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Characterization of Countably Normed Nuclear Spaces

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Abstract: *Every count ably normed nuclear space is isomorphic to a subspace of a nuclear Frechet space with basis and a continuous norm. The proof as given in section 2 is a modification of the Komura-Komura inbedding theorem .In this paper, we shall show that a nuclear Frechet space with a continuous norm is isomorphic to a subspace of a nuclear Frechet space with basis and a continuous norm if and only if it is countably normed.The concept of countably normedness is very important in constructing the examples of a nuclear Frechet space. Moreover, the space with basis can be chosen to be a quotient of (s).*

Key words and phrases: *Nuclear frechet space, Countably normed and Nuclear Kothe space.*

I. Normed Spaces:

Let E be a Frechet space which admits a continuous norm. The topology of E can then be defined by an increasing sequences $\left(\|.\|_k \right)$of norms (the index set is N= $\{1,2,3,........\}$).Let K_k denotes equipped with the norm $\|.\|_k$ only and let E_k be the completion of E_k.The identity mapping $E_{k+1} \rightarrow E_k$ has a unique extension $\phi_k : E_{k+1} \rightarrow E_k$ and this latter mapping is called canonical. The space E is said to be countably normed if the system $\left(\|.\|_k \right)$ can be chosen in such a way that each ϕ_k is injective.

To give an example of a countably normed space,assume that E has an absolute basis i.e. there is a sequence (X_n) in E such that every(ξ_n) is a sequence of scalars. Then E is isomorphic to the Kothe sequence space

$$K(a)=K(a_n^k)=\left\{(\xi_n)\middle|\|(\xi_n)\|_k = \sum_n |\xi_n| a_n^k \langle \infty \forall K \right\} \quad (1)$$

Where $a_n^k = \|X_n\|_k$ .The topology of K(a) is defined by the norms $|.|_k$ .The completions

$(K(a)_k)$ Can be isometric ally identified with ℓ_1 and then the canonical mapping $\phi_k : \ell_1 \rightarrow \ell_1$ is the diagonal transformation $(\xi_n)_n \rightarrow ((a_n^k / a_n^{k+1})\xi_n)_n$ which is clearly injective. Therefore E is countably normed. Consider now a nuclear Frechet space E which admits a continuous norm.The topology of E can be defined by a sequence $(\|.\|_k)$ of Hilbert norms, that is, $\|X\|_k = \langle X, X \rangle_k$ $X \in E$ .where $\langle .,. \rangle_k$ is an inner product of E.

Theorem (1.1):

If a nuclear Frechet space E is countably normed, then the topology of E can be defined by a sequence of Hilbert norms such that the canonical mappings $\phi_k : E_{k+1} \rightarrow E_k$ are injective.

Suppose finally that (X_n) is a basis of E. Since (X_n) is necessarily absolute.E can be identified with a Kothe space K(a).By the Grothendieck-Pietch nuclearity criterion, for every K there is ℓ with $(a_n^k / a_n^\ell) \in \ell_1$ .Conversely, if the matrix (a_n^k) with $0<a_n^k \leq a_n^{k+1}$ satisfies this criterion ,then the Kothe space K(a) defined through(1) is a nuclear Frechet space with a continuous norm and the sequence of coordinate vectors constitutes a basis.In particular,(s)=K(n^k).The topology of such a nuclear Kothe space can also defined by the sub-norms, $|(\xi_n)|_{k,\infty} = \sup_n |\xi_n| a_n^k$.

An Imbedding Theorem:

We are now ready to prove the following two conditions are equivalent:

(1) E is countably normed,
(2) E is isomorphic to a subspace of a nuclear Kothe space which admits a continuous norm.
Moreover, the Kothe space in (2) can be chosen to be a quotient of (s).
Proofs:
As explained before, we know that a nuclear Kothe space with a continuous norm is countably normed. Since countably normedness is inherited by subspaces the implication $(2) \Rightarrow (1)$ is clear.

To prove $(1) \Rightarrow (2)$ we choose a sequence $(\|.\|_k)$ of Hilbert norms defining the topology of E such that each canonical mapping $\phi_k : E_{k+1} \to E_k$ is injective (Theorem1).Let $U_k = \{ X \in E \| X \|_k \le 1 \}$ and identity(E_k)'with

$E'_k = \{ f \in E | \|f\|'_k = \sup |< x, f > | < \infty \}$

Then $\phi'_k : E'_k \to E'_{k+1}$ is simply the inclusion mapping. As a Hilbert space, E'_{k+1} is reflexive. Using this and the fact that $\phi_k : E_{k+1} \to E_k$ is injective ,one sees easily that $\phi'_k (E'_k) = E'_k$ is dense in , E'_{k+1}.

We can construct in each E'_k a sequence $f_n^{(k)}$ of functional with the following properties:

$U_k^o \subset \{ f_n^{(k)} n \in N \}^{oo}$ (2)

$\{ n^1 f_n^{(k)} n \in N \}$ (3) is equicontinuous for every ℓ.

