IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 7, Issue 3 (Jul. - Aug. 2013), PP 12-18
www.iosrjournals.org

Cash Flow Valuation Model in Continuous Time

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Abstract: *This paper present valuation models which is define as the expected discounted value of a stream of each flows at a time. Three equivalent forms of this value process is established with each of which has its own merits. Local dynamics of the values process is also considered.*
Keywords: *Cash Flow, Valuation, Continuous, Deflator, Filtration.*

I. Introduction

The advantage of the stochastic calculus of semimartigales, especially on Ito diffusion model gives us an opportunity to consider the cash flow in continuous time. When an individual or a firm is faced with a stream of cash flows $(C_t)_{t\geq 0}$, we define the value of such a stream at time t as

$$V_t = E_t\left[\int_t^{\infty} C_s e^{-r(s-t)} d_s\right]$$

where $E_t[\cdot]$ denotes the expectation with respect to all known information up to time t, and r is some constant discount rate. We can re-write the above V_t as follow:

$$V_t = E_t\left[\int_t^{\infty} C_s e^{-r(s-t)} ds\right] \qquad (1)$$

$$V_t = E_t\left[e^{rt}\int_0^{\infty} C_s e^{-rs} ds\right]$$

$$V_t e^{-rt} = E_t\left[\int_t^{0} C_s e^{-rs} d_s + \int_0^{\infty} C_s e^{-rs} ds\right]$$

$$V_t e^{-rt} = E_t\left[\int_0^{\infty} C_s e^{-rs} d_s - \int_0^{t} C_s e^{-rs} ds\right]$$

Thus,

$$V_t e^{-rt} = E_t\left[\int_0^{\infty} C_s e^{-rs} d_s\right] - \int_0^{\infty} C_s e^{-rs} ds \qquad (2)$$

Assuming $E\left[\left|\int_0^{\infty} C_s e^{-rs} ds\right|\right] < \infty$, then this is a decomposition where the discounted present value is the sum of a uniformly integrable martingale and a predictable process. If we let M denote the martingale then (2) becomes

$$d(V_t e^{-rt}) = dM_t - C_t e^{-rt} dt$$

Using differential rule for products, the dynamic of the present value V, is given as

$$e^{-rt} dV_t + V_t de^{-rt} = dM_t - C_t e^{-rt} \mathrm{dt}$$

$$e^{-rt} dV_t - re^{-rt} V_t dt = dM_t - C_t e^{-rt} \mathrm{dt}$$

$$\Rightarrow \quad dV_t - rV_t dt = e^{rt} dM_t - C_t \mathrm{dt}$$

$$dV_t = e^{rt} dM_t + (rV_t - C_t) dt$$

Let $\Lambda_t = e^{-rt}$ then (1) becomes

$$V_t = E_t\left[\int_t^{\infty} C_s \frac{\Lambda_s}{\Lambda_t} ds\right]$$

Thus,

$$V_t \Lambda_t = E_t\left[\int_t^{\infty} C_s \Lambda_s ds\right]$$

for $h > 0$, we have

$$V_t \Lambda_t = E_t\left[\int_t^{t+h} C_s \Lambda_s ds + \int_{t+h}^{\infty} C_s \Lambda_s ds\right]$$

$$\Rightarrow \quad V_t \Lambda_t = E_t\left[\int_t^{t+h} C_s \Lambda_s ds + \int_{t+h}^{\infty} C_s \Lambda_s ds\right]$$

$$0 = E_t\left[\int_t^{t+h} C_s \Lambda_s ds + V_{t+h} \Lambda_{t+h} - V_t \Lambda_t\right]$$

$$0 = E_t\left[\int_t^{t+h} C_s \Lambda_s ds\right] + E_t[V_{t+h} \Lambda_{t+h} - V_t \Lambda_t]$$

let $h \to 0$ then

$$0 = C_t \Lambda_t dt + E_t[d(V_t \Lambda_t)]$$

The introduction of Λ_t is to allow for more general discount factors, especially stochastic ones. Also, we want to generalize the cash flows, allowing process of finite variation as integrators with which we integrate deflator.

