beta H = 4T_n(H\alpha x)Y(H\beta y)$, which implies
$2T_n(x\alpha yYH\beta H + H\beta HYx\alpha y) = 2T_n(H\alpha xYH\beta y + H\beta yYH\alpha x)$.
The last relation along with (2) gives

$$2\sum_{i+j+s=n} t_i(x\alpha y)Yk_j(H)\beta k_s(H) + 2\sum_{i+j+s+t=n} t_i(H)\beta k_j(H)Yk_s(x)\alpha k_t(y) = 2\sum_{i+j+s=n} t_i(H\alpha x)Yk_j(H)\beta k_s(Y) + 2\sum_{i+j+s=n} t_i(H\beta y)Yk_j(H)\alpha k_s(x)$$

Which implies

$$2\sum_{i+j+s=n} t_i(x\alpha y)Yk_j(H)\beta k_s(H) + 2\sum_{i+j+s+t=n} t_i(H)\beta k_j(H)Yk_s(x)\alpha k_t(y) = 2\sum_{i+j+s=n} t_i(x\alpha H + H\alpha x)Yk_j(H)\beta k_s(H) + \sum_{i+j+s=n} t_i(y\beta H + H\beta y)Yk_j(H)\alpha k_s(x)$$

Which further gives

$$2\sum_{i+j+s=n} t_i(x\alpha y)Yk_j(H)\beta k_s(H) + 2\sum_{i+j+s+t=n} t_i(H)\beta k_j(H)Yk_s(x)\alpha k_t(y) =$$

$$\sum_{r+t+j+s=n} t_r(x)\alpha k_t(H)Yk_j(H)\beta k_s(y) + t_r(H)\alpha k_t(H)Yk_j(H)\beta k_s(y) +$$

$$\sum_{r+t+j+s=n} t_r(y)\beta k_t(H)Yk_j(H)\alpha k_s(x) + t_r(H)\alpha k_t(y)Yk_j(H)\beta k_s(y)$$

Replacing H by $k_w^{-1}(H)$ in thae last relation, where w= j or s or t, we get

$$2\sum_{i+j+s=n} t_i(x\alpha y)YH\beta H + 2\sum_{i+s+t=n} t_i(k_i^{-1}(H))\beta HYk_s(x)\alpha k_t(y) =$$

$$\sum_{r+s=n} t_r(x)\alpha HYHk_j\beta k_s(y) + \sum_{r+t+s=n} t_r(k_i^{-1}(H))\beta\alpha k_t(x)YH\beta k_s(y) +$$

$$\sum_{r+s=n} t_r(y)\beta HYH\alpha k_s(x) + \sum_{r+t+s=n} t_r(H)\beta k_t(x)YH\alpha k_s(x)$$

Since $H \in U \subseteq Z(M)$ and $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, therefore

$$2\sum_{i=n} t_i(x\alpha y)YH\beta H = \sum_{r+s=n} t_r(x)\alpha k_s(y)YH\beta H + \sum_{r+s=n} t_r(y)YH\beta H\alpha k_s(x)$$

The last relation along with (11) gives

$$\sum_{i=n} t_i(x\alpha y)YH\beta H = \sum_{r+s=n} t_r(x)\alpha k_s(y)YH\beta H$$

That is HYHβH = 0.
Using lemma 3.4 in the last relation we get
HβH = 0
Now HβMαH = (HβH) αM = {0}.
Thus H = 0, that is

$T_n(x\alpha y) - \sum_{i+j=n} t_i(x)\alpha k_j(y) = 0$. So, $T_n(x\alpha y) = \sum_{i+j=n} t_i(x)\alpha k_j(y)$.

Lemma (3.11):

Let M be a semiprime Γ-ring satisfying the assumption $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ and for some fixed element $m \in M$ if $T_n(x) = \sum_{i=1}^{n} m\alpha k_i(x) + k_i(x)\alpha m$ is a Jordan higher K-centralizer, then $m \in Z(M)$.

