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L^p Inequalities Concerning Polynomials Having Zeros in Closed Interior of A Circle

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Abstract: *Let* $p(z) = \sum_{j=0}^{n} a_i z^j$ *be a polynomial of degree* n *and* $p'(z)$ *be its derivative, then Zygmund [9] proved that*

$$\left(\int_0^{2\pi} |p'(e^{i\theta})|^r \, d\theta \right)^{\frac{1}{r}} \le n \left(\int_0^{2\pi} |p(e^{i\theta})|^r \, d\theta \right)^{\frac{1}{r}}, \quad r \ge 1$$

In this paper we shall obtain similar type of inequalities in reverse order for the polynomials having r *fold zeros at origin and rest of the zeros in* $|z| \le k$, $k \le 1$.

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I. Introduction And Statement Of The Results

Let $p(z)$ be a polynomial of degree n and $p'(z)$ its derivative. Then the following well known inequality is due to Bernstein [2].

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)| \qquad (1.1)$$

L^p analogue of (1.1) was obtained by Zygmund [9]. He proved that
If $p(z)$ is a polynomial of degree n and $p'(z)$ its derivative then for $r \ge 1$,

$$\left(\int_0^{2\pi} |p'(e^{i\theta})|^r \, d\theta \right)^{\frac{1}{r}} \le n \left(\int_0^{2\pi} |p(e^{i\theta})|^r \, d\theta \right)^{\frac{1}{r}} \qquad (1.2)$$

In this paper we obtain integral mean estimates for polynomials having r fold zeros at origin and rest of the zeros in $|z| \le k$, $k \le 1$. For the same class of polynomials we shall also obtain L^p inequalities for polar derivative of a polynomial.

THEOREM 1. *Let* $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$ *be a polynomial of degree* n, *having zero of order* s *at origin and rest of the zero in* $|z| \le k$, $k \le 1$. *Then for* $r \ge 1$,

$$\left[\int_0^{2\pi} |p'(e^{i\theta})|^r \, d\theta \right]^{\frac{1}{r}} \ge \{n - (n-s)E_K\} \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r \, d\theta \right\}^{\frac{1}{r}} \qquad (1.3)$$

where $E_k = k^\mu / \left[\frac{1}{2\pi} \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^r \, d\theta \right]^{\frac{1}{r}}$ *and* $0 \le s \le n - \mu$.

Letting $r \to \infty$ in (1.3) and making use of the fact from analysis [7], [8] that

$$\left\{ \int_0^{2\pi} |p(e^{i\theta})|^r \, d\theta \right\}^{\frac{1}{r}} \to \max_{0 \le \theta \le 2\pi} |p(e^{i\theta})| \quad \text{as} \quad r \to \infty$$

we obtain the following inequality

$$\max_{|z|=1} |p'(z)| \ge \frac{n + sk^\mu}{1 + k^\mu} \max_{|z|=1} |p(z)| \qquad (1.4)$$

Next we obtain the following improvement of Theorem 1 which also generalizes a result due to Jain [5].

THEOREM 2. *Let* $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \le \mu \le n$, *be a polynomial of degree* n, *having all its zeros in* $|z| \le k$, $k \le 1$, *with a zero of order* s *at origin. Then for* β *with* $|\beta| < k^{n-s}$ *and* $r \ge 1$,

$$\left\{\int_0^{2\pi}\left|p'(e^{i\theta}) + \frac{sm}{k^n}\bar{\beta}e^{i(s-1)\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \tag{1.5}$$

$$\ge \{n-(n-s)c_k\}\left\{\int_0^{2\pi}\left|p(e^{i\theta}) + \frac{m}{k^n}\bar{\beta}e^{is\theta}\right|^r d\theta\right\}^{\frac{1}{r}}$$

where $m = \min_{|z|=k} |p(z)|$, $0 \le s \le n-\mu$ *and* $c_k = k^\mu \Big/ \left[\frac{1}{2\pi}\int_0^{2\pi} |1+k^\mu e^{i\theta}|^r \, d\theta\right]^{\frac{1}{r}}$.

For $\mu = 1$ Theorem 2 reduces to a result due to Jain [5].

Letting $r \to \infty$ in (1.5) we obtain the following inequality

$$\max_{|z|=1}\left|p'(z) + \frac{sm}{k^n}\bar{\beta}z^{s-1}\right| \ge \frac{n+sk^\mu}{1+k^\mu}\max_{|z|=1}\left|p(z) + \frac{m}{k^n}\bar{\beta}z^s\right| \tag{1.6}$$

where m is same as in Theorem 2 and $0 \le s \le n-\mu$.

By choosing argument of β suitably and letting $|\beta| \to k^{n-s}$ in (1.6) we get

COROLLARY 1. *If* $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$ *is a polynomial of degree* n, $1 \le \mu \le n$, *having all zeros in* $|z| \le k$, $k \le 1$, *with a zero of order* s *at origin then*

$$\max_{|z|=1}|p'(z)| \ge \frac{n+sk^\mu}{1+k^\mu}\max_{|z|=1}|p(z)| + \frac{(n-s)m}{(1+k^\mu)k^s} \tag{1.7}$$

where $m = \min_{|z|=k}|p(z)|$.