1.1 Preamble

Let $(\Omega, \mathcal{F}_t, (\mathcal{F}_t)\, P)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$. The filtration is assumed to be right continuous and complete. Any adapted process will be adapted with respect to the filtration $\mathcal{F}_t$. Let

$\mathcal{F}_\infty$ be σ-algebra $V_t \geq 0\, \mathcal{F}$, and assume that $\mathcal{F}_\infty = \mathcal{F}$. Thus a process is cadlag, if almost every samples path of the process is right continuous with left limits.

A process where path are a.s positive increasing and right continuous is said to be an increasing process. An increasing function has left limits and thus any increasing process is endlag.

For every increasing process A we use $A_0 - = 0$ a.s but not necessary that $A_0 = 0$ a.s. A process is of finite variation if it is cadlag and adapted and if almost every sample path is of finite variation on each compact subset of $[0, \infty)$.

A process is of finite variation if it is the difference between two increasing processes. A process X is said to be optional if the mapping $X: [0, \infty) \rightarrow \mathcal{R}$ is measureable whenever $[0, \infty) \times \Omega$. is given as the optional σ $-$algebra.

Since the optional σ $-$algebra is generated by the family of all adapted process which are cadlag. Thus, every finite variation process (FV), A is optional. Assuming A is a finite variation process and X is a real valued process on $[0, \infty) \times \Omega$ that is $B \times \mathcal{F}$ - measurable (where $\mathcal{B}$ denotes the Borel σ $-$algebra on $[0, \infty)$ then stiettjes integral is defined as

$$(X \cdot A)_t(\omega) = \int_{[0,t]} X_s(\omega)\, dA_s(\omega)$$

whenever it exists. If $X \cdot A$ exists for all $t \in [0, \infty)$ and almost all $\omega \in \Omega$ then $(X.A)_t$ defines a process with $(X.A)_0 = X_0 A_0$. If X is optional, then there is an optional version of $(X.A)$. If A is an FV process, then $|A|_t(\omega) = \int_0^t |dA_s(\omega)|$ denotes the total variation of A. $|A|$ is adapted and cadlag and $|A|_0 = |A|_0$. If A is an FV process and X is measurable process then the integral $\int_0^t X_s(\omega)\, d\,|\, A_s(\omega)|$ denotes integration with respect to $d|A|$. A seminartingale is an adapted and cadlag process $(X_t)_{t \geq 0}$ having a decomposition $X_t = X_0 + M_t + A_t$ where M is a local martingale and A is an FV process.

II. Cash flows and deflators

Definition 1: *A cash flow process is a* $C_t(\omega)_{t \geq 0}$ *is a finite variation process.*

This definition of the cash flow process makes it trivially a seminartingale. It can also be used as integrator.

Definition 2: *A deflator is a strictly positive seminartingale that is finite a.s.*

Deflator is considered to be a semimartingale in order to make differentiation rule valid for semimartingales. It is noted that if Λ is a deflator then both $1/\Lambda$ and $ln\Lambda$ are well defined, and since $1/x$ and $In\ x$ are twice continuously differentiable on

(0, ∞), both $1/\Lambda$ and $In\ \Lambda$ are semiartingales

Definition 3: *Given a deflator* Λ, *the discount process implied by* Λ *is defined by*

$$m(s,t) = \frac{\Lambda(t)}{\Lambda(s)} \qquad s, t \geq 0$$

The proposition below presents some important properties of the discount process. It is easier to work with deflator as defined above in continuous time.

Proposition 1: *Let m be a discount process implied by the deflator* Λ. *Then*

i. $m(s,t,\omega)\ is\ \mathcal{F}_{\max(s,t)} - measurable\ for\ every\ s,t \in [0, \infty]$
ii. $0 < m(s,t) < \infty\ a.s\ for\ every\ s,t \in [0, \infty)$
iii. $m(s,t) = m(s,u)\ m(u,t)\ a.s\ for\ every\ 0 \leq s \leq u \leq t$

A discount process fulfilling $0 < m(s,t) < 1$ a.s. for every $s,t \in [0, \infty)$ will be termed as normal discount process. Hence, m is normal discount process. Thus, m is normal if and only if Λ is non-decreasing. Thus, a discount process m, with deflator Λ can be written in the form

$$m(s,t,\omega) = \exp\left(-\int_s^t \lambda(\omega) du\right)$$

If and only if $In\ \Lambda(t, \omega)$ is absolutely continuous in t for almost every $\omega \in \Omega$ with density $-\lambda(t, \omega)$.