Proof:

By hypothesis

(12) $$T_n(x) = \sum_{i=1}^{n} m\alpha k_i(x) + k_i(x)\alpha m$$

Since T_n is a Jordan higher K-centralizer, therefore

(13) $$T_n(x\beta y + y\beta x) = \sum_{i+j=n} t_i(x)\beta k_j(y) + k_j(y)\beta t_i(x)$$

Using (12) in (13), we get

$$\sum_{i=1}^{n} m\alpha k_i(x\beta y + y\beta x) + k_j(x\beta y + y\beta x)\alpha m = \sum_{i+j=n} (m\alpha k_i(x) + k_i(x)\alpha m)\beta k_j(y) +$$
$$k_j(y)\beta(m\alpha k_i(x) + k_i(x)\alpha m)$$

Which implies

$$\sum_{i+j=n} m\alpha k_i(x)\beta k_j(y) + m\alpha k_i(y)\beta k_j(x) + k_j(x)\beta k_j(y)\alpha m + k_i(y)\beta k_j(x)\alpha m$$

$$= \sum_{i+j=n} (m\alpha k_i(x) + k_i(x)\alpha m)\beta k_j(y) + k_j(y)\beta(m\alpha k_i(x) + k_i(x)\alpha m)$$

So, we have

$$\sum_{i+j=n} m\alpha k_i(x)\beta k_j(y) + m\alpha k_i(y)\beta k_j(x) + k_i(x)\beta k_j(y)\alpha m + k_i(y)\beta k_j(x)\alpha m$$

$$= \sum_{i+j=n} m\alpha k_i(x)\beta k_j(y) + k_i(x)\alpha m\beta k_j(y) + k_j(y)\beta m\alpha k_i(x) + k_j(y)\beta k_i(x)\alpha m$$

Which further gives

$$\sum_{i+j=n} m\alpha k_i(y)\beta k_j(x) + k_i(x)\beta k_j(y)\alpha m = \sum_{i+j=n} k_i(x)\alpha m\beta k_j(y) + k_j(y)\beta m\alpha k_i(x)$$

Using the assumption $x\alpha y\beta z = x\beta y\alpha z$ in the last relation, we get

$$\sum_{i+j=n} (m\alpha k_i(y)\beta k_j(x) - k_j(y)\alpha m\beta k_j(x)) - \sum_{i+j=n} k_i(x)\beta m\alpha k_j(y) - k_i(x)\beta k_j(y)\alpha m$$

$$= \sum_{i+j=n} (m\alpha k_i(y) - k_i(y)\alpha m)\beta k_j(x) - \sum_{i+j=n} k_i(x)\beta(m\alpha k_j(y) - k_j(y)\alpha m) = 0$$

Which implies

$$\sum_{i=1}^{n} m\alpha k_j(y) - k_i(y)\alpha m \in Z(M).$$

The last relation along with lemma 3.7 implies $m \in Z(M)$.

Lemma (3.12):

Let M be a semiprime Γ-ring satisfying the assumption $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then every Jordan higherK-centralizer of M maps Z(M) into Z(M).

Proof:

Let $m \in Z(M)$. Then

(14) $$T_n(m\alpha x) = T_n(m\alpha x + x\alpha m) = \sum_{i+j=n} t_i(m)\alpha k_j(x) + k_j(x)\alpha t_i(m)$$

Let $S_n(x) = 2T_n(m\alpha x)$. Then

$S_n(x\beta y + y\beta x) = 2T_n(m\alpha(x\beta y + y\beta x)) = 2T_n(m\alpha x\beta y + m\alpha y\beta x)$.

Since $m \in Z(M)$ and $x\alpha y\beta z = x\beta y\alpha z$, one has

$S_n(x\beta y + y\beta x) = 2T_n((x\alpha m)\beta y + y\beta(x\alpha m))$

$$\sum_{i+j=n} 2t_i(x\alpha m)\beta k_j(y) + 2k_j(y)\beta t_i(x\alpha m) = \sum_{i+j=n} S_i(x)\beta k_j(y) + k_j(y)\beta S_i(x).$$

Hence S_n is a Jordan higher K-centralizer. So (14) along with lemma 3.11.2013 $T_n(m) \in Z(M)$.

Theorem (3.13):

Every Jordan higher K-centralizer of a 2-torsion free semiprime Γ-ring M satisfying the assumption $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ is a higher K-centralizer.