For $\mu = 1$ inequality (1.7) improves upon a result proved by Aziz and Shah [1].

Let $D_\alpha P(z)$ denote the polar differentiation of the polynomial $P(z)$ of degree n with respect to the point α. Then

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha} = p'(z).$$

Now we obtain L^p inequality for the polar derivative of a polynomial. Our result generalizes a result due to Dewan et al. [4]. More precisely we prove:

THEOREM 3. *If* $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \le 1$ *with* s*-fold zeros at the origin, then for every real or complex number* $\alpha \ge k^\mu$ *and for each* $r > 0$,

$$\left\{\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}} \ge \frac{(|\alpha| - k^\mu)(n+sk^\mu)}{1+k^\mu}\left\{\int_0^{2\pi}|p(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}}. \tag{1.8}$$

If in (1.8) $r \to \infty$ we get the following result.

COROLLARY 2. *If* $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \le 1$, *with* s *fold zeros at the origin, then for every real or complex number* $\alpha \ge k^\mu$

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{(|\alpha|-k^\mu)(n+sk^\mu)}{1+k^\mu} \max_{|z|=1} |p(z)|. \quad (1.9)$$

II. Lemmas

We will need following lemmas to prove our theorems.

LEMMA 1. *If* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$, *is a polynomial of degree* n, *having no zeros in the disk* $|z| < k$, $k \geq 1$, *then*

$$k^\mu |p'(z)| \leq |q'(z)| \quad \text{for} \quad |z| = 1. \quad (2.1)$$

The above lemma is due to Chan and Malik [3].

LEMMA 2. *If* $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, *is a polynomial of degree* n, *having all its zero in* $|z| \leq k \leq 1$, *then*

$$|q'(z)| \leq k^\mu |p'(z)| \quad \text{for } |z| = 1. \quad (2.2)$$

PROOF OF LEMMA 2. Since all the zeros of $p(z)$ lie in $|z| \leq k \leq 1$, all the zeros of $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$ lie in $|z| \geq \frac{1}{k}$, $\frac{1}{k} \geq 1$. Hence applying Lemma 1 to the polynomial $q(z)$, we get inequality (2.2). This proves Lemma 2. □

LEMMA 3. *If* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *is a polynomial of degree* n *such that* $p(z) \neq 0$ *in* $|z| < k$, $k \geq 1$, *then for* $r > 0$,

$$\left\{\int_0^{2\pi} |p'(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}} \leq nE_r \left\{\int_0^{2\pi} |p(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}} \quad (2.3)$$

where $E_r = \left\{\frac{1}{2\pi}\int_0^{2\pi} |k^\mu + e^{i\theta}|^r \, d\theta\right\}^{\frac{-1}{r}}$ *and* $1 \leq \mu \leq n$.

The above lemma is due to Rather [6].

III. Proofs Of The Theorems

PROOF OF THEOREM 1. Let $p(z) = z^s h(z)$ where $h(z)$ is a polynomial of degree $n-s$, having all its zeros in $|z| \leq k$, $k \leq 1$ and $h(0) \neq 0$. Then

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)} = z^{n-s} \overline{h\left(\frac{1}{\bar{z}}\right)}, \quad (3.1)$$

is also a polynomial of degree $n-s$.

Now $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$ for $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$ gives

$$|q'(e^{i\theta})| = |np(e^{i\theta}) - e^{i\theta} p'(e^{i\theta})|. \quad (3.2)$$

Also we have

$$|q(e^{i\theta})| = |p(e^{i\theta})|. \quad (3.3)$$

The polynomial $q(z)$, given by (3.1) will have no zeros in $|z| < \frac{1}{k}$, $\frac{1}{k} \geq 1$. Applying Lemma 3 to $q(z)$ for $r \geq 1$, we have

$$\left\{\int_0^{2\pi} |q'(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}} \le (n-s)E_k \left\{\int_0^{2\pi} |q(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}} \qquad (3.4)$$

where $E_k = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{k^{\mu}}+e^{i\theta}\right|^r d\theta\right\}^{-\frac{1}{r}}$. Now

$$\left\{\int_0^{2\pi} |np(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}} = \left\{\int_0^{2\pi} |np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta}) + e^{i\theta}p'(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}}.$$

By Minkowski's inequality for $r \ge 1$ we have

$$\left\{\int_0^{2\pi} n|p(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}} \qquad (3.5)$$

$$\le \left\{\int_0^{2\pi} n|p(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}} + \left\{\int_0^{2\pi} |e^{i\theta}p'(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}}.$$

Using (3.2), (3.3) and (3.4) in (3.5) we get

$$\left\{\int_0^{2\pi} |p'(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}} \ge (n-(n-s)E_k)\left\{\int_0^{2\pi} |p(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}}.$$

This completes the proof of Theorem 1. □

PROOF OF THEOREM 2. Let $p(z) = z^s h(z)$, where $h(z)$ is a polynomial of degree $(n-s)$ having all its zeros in $|z| \le k$, $k \le 1$ and $h(0) \ne 0$. Then

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)} = z^{n-s} \overline{h\left(\frac{1}{\overline{z}}\right)} \quad (3.6)$$

is a polynomial of degree $(n-s)$.