III. Valuation

Definition 4: *Given a cash flow process* C *and a deflator* Λ *such that* $E[\int \Lambda_s \mid dc_s \mid] < \infty$ *the value process is defined for* $t \in [0, \infty)$ *as*

$$V_t = \frac{1}{\Lambda_t} E\left[\int_{(0,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t\right]$$

By noting that $\int_{[0,\infty)} = \int_{[0,t]} + \int_{(t,\infty)}$ $and\ using\ the\ fact\ that\ every\ optimal\ we$ have from

$$V_t \Lambda_t = E\left[\int_{[t,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t\right]$$

$$V_t\Lambda_t = E[\int_{[0,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t] - E[\int_{[0,t]} \Lambda_s dC_s \mid \mathcal{F}_t]$$
$$= E[\int_{[0,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t] - \int_{[0,t)} \Lambda_s dC_s$$
$$= M_t - (\Lambda \cdot C)_t \qquad (3)$$

Since $E \mid M_t \mid = E \mid E[\int_{[0,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t] \mid \leq E[\int_{[0,\infty)} \Lambda_s \mid dC_s \mid \mathcal{F}_t] < \infty$

for every $t \in [0,\infty)$, M_t is a uniformly integrable martingale. The filtration $(\mathcal{F}_t)$ is right continuous, thus there exists a modification of M that is right continuous. Recall, $M_\infty = \lim_{t\to\infty} M_t = \int_0^\infty \Lambda dC_s$ a.s. Since M and $\Lambda \cdot C$ are right continuous and adapted the value process is also continuous and adapted. Thus, the value process is optional. From (3) we have

$$V_t = \frac{M_t}{\Lambda_t} - \frac{(\Lambda \cdot C)_t}{\Lambda_t}$$

Since M is a (true) Martingale, it is especially a semimartingale. $\Lambda \cdot C$ is a process of finite variation and is thus also a seminartingale. Since Λ is a strictly positive seminartingale hence $1/\Lambda$ is also a strictly positive seminartingale and since the product of two seminartingales is a seminmartingale. Hence, V is a semimartingale.

As in discrete time case there exists three equivalent ways of writing value process. Assumption made here are measurability condition on C and Λ and integrability condition making M into a uniformly integrable martingale. In addition to this, there must be a condition stating that the discounted value goes to zero as t tends to infinity (∞) i.e $V_t m_t \to 0$ as $t \to \infty$ a.s.

Theorem 2: *Let C and Λ be a cash flow process and deflator respectively such that* $E[\int_{[0,\infty)} \Lambda_s \mid dC_s \mid] < \infty$ *Then the following three statement are equivalent*

(i) *for every* $t \in [0,\infty)$

$$V_t = \frac{1}{\Lambda_t} E[\int_{(t,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t] \qquad (4)$$

(ii)(a) *for every* $t \in [0,\infty)$

$$M_t = V_t\Lambda_t + \int_{[0,t]} \Lambda_s dC_s \qquad (5)$$

is a uniformly integrable martingale and

(b) $V_t\Lambda_t \to 0$ *a.s when* $t \to \infty$

(iii) *for each* $t \in [0,\infty)$ *we have*

(a) *for every* $h > 0$

$$V_t\Lambda_t = E[V_{t+h}\Lambda_{t+h} + \int_{(t,t+h]} \Lambda_s dC_s \mid \mathcal{F}_t] \qquad and$$

(b) $\lim_{T\to\infty} E[V_{t+T}\, \Lambda_{t+T} \mid \mathcal{F}_t] = 0$

Proof:

We begin by showing that (i) ⇔ (ii) and (i) ⇔ (iii)

(i) ⇔ (ii) To prove the if part

(ii) $V_t = \frac{1}{\Lambda_t} E[\int_{(t,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t]$ Using the fact that