Proof:

Suppose that T_n is a Jordan higher K-centralizer, then

$$T_n(x\alpha y + y\alpha x) = \sum_{i+j=n} t_i(x)\alpha k_j(y) + k_i(y)\alpha t_j(x)$$

$$= \sum_{i+j=n} k_i(x)\alpha t_j(y) + t_j(y)\alpha k_j(x)$$

Replacing y by $x\beta y + y\beta x$ in the last relation we get

$$\sum_{i+j=n} t_i(x)\alpha k_j(x\beta y + y\beta x) + k_i(x\beta y + y\beta x)\alpha t_j(x)$$

$$= \sum_{i+j=n} t_i(x\beta y + y\beta x)\alpha k_j(x) + k_j(x)\alpha t_i(x\beta y + y\beta x)$$

$$= \sum_{r+s+j=n} t_r(x)\beta k_s(y)\alpha k_j(x) + k_r(y)\beta t_s(x)\alpha k_j(x) + k_j(x)\alpha t_r(x)\beta k_s(y) + k_j(x)\alpha k_s(y)\beta t_r(x)$$

Which implies

$$\sum_{i+t+u=n} t_i(x)\alpha k_t(x)\beta k_u(y) + t_i(x)\alpha k_t(y)\beta k_u(x) + k_t(x)\beta k_u(y)\alpha t_i(x) + k_t(y)\beta k_u(x)\alpha t_i(x)$$

$$= \sum_{r+s+j=n} t_r(x)\beta k_s(y)\alpha k_j(x) + k_s(y)\beta t_r(x)\alpha k_j(x) + k_j(x)\alpha t_r(x)\beta k_s(y) + k_j(x)\alpha k_s(y)\beta t_r(x)$$

Using the assumption $x\alpha y\beta z = x\beta y\alpha z$, from the last relation, we get

$$\sum_{i+t+u=n} t_i(x)\alpha k_t(x)\beta k_u(y)+k_t(y)\beta k_u(x)\alpha t_i(x)$$

$$=\sum_{r+s+j=n} k_s(y)\beta t_r(x)\alpha k_j(x)+k_j(x)\alpha t_r(x)\beta k_s(y)$$

So, we have

$$\sum_{i+t+u=n}(t_i(x)\alpha k_t(x)-k_t(x)\alpha t_i(x))\beta k_u(y)=\sum_{i+t+u=n} k_u(y)(t_i(x)\alpha k_t(x)-k_t(x)\alpha t_i(x))$$

That is, $[t_i(x),k_t(x)]_\alpha\beta k_u(y) = k_u(y)\beta[t_i(x),k_t(x)]_\alpha$, which implies $[t_i(x),k_t(x)] \in Z(M)$.

Now we prove that $[t_i(x),k_t(x)]_\alpha = 0$.

Let $m \in Z(M)$, lemma 3.12 implies that $T_n(m) \in Z(M)$. Thus

$$2T_n(m\alpha x)=T_n(m\alpha x+x\alpha m)$$

$$=\sum_{i+j=n} t_i(m)\alpha k_j(x)+k_j(x)\alpha t_i(m)$$

$$=2t_i(x)\alpha k_j(m)$$

Which implies

(15) $$T_n(m\alpha x)=\sum_{i+j=n} t_i(x)\alpha k_j(m)$$

$$=\sum_{i+j=n} t_i(m)\alpha k_j(x)$$

Now

$[t_i(x),k_t(x)]_\alpha\beta k_u(m) = t_i(x)\alpha k_t(x)\beta k_u(m) - k_t(x)\alpha t_i(x)\beta k_u(m)$.

The last relation along with (15) implies

$[t_i(x),k_t(x)]_\alpha\beta k_u(m) = 0$

Since $[t_i(x),k_t(x)]_\alpha$ itself is a contral element one has $[t_i(x),k_t(x)]_\alpha = 0$. Now

$$2T_n(x\alpha x)=T_n(x\alpha x+x\alpha x)$$

$$=\sum_{i+j=n} t_i(x)\alpha k_j(x)+k_j(x)\alpha t_i(\alpha)$$

$$=2\sum_{i+j=n} t_i(x)\alpha k_j(x)$$

$$=2\sum_{i+j=n} k_j(x)\alpha t_i(x)$$

That is, $T_n(x\alpha x)=\sum_{i+j=n} k_j(x)\alpha t_i(x)$.

Hence, T_n is a Jordan left higher K-centralizer. By theorem 3.10, T_n is a left higher K-centralizer. Similarly, we can prove that T_n is a right higher K-centralizer. Therefore T_n is a higher K-centralizer.

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