The polynomial $q(z)$ given by (3.6) will have no zeros in $|z| \le \frac{1}{k}, \frac{1}{k} \ge 1$. Now if

$$m_1 = \min_{|z|=\frac{1}{k}} |q(z)|$$

then

$$m_1 = \min_{|z|=\frac{1}{k}} \left| z^n \overline{p\left(\frac{1}{\overline{z}}\right)}\right| = \frac{m}{k^n} .$$

By Rouche's theorem the polynomial

$$q(z) + m_1\beta z^{n-s}, \quad |\beta| < k^{n-s},$$

of degree $(n-s)$ will also no zero in $|z| < \frac{1}{k}, \frac{1}{k} \ge 1$. Hence by applying Lemma 3 to the polynomial $q(z) + m_1\beta z^{n-s}$ for $r \ge 1$ and $|\beta| < k^{n-s}$ we have

$$\left\{\int_0^{2\pi}\left|q'(e^{i\theta}) + \frac{m}{k^n}(n-s)\beta e^{i(n-s-1)\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \qquad (3.7)$$

$$\le (n-s)c_k \int_0^{2\pi}\left|q(e^{i\theta}) + \frac{m}{k^n}\beta e^{i(n-s)\theta}\right|^r d\theta$$

where $c_k = k^{\mu} / \left[\frac{1}{2\pi}\int_0^{2\pi} |1+k^{\mu}e^{i\alpha}|^r d_{\alpha}\right]^{\frac{1}{r}}$.

Now using (3.2) and (3.3) in the above inequality we get

$$\left[\int_0^{2\pi}\left|np(e^{i\theta})-e^{i\theta}p'(e^{i\theta})+\frac{\bar{\beta}m}{k^n}(n-s)e^{is\theta}\right|^r d\theta\right]^{\frac{1}{r}} \qquad (3.8)$$

$$\le (n-s)c_k\left(\int_0^{2\pi}\left|p(e^{i\theta})+\frac{\bar{\beta}m}{k^n}e^{is\theta}\right|^r d\theta\right)^{\frac{1}{r}}$$

Now by Minkowski's inequality for $r\ge 1$ and $|\beta|<k^{n-s}$, we have

$$n\left(\int_0^{2\pi}\left|p(e^{i\theta})+\frac{m}{k^n}\bar{\beta}e^{is\theta}\right|^r d\theta\right)^{\frac{1}{r}} \qquad (3.9)$$

$$\le\left(\int_0^{2\pi}\left|np(e^{i\theta})+\frac{m}{k^n}\bar{\beta}(n-s)e^{is\theta}-e^{i\theta}p'(e^{i\theta})\right|^r d\theta\right)^{\frac{1}{r}}$$

$$+\left(\int_0^{2\pi}\left|e^{i\theta}p'(e^{i\theta})+s\frac{m}{k^n}\bar{\beta}e^{is\theta}\right|^r d\theta\right)^{\frac{1}{r}}.$$

Combining (3.8) and (3.9) we get

$$\left(\int_0^{2\pi}\left|p'(e^{i\theta})+\frac{sm}{k^n}\bar{\beta}e^{i(s-1)\theta}\right|^r d\theta\right)^{\frac{1}{r}}$$

$$\ge[n-(n-s)c_k]\left(\int_0^{2\pi}\left|p(e^{i\theta})+\frac{m}{k^n}\bar{\beta}e^{is\theta}\right|^r d\theta\right)^{\frac{1}{r}},$$

which is the desired result. This completes the proof of the Theorem 2. □

PROOF OF THEOREM 3. Since $p(z)$ has all its zeros in $|z|\le k\le 1$ with s-fold zeros at the origin, we can write

$$p(z)=z^s h(z),$$

where $h(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z|\le k\le 1$.

Now for every real or complex number α with $|\alpha|\ge k^{\mu}$, we have

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z).$$

Which implies

$$|D_\alpha p(z)|\ge|\alpha p'(z)|-|np(z)-zp'(z)|. \qquad (3.10)$$

Using (3.2) and Lemma 2 in (3.10) we get

$$|D_\alpha p(z)|\ge|\alpha||p'(z)|-|q'(z)|$$

$$\ge(|\alpha|-k^{\mu})|p'(z)| \quad \text{for } |z|=1. \qquad (3.11)$$

Inequality (3.11) in conjunction with (1.4) gives

$$|D_\alpha p(z)|\ge(|\alpha|-k^{\mu})\frac{n+sk^{\mu}}{1+k^{\mu}}|p(z)| \quad \text{for } |z|=1. \qquad (3.12)$$

From (3.12) we deduce that for each $r>0$

$$\left\{\int_0^{2\pi}|D_\alpha(pe^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}\ge\frac{(|\alpha|-k^{\mu})(n+sk^{\mu})}{1+k^{\mu}}\left\{\int_0^{2\pi}|p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}$$

which proves the desired result.

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