$\int_{[0,\infty)} = \int_{[0,t]} + \int_{(t,\infty)}$ *Thus,*

$$V_t = \frac{1}{\Lambda_t} E[\int_{[0,\infty)} \Lambda_s dC_s - \frac{1}{\Lambda_t}\int_{[0,t]} \Lambda_s dC_s \mid \mathcal{F}_t]$$
$$V_t = \frac{1}{\Lambda_t} E[\int_{[0,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t] - \frac{1}{\Lambda_t}\int_{[0,t]} \Lambda_s dC_s$$

Note that $E[\int_{(0,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t] = M_t$ *It implies that*

$$V_t = \frac{1}{\Lambda_t} M_t - \frac{1}{\Lambda_t}\int_{[0,t]} \Lambda_s dC_s$$
$$V_t\Lambda_t = M_t - \int_{[0,t]} \Lambda_s dC_s$$
$$M_t = V_t\Lambda_t + \int_{[0,t]} \Lambda_s dC_s$$

Since M is uniformly integrable martingale and the fact that

$M_t \to M_\infty = \int_0^\infty \Lambda_s dC_s$ a.s as $t \to \infty$

Thus,

$\lim_{t\to\infty} V_t \Lambda_t = \lim_{t\to\infty}(M_t - \int_{[0,t]} \Lambda_s dC_s) = 0 \quad a.s$

Turning to the only if 'part'. That is (ii) ⇒ (i). Let $t \to \infty$ in equation (5)

As $t \to \infty$, $M_t \to M_\infty$, $V_t\Lambda_t \to 0$

Thus, $M_\infty = \int_{[0,\infty)} \Lambda_s dC_s$

Taking conditional expectation with respect to the $\sigma -$ algebra $\mathcal{F}_t$

i.e. $E[M_\infty \mid \mathcal{F}_t] = E[\int_{[0,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t]$

$$E[M_\infty \mid \mathcal{F}_t] = M_t = E[\int_{[0,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t]$$

$$= E[\int_{[0,t]} \Lambda_s dC_s + \int_{(t,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t]$$

$$M_t = \int_{[0,t]} \Lambda_s dC_s + E[\int_{(t,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t]$$

Thus,

$$V_t\Lambda_t + \int_{[0,t]} \Lambda_s dC_s = M_t = \int_{[0,t]} \Lambda_s dC_s + E[\int_{(t,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t]$$

$$V_t\Lambda_t = E[\int_{(t,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t]$$

which is our desire result.

(i) ⇔ (ii). For 'if' part take $h > 0$. From equation (4)

$$V_t\Lambda_t = E[\int_{(t,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t]$$

$$= E[\int_{(t,t+h)} \Lambda_s dC_s + \int_{(t+h,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t]$$

$$= E[\int_{(t,t+h)} \Lambda_s dC_s + E[\int_{(t+h,\infty)} \Lambda_s dC_s \mid \mathcal{F}_{t+h}] \mid \mathcal{F}_t]$$

$$= E[\int_{(t,t+h)} \Lambda_s dC_s + V_{t+h}\Lambda_{t+h} \mid \mathcal{F}_t]$$

Since

$\mid \int_{(t,\infty)} \Lambda_s dC_s \mid \le \int_{(t,\infty)} \Lambda_s |dC_s| \le \int_{[0,\infty)} \Lambda_s |dC_s|$

and $\int_{[0,\infty)} \Lambda_s |dC_s|$ is integrable, thus, using the dominated convergence theorem to get for every $A \in \mathcal{F}_t$

$$\lim_{T\to\infty} E[V_{t+T}\Lambda_{t+T} 1_A] = \lim_{T\to\infty} E[\int_{(t+T,\infty)} \Lambda_s dC_s 1_A]$$

$$= \lim_{T\to\infty} E[\int_{(t+T,\infty)} \Lambda_s dC_s 1_A] = 0$$

where the last equality follows from the fact that $\int_{[0,\infty]} \Lambda_s dC_s$ is finite a.s.