Now set $g_n^{(1)} = f_n^{(1)}$,$n \in N$ and using the fact that E_k is dense in every E'_k choose $g_n^{(k)} \in E'_1$, $k \ge 2, n \in N$ with

$\| f_n^{(k)} - g_n^{(k)} \| < 2^{-n}$ (4)

In the construction of the desired Kothe space K(a) we will use two indices k and n to enumerate the coordinate basis vectors.
First,set

$a_{kn}^f = 2^k n^{2\ell}, k, n \in N, \ell > k$ (5)

Then choose $a_{kn}^k, a_{kn}^{k-1}, \ldots. a_{kn}^1$ so that

$1 > a_{kn}^k \ge a_{kn}^{k-1} \ge \ldots\ldots \ge a_{kn}^1 > 0, k, n \in N$ (6)

$\frac{a_{kn}^{\ell+1}}{a_{kn}^{\ell+2}} \ge \frac{a_{kn}^{\ell}}{a_{kn}^{\ell+1}}, k, n \in N, \ell \le k$ (7)

$a_{kn}^{\ell} \le \frac{1}{\| g_n^{(k)} \|'_{\ell}}, k, n \in N, \ell \le k$ (8)

Note that holds trivially for $\ell > k$ .Consequently, if $K(a_{kn}^{\ell}) = K(a)$ is nuclear ,then it is also isomorphic to a quotient space of (s).But by for every $\ell \ge 2$

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{kn}^{\ell}}{a_{kn}^{\ell+1}} = \sum_{k=1}^{\ell-1}\sum_{n=1}^{\infty}\frac{a_{kn}^{\ell}}{a_{kn}^{\ell+1}} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{kn}^{\ell}}{a_{kn}^{\ell+1}} \le \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{kn}^{\ell}}{a_{kn}^{\ell+1}} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{kn}^{k}}{a_{kn}^{k+1}} < (\ell-1)\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{k=1}^{\infty}\sum_{n==1}^{\infty}\frac{1}{2^k n^{2(k+1)}} < \infty$$

To imbed E into K(a) we set $Ax = (<x, g_n^{(k)}>)$,$x \in E$ .We have to show that $Ax \in K(a)$, $A : E \to K(a)$ is a continuous injection and that $A^{-1} : A(E) \to E$ is also continuous.

Fix $\ell \ge 2$. Applying (3) to the sequence $(f_n^{(k)})_n$,$k = 1 \ldots\ldots \ell - 1$, we can find an index an index $p \ge \ell$ and a constant C such that

$$\sup_{k<1,n} 2^k n^{2\ell} \left| < X, f_n^{(k)} > \right| \le C \|X\|_p, X \in E$$

From (4), (5) and (8) we then get for every $X \in E$,

$$|AX|_{\ell,\infty} = \sup_{k,n} a_{kn}^{\ell} \left| < X, g_n^{(k)} > \right| \le \sup_{k<\ell,n} a_{kn}^{\ell} \left| < X, g_n^{(k)} > \right| + \sup_{k\ge\ell,n} a_{kn}^{\ell} \left| < X, g_n^{(k)} > \right|$$

$$\le \sup_{k,n} 2^k n^{2\ell} \left| < X, g_n^{(k)} > \right| + \sup_{k\ge\ell,n} \frac{1}{\left\| g_n^{(k)} \right\|_{\ell}^{'}} \left| < X, g_n^{(k)} > \right|$$

$$\le \sup_{k<\ell,n} 2^k n^{2\ell} \left\| g_n^{(k)} - f_n^{(k)} \right\|_k^{'} \|X\|_k$$

$$+ \sup_{k<\ell,n} 2^k n^{2\ell} \left| < X, f_n^{(k)} > \right| + \|X\|_{\ell} \le C^{'} \|X\|_{p}.$$

Where $C = Sup_{\,n} n^{2\ell} 2^{\ell-n} + C + 1 < \infty$

Consequently, $AX \in K(a)$ and $A:E \to k(a)$ is continuous. From (2) it follows that for every $X \in E$.

$$\|X\|_{\ell} = \sup_{f \in U_{\ell}^{o}} \left| < X, f > \right| \le \sup_{n} \left| < X, f_n^{(\ell)} > \right| \tag{9}$$

Further since $a_{\ell n}^{\ell+1} > 1$

$$\sup_n \left| < X, f_n^{(\ell)} > \right| \le \sup_n \left| < X, f_n^{(\ell)} - g_n^{(\ell)} > \right| + \sup_n \left| X, g_n^{(\ell)} \right| \tag{10}$$

$$\le \sup_n \left\| f_n^{(\ell)} - g_n^{(\ell)} \right\|_{\ell}^{'} \|X\|_{\ell} + \sup_{k,n} a_{kn}^{\ell+1} \left| < X, g_n^{(k)} > \right|$$

$$\le \frac{1}{2} \|X\|_{\ell} + |AX|_{\ell+1,\infty}.$$

Thus,by (9)and(10) we have for every $X \in E$.

$$\|X\|_{\ell} \le |AX|_{\ell+1,\infty}$$

Since ℓ is arbitrary ,this shows that A is injective and that $A^{-1} : A(E) \to E$ is continuous.
Finally, we remark that is not possible to find a single nuclear Frechet space with basis and a continuous norm containing all countably normed nuclear spaces and subspaces.

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