To prove the other direction of the equivalence for $T > 0$ from

$$V_t\Lambda_t = E[\int_{(t,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t] = E[\int_{(t,t+T)} \Lambda_s dC_s + \int_{(t+T,\infty)} \Lambda_s dC_s \mid \mathcal{F}_t]$$

$$= E[\int_{(t,t+T)} \Lambda_s dC_s + E[\int_{(t+T,\infty)} \Lambda_s dC_s \mid \mathcal{F}_{t+T}] \mid \mathcal{F}_t]$$

$$= E[\int_{(t,t+T)} \Lambda_s dC_s + V_{t+T}\Lambda_{t+T} \mid \mathcal{F}_t]$$

Taking limit as $T \to \infty \quad a.s$

$$V_t\Lambda_t = \lim_{T\to\infty} E[V_{t+T}\Lambda_{t+T} \mid \mathcal{F}_t] + \lim_{T\to\infty} E[\int_{(t,t+T)} \Lambda_s dC_s \mid \mathcal{F}_t]$$

Using the theorem of dominated convergence we have

$$V_t\Lambda_t = \lim_{T\to\infty} E[\int_{(t,t+T)} \Lambda_s dC_s \mid \mathcal{F}_t]$$

IV. Local dynamics of the value process

To begin on the local dynamics of the value process, we start from the relation

$V_t\Lambda_t = M_t - (\Lambda \cdot C)_t$

Since all the process in the above expression are seminartingales thus differentiation rule for product of seminartingale can therefore be used.

Hence, we have from

$$d(\Lambda_t V_t) = V_t d\Lambda_t + \Lambda_t dV_t + d[\Lambda, V]_t \qquad (6)$$

Note that

$$d(V_t\Lambda_t) = dM_t - \Lambda_t dC_t \qquad (7)$$

$$\Rightarrow \quad dM_t + \Lambda_t dC_t = V_{t^-} d\Lambda_t + \Lambda_{t^-} dV_t + d[V, \Lambda]_t \qquad (8)$$

Note also that, $\Delta\Lambda_t \Delta C_t = \Lambda_t dC_t - \Lambda_{t^-} dC_t$

$$\Lambda_t dC_t = \Delta\Lambda_t \Delta C_t + \Lambda_{t^-} dC_t \qquad (9)$$

Divide equation (6) by Λ_{t^-} we get

$$\frac{1}{\Lambda_{t^-}} d[V_t\Lambda_t] = V_{t^-} \frac{d\Lambda_t}{\Lambda_{t^-}} + dV_t + \frac{d[V,\Lambda]_t}{\Lambda_{t^-}}$$

$$dV_t = V_{t^-}\left(-\frac{d\Lambda_t}{\Lambda_{t^-}}\right) + \frac{1}{\Lambda_{t^-}} d[V_t\Lambda_t] - \frac{1}{\Lambda_{t^-}} d(V, \Lambda)_t \qquad (10)$$

Substitute for (7) and (9) in (10)

$dV_t = V_{t^-}\left(-\frac{d\Lambda_t}{\Lambda_{t^-}}\right) + \frac{1}{\Lambda_{t^-}}(dM_t - \Lambda_t dC_t) - \frac{1}{\Lambda_{t^-}}d(V,\Lambda)_t$
$dV_t = V_{t^-}\left(-\frac{d\Lambda_t}{\Lambda_{t^-}}\right) + \frac{1}{\Lambda_{t^-}}(dM_t - \Delta\Lambda_t\Delta C_t - \Lambda_{t^-}dC_t) - \frac{1}{\Lambda_{t^-}}d(V,\Lambda)_t$
$dV_t = V_{t^-}\left(-\frac{d\Lambda_t}{\Lambda_{t^-}}\right) - dC_t + \frac{1}{\Lambda_{t^-}}(dM_t - \Delta\Lambda_t\Delta C_t) - \frac{1}{\Lambda_{t^-}}d(V,\Lambda)_t$
Let $dR_t = -\frac{d\Lambda_t}{\Lambda_{t^-}}$ $\quad and\ N_t = \int_0^t \frac{1}{\Lambda_{s^-}}dM_s$
$dV_t = V_{t^-}dR_t - dC_t + dN_t - \frac{1}{\Lambda_{t^-}}(\Delta\Lambda_t\Delta C_t + d[V,\Lambda])_t$
Λ becomes the stochastic exponential of $\mathcal{R}$ if $\Lambda_o = 1$. Hence R can be referred to as discount rate associated with Λ. For an improvement of economical interpretation of equation (6) note that if given a cash flow $r_t^f\,dt$ for $t \in [0,\infty)$, where r_t^f is measurable and adapted and such that for almost everywhere $\omega \in \Omega$ we have $0 \le r_t^f(\omega)$ for every $t \in [0,\infty)$ and if the value process of this cash flow stream fulfills $V_t \equiv 1$, Then r_t^f can be referred to as a locally risk free interest rate if all these specification are inserted into equation (7)
We have from
$d(V_t\Lambda_t) = dM_t - \Lambda_t dC_t$

$$d\Lambda_t = dM_t - \Lambda_t\, r_t^f\, dt$$

$$r_t^f\,dt = -\frac{d\Lambda_t}{\Lambda_{t^-}} + \frac{1}{\Lambda_t}dM_t$$

Where M^* is a martingale. Thus, if the deflator Λ assigns the cash flow stream given by $r_t^f\,dt$ the value 1 for all $t \in [0,\infty)$, then in using the same Λ for valuing another cash flow stream C, the differential of V can be expressed in terms of the risk free rate.
From $M_t = E[\int_0^\infty \Lambda_s dC_s \mid \mathcal{F}_t]$ inserting $r_t^f\,dt$ we have

$$M^* = E[\int_0^\infty r_t^f\,\Lambda_s dC_s \mid \mathcal{F}_t]$$

With $V_t \equiv 1$ and inserting r_t^f in equation (7) and integrate we have
$\Lambda_t = E[\int_{(0,\infty)} r_t^f\,\Lambda_s dC_s \mid \mathcal{F}_t] - \int_{[0,t]}\Lambda_s r_t^f\,ds$

$$\Lambda_t = M^* - \int_{[0,t]}\Lambda_s r_t^f\,d\,s \qquad (11)$$

Since $r_t^f \ge 0$ a.s. thus $A_t = \int_{[0,t]}\Lambda_s r_t^f\,d\,s$ is an increasing process. That means Λ is a non-negative supermartingale. An adapted and cadlag process is said to be potential if it is non-negative super martingale that tends to 0 a. s. as $t \to \infty$
Since
$E[\Lambda_t] = E[\int_0^\infty r_s^f\,\Lambda_s dC_s \mid \mathcal{F}_t] - \int_0^t E[r_s^f\,\Lambda_s]ds \to 0 \quad a.s$
As $t \to \infty$ then the proposition below is proved.
Proposition 3: *If there exists a risk-less asset, then the deflator pricing this asset is a potential.*
Equation (11) can be termed equation for deflator Λ. If the filtration $\mathcal{F}_t$ is generated by a Brownian motion, then every square integrable martingale can be written as Ito integral. Here, one more asset priced by Λ is needed to determine Λ.
To express the differential of V in terms of the rate r_t^f. Assuming Λ is a continuous process then
$V_{t^-}d\Lambda_t + \Lambda_{t^-}dV_t + d(V,\Lambda)_t = dM_t - \Lambda_t dC_t$
$dV_t = V_{t^-}\left(-\frac{d\Lambda_t}{\Lambda_t}\right) - dC_t - \frac{1}{\Lambda_t}d(V,\Lambda)_t + \frac{1}{\Lambda_t}dM_t$

Substitute for $d\Lambda_t = dM_t^* - r_t^f\,\Lambda_t dt$

$$dV_t = \frac{-V_t}{\Lambda_t}dM_t^* + V_t r_t^f\,dt - dC_t - \frac{1}{\Lambda_t}d(V,\Lambda)_t + \frac{1}{\Lambda_t}dM_t$$

$dV_t + dC_t = V_t r_t^f\,dt - \frac{1}{\Lambda_t}d(V,\Lambda)_t + \frac{1}{\Lambda_t}dM_t - \frac{V_t}{\Lambda_t}dM_t^*$
Thus, $\frac{dV_t + dC_t}{V_t} = r_t^f\,dt - \frac{1}{V_t\Lambda_t}d(V,\Lambda)_t + \frac{V_t}{\Lambda_t}dM_t - \frac{1}{\Lambda_t}dM_t^*$
Taking conditional expectations
$E\left[\frac{dV_t + dC_t}{V_t}\,\middle|\,\mathcal{F}_t\right] = r_t^f\,dt - E[\frac{d(V,\Lambda)_t}{V_t\Lambda_t}]$
Assuming V is strictly positive and write $d(V,\Lambda)_t = dV_t d\Lambda_t$
Then $E\left[\frac{dV_t + dC_t}{V_t}\,\middle|\,\mathcal{F}_t\right] = r_t^f\,dt - E[\frac{dV_t}{V_t}\frac{d\Lambda_t}{\Lambda_t} \mid \mathcal{F}_t]$

The left handside is known as the instantaneous net return of the value process at time t. The expected return of the value process has now been decomposed into risk – free part $r_t^f\,dt$ and risk premium part $-E\left[\frac{dV_t}{V_t}\frac{d\Lambda_t}{\Lambda_t}\,\middle|\,\mathcal{F}_t\right]$. If $\frac{dV_t}{V_t}$ *and* $\frac{d\Lambda_t}{\Lambda_t}$ are negatively correlated, then there is a positive risk premium and if they are positively correlated then the risk premium is negative. Since the of a risky investment giving us the cash flow C is expected to have a return strictly greater than the risk-free rate, it is seen that $\frac{dV_t}{V_t}$ *and* $-\frac{d\Lambda_t}{\Lambda_t}$ is expected to be positively correlated. The intuition is that a risky investment is desirable if its value is high in 'bad' states of the world (when we really need money) and low in 'good' states of the economy (where everything else is good). An investment with such properties will have a high price (since demand for this desirable investment opportunity is high) and thus, a low expected return. Hence, $\frac{d\Lambda_t}{\Lambda_t}$ is interpreted as a measure of how 'bad' a state of economy is.

To solve equation (11) when the risk – less rate is equal to the constant $r_f > 0$. Inserting this in equation (4) of theorem 2.

We have for $V_t \equiv 1$

$$\Lambda_t = E[\int_{(0,\infty)} r_f \Lambda_s ds \mid \mathcal{F}_t] \qquad (12)$$

This is regard as stochastic differential equation for discount factor Λ. substituting for $\Lambda_s = e^{-r_f s}$ on the RHS

$$E[\int_t^{\infty} r_f e^{-r_f s} ds \mid \mathcal{F}_t] = [e^{-r_f s}]_t^{\infty} = e^{-r_f s} = \Lambda_t$$

Hence Λ is also equivalent to equation (12).

Assuming $K \epsilon\, IR$, then also $Ke^{-r_f s}$ is a solution. The proposition below shows that there exist other solution to equation (12)

Proposition 3: *Let $r_f > 0$ be a given real number and consider*

$$X_t = E[\int_{(t,\infty)} r_f X_s ds \mid \mathcal{F}_t] \qquad (13)$$

Then X is a solution to (13) if and only if it can be written as

$$X_t = e^{-r_f t}(X_o + \int_0^t e^{-r_f s} dMs \mid \mathcal{F}_t] \qquad (14)$$

for some uniformly integrable martingale M.

The following Lemma is needed for the prove

Lemma 4: *Let N be a local martingale that is cadlag and fulfils $N_o = 0$ a.s and let U be a predictable increasingly process with $U_o > 0$ a.s if*

$\int_0^t (\frac{1}{U_s}) dN_s$ converges a.s as $t \to \infty$ and the limit is finite, then $\frac{N_t}{U_t} \to \infty$ a.s as $t \to \infty$ on $\{U_\infty = \infty\}$

Now back to the proof of the above proposition.

Proof: consider a discount factor Λ of the form given in equation (14) and define the local martingale as

$N_t = \int_0^t e^{-r_f s} dM_s$ *with N and* $U_t = e^{-r_f t}$ *we have*

$dN_t = e^{-r_f t} dM_t$

$dM_t = \frac{1}{e^{-r_f t}} dN_t$

$M_t = \int_0^t (\frac{1}{U_s}) dN_s$

As $t \to \infty$, $\int_0^t \left(\frac{1}{U_s}\right) dN_s = M_t \xrightarrow{a.s} M_\infty \epsilon\, \mathcal{L}^1$

Since $P(U_\infty = \infty) = 1$ the Lemma above can be concluded that

$e^{-r_f t} N_t \to \infty$ *a.s, from* $\frac{N_t}{U_t} \to 0$

With the use of differentiation rule for products

$d[e^{-r_f t}(\Lambda_o + N_t)] = -r_f\, t e^{-r_f t}(\Lambda_o + N_t)dt + dM_t$

Using integration by parts

$\int_t^T r_f e^{-r_f s}(\Lambda_o + N_s)ds = [-e^{-r_f s}(\Lambda_o + N_s)]_t^T + [M_t]_t^T$

$= M_t - M_t - e^{-r_f T}(\Lambda_o + N_T) + e^{-r_f t}(\Lambda_o + N_s)$

as $T \to \infty$, $e^{-r_f T}(\Lambda_o + N_T) \to 0$ *a.s Then we have*

$\int_t^{\infty} r_f e^{-r_f s}(\Lambda_o + N_s)ds = M_\infty - M_t + e^{-r_f t}(N_o + N_s)$

$\Rightarrow \; e^{-r_f t}(\Lambda_o + N_s) = \int_t^{\infty} r_f e^{-r_f s}(\Lambda_o + N_s)ds - M_\infty + M_t$

Taking conditional expectation with respect to $\mathcal{F}_t$

$E[e^{-r_f t}(\Lambda_o + N_t) \mid \mathcal{F}_t] = E[\int_t^\infty r_f e^{-r_f s}(\Lambda_o + N_s)ds \mid \mathcal{F}_t] - E[M_\infty \mid \mathcal{F}_t] + E[M_t \mid \mathcal{F}_t]$

$e^{-r_f t}(\Lambda_o + N_s) = E[\int_t^\infty r_f e^{-r_f s}(\Lambda_o + N_s)ds \mid \mathcal{F}_t] - M_t + M_t$

$e^{-r_f t}(\Lambda_o + N_s) = E[\int_t^\infty r_f e^{-r_f s}(\Lambda_o + N_s)ds \mid \mathcal{F}_t]$

$Or \quad \Lambda_t = E[\int_0^\infty r_f(\Lambda_s)ds \mid \mathcal{F}_t]$

By letting $t = 0$

$$E[\mid \int_t^\infty r_f e^{-r_f s}(\Lambda_o + N_s)ds \mid] \leq \mid \Lambda_o \mid + E[\mid M_\infty \mid] + \mid M_o \mid < \infty$$

In conclusion $re^{-r_f t}(\Lambda_o + N_t)$ is a solution.

Assuming Λ is a solution

$\Lambda_t = E[\int_t^\infty r_f(\Lambda_s)ds \mid \mathcal{F}_t]$ is written as

$\Lambda_t = E[-\int_0^t r_f(\Lambda_s)ds + \int_0^\infty r_f(\Lambda_s)ds \mid \mathcal{F}_t]$

$\Lambda_t = E[\int_0^\infty r_f(\Lambda_s)ds \mid \mathcal{F}_t] - \int_0^t r_f(\Lambda_s)ds$

Introducing U. I martingale $M_t = E[\int_0^\infty r_f(\Lambda_s)ds \mid \mathcal{F}_t]$ we have

$\Lambda_t e^{r_f t} = \Lambda_o + \int_0^t r_f \Lambda_s e^{r_f s} ds$

which is our desired result.

V. Conclusion:

We have been able to write out the three equivalent ways of cash flow each of which has its own merit. While considering the local dynamics of the value process it's discovered that to have a locally risk free interest rate the value process of the cash flow stream must fulfill $V_t = 1$

REFERENCES

Armerin, F. (2002), 'Valuation of Cash Flows in Continuous Time', *Working paper*

Bias, B., et al (1997), 'Financial Mathematics', Lecture Notes in Mathematics, 1656, *Springer – Verlag*

Delbean, F., et al (1994), 'A General Version of the Fundamental Theorem of Asset Pricing', *Math. Annalen*, Vol. 300, 463-520

Duffie, D., et al (1995), 'Black's consol rate conjecture', *The Annals of Applied Probability*, Vol. 5, No. 2, 356-382

Harrison, J.M., et al (1979), 'Martingales and Arbitrage in Multi-period Securities Markets', *Journal of Economic Theory*

Hubalek F., (2001), 'The Limitations of No-Arbitrage Arguments for Real Options', *International Journal of Theoretical and Applied Finance* Vol. 4, No